Positive Scalar Curvature on Foliations: The Enlargeability

Weiping Zhang

Abstract. We generalize the famous result of Gromov and Lawson on the nonexistence of metric of positive scalar curvature on enlargeable manifolds to the case of foliations, without using index theorems on noncompact manifolds.

Mathematics Subject Classification (2010). 57R30, 58J20.

Keywords. Foliation, positive scalar curvature, enlargeability, relative index theorem.

0. Introduction

It has been an important subject in differential geometry to study when a smooth manifold carries a Riemannian metric of positive scalar curvature (cf. [8, Chap. IV]). A famous result of Gromov and Lawson [6], [7] states that an enlargeable manifold (in the sense of [7, Definition 5.5]) does not carry a metric of positive scalar curvature. In particular, there is no metric of positive scalar curvature on any torus, which is a classical result of Schoen–Yau [10] and Gromov–Lawson [6]. A generalization to foliations of the Schoen–Yau and Gromov–Lawson result on torus has been given in [11, Corollary 0.5]. In this paper, we further extend the above result of Gromov–Lawson on enlargeable manifolds to the case of foliations.

Let F be an integrable subbundle of the tangent vector bundle TM of a closed smooth manifold M. Let g^F be a Euclidean metric on F, and $k^F \in C^\infty(M)$ be the associated leafwise scalar curvature (cf. [11, (0.1)]). For any covering manifold $\pi : \widetilde{M} \to M$, one has a lifted integrable subbundle with metric $(\widetilde{F}, g^{\widetilde{F}}) = (\pi^* F, \pi^* g^F)$.

Definition 0.1. One calls (M, F) an enlargeable foliation if for any $\varepsilon > 0$, there is a covering manifold $\widetilde{M} \to M$ and a smooth map $f : \widetilde{M} \to S^{\dim M}(1)$ (the standard unit sphere), which is constant near infinity and has non-zero degree, such that for any $X \in \Gamma(\widetilde{F})$, $|f_*(X)| \leq \varepsilon |X|$.

When $F = TM$ and M is spin, this is the original definition of the enlargeability of M due to Gromov and Lawson [6], [7].

The main result of this paper can be stated as follows.

Theorem 0.2. *Let (M, F) be an enlargeable foliation. Then*

(i) *if TM is spin, then there is no g^F such that $k^F > 0$ over M;*

(ii) *if F is spin, then there is no g^F such that $k^F > 0$ over M.*

When $F = TM$, one recovers the classical theorem of Gromov–Lawson [6], [7] mentioned at the begining. In a recent paper [2], Benameur and Heitsch proved Theorem 0.2(ii) under the condition that (M, F) has a Hausdorff homotopy groupoid.

As a direct consequence of Theorem 0.2(i), one obtains an alternate proof, without using the families index theorem, of [11, Corollary 0.5] mentioned above (for the special case where the integrable subbundle on torus is spin, this result is due to Connes, as was stated in [5, p. 192]).

If M is enlargeable and (M, F) carries a transverse Riemannian structure, then Theorem 0.2(i) is trivial, as in this case, if there is g^F with $k^F > 0$ over M, then one can construct g^{TM} with $k^{TM} > 0$ over M, which contradicts with the Gromov–Lawson theorem. Thus, the main difficulty for Theorem 0.2 is that there might be no transverse Riemannian structure on (M, F). This is similar to what happens in [4] and [11], where one adapts the Connes fibration constructed in [4] to overcome this kind of difficulty.

Recall that we have proved geometrically in [11] that if M is oriented and there exists g^F with $k^F > 0$ over M, then under the condition that either TM or F is spin, one has $\widehat{A}(M) = 0$. The case where F is spin is a famous result of Connes [4, Theorem 0.2].

Our proof of Theorem 0.2 combines the methods in [6], [7] and [11]. It is based on deforming (twisted) sub-Dirac operators on the Connes fibration over $\widetilde{M}$. A notable difference with respect to [7], where the relative index theorem on noncompact manifolds plays an essential role, is that we will work with compact manifolds even for the noncompactly enlargeable situation. It will be carried out in Section 1.

1. Proof of Theorem 0.2

In this section, we first prove in Section 1.1 the easier case where (M, F) is a compactly enlargeable foliation, i.e., the covering manifold $\widetilde{M}$ in Definition 0.1 is compact. Then in Section 1.2 we show how to extend the arguments in Section 1.1 to the case where $\widetilde{M}$ is noncompact.

1.1. The case of compactly enlargeable foliations

Let F be an integrable subbundle of the tangent bundle TM of an oriented closed manifold M.

Let g^F be a metric on F and k^F be the scalar curvature of g^F. Let (E, g^E) be a Hermitian vector bundle on M carrying a Hermitian connection ∇^E.

Let $R^E = (\nabla^E)^2$ be the curvature of ∇^E.

For any $\varepsilon > 0$, we say (E, g^E, ∇^E) verifies the leafwise ε-condition if for any $X, Y \in \Gamma(F)$, the following pointwise formula holds on M,

$$\left| R^E(X,Y) \right| \leq \varepsilon \, |X| \, |Y|. \tag{1.1}$$

The following result extends slightly [11, Theorem 0.1] and [4, Theorem 0.2].[1]

Theorem 1.1. *If $k^F > 0$ over M and either TM or F is spin, then there exists $\varepsilon_0 > 0$ such that if (E, g^E, ∇^E) verifies the leafwise ε_0-condition, then*

$$\langle \widehat{A}(TM)\mathrm{ch}(E), [M] \rangle = 0.$$

Proof. The proof of this theorem is an easy modification of the proof given in [11] for the case of $E = \mathbf{C}|_M$. We only give a brief description, by following the notations given in [11]. Let $\delta > 0$ be such that $k^F \geq \delta$ over M. Without loss of generality, we may well assume that $\dim M$, $\mathrm{rk}(F)$ are divisible by 4, and that TM, F and TM/F are oriented with compatible orientations.

We assume first that TM is spin.

Following [4, §5] (cf. [11, §2.1]), let $\pi : \mathcal{M} \to M$ be the Connes fibration over M such that for any $x \in M$, $\mathcal{M}_x = \pi^{-1}(x)$ is the space of Euclidean metrics on the linear space $T_x M / F_x$. Let $T^V \mathcal{M}$ denote the vertical tangent bundle of the fibration $\pi : \mathcal{M} \to M$. Then it carries a natural metric $g^{T^V \mathcal{M}}$.

By using the Bott connection on TM/F, which is leafwise flat, one lifts F to an integrable subbundle $\mathcal{F}$ of $T\mathcal{M}$. Then g^F lifts to a Euclidean metric $g^{\mathcal{F}} = \pi^* g^F$ on $\mathcal{F}$.

Let $\mathcal{F}_1^\perp \subseteq T\mathcal{M}$ be a subbundle, which is transversal to $\mathcal{F} \oplus T^V \mathcal{M}$, such that we have a splitting $T\mathcal{M} = (\mathcal{F} \oplus T^V \mathcal{M}) \oplus \mathcal{F}_1^\perp$. Then $\mathcal{F}_1^\perp$ can be identified with $T\mathcal{M}/(\mathcal{F} \oplus T^V \mathcal{M})$ and carries a canonically induced metric $g^{\mathcal{F}_1^\perp}$. We denote $\mathcal{F}_2^\perp = T^V \mathcal{M}$.

Let $\mathcal{E} = \pi^* E$ be the lift of E which carries the lifted Hermitian metric $g^{\mathcal{E}} = \pi^* g^E$ and the lifted Hermitian connection $\nabla^{\mathcal{E}} = \pi^* \nabla^E$. Let $R^{\mathcal{E}} = (\nabla^{\mathcal{E}})^2$ be the curvature of $\nabla^{\mathcal{E}}$.

For any β, $\varepsilon > 0$, following [11, (2.15)], let $g_{\beta,\varepsilon}^{T\mathcal{M}}$ be the metric on $T\mathcal{M}$ defined by the orthogonal splitting,

$$T\mathcal{M} = \mathcal{F} \oplus \mathcal{F}_1^\perp \oplus \mathcal{F}_2^\perp, \qquad g_{\beta,\varepsilon}^{T\mathcal{M}} = \beta^2 g^{\mathcal{F}} \oplus \frac{g^{\mathcal{F}_1^\perp}}{\varepsilon^2} \oplus g^{\mathcal{F}_2^\perp}. \tag{1.2}$$

Now we replace the sub-Dirac operator constructed in [11, (2.16)] by the obvious twisted (by $\mathcal{E}$) analogue

$$
\begin{aligned}
D_{\mathcal{F}\oplus\mathcal{F}_1^\perp,\beta,\varepsilon}^{\mathcal{E}} : \Gamma\left(S_{\beta,\varepsilon}(\mathcal{F}\oplus\mathcal{F}_1^\perp)\widehat{\otimes}\Lambda^*\left(\mathcal{F}_2^\perp\right) \otimes \mathcal{E} \right) \\
\longrightarrow \Gamma\left(S_{\beta,\varepsilon}(\mathcal{F}\oplus\mathcal{F}_1^\perp)\widehat{\otimes}\Lambda^*\left(\mathcal{F}_2^\perp\right) \otimes \mathcal{E} \right),
\end{aligned}
\tag{1.3}
$$

where $S_{\beta,\varepsilon}(\cdot)$ is the notation for the spinor bundle determined by $g_{\beta,\varepsilon}^{T\mathcal{M}}$.

---

[1] The case where F is spin is due to Connes, cf. [5, p. 192].

The analogue of [11, (2.28)] now takes the form

$$\left(D_{\mathcal{F}\oplus\mathcal{F}_1^\perp,\beta,\varepsilon}^{\mathcal{E}}\right)^2 = -\Delta^{\mathcal{E},\beta,\varepsilon} + \frac{k^{\mathcal{F}}}{4\beta^2} + \frac{1}{2\beta^2}\sum_{i,\,j=1}^{\mathrm{rk}(F)} R^{\mathcal{E}}(f_i,f_j)c_{\beta,\varepsilon}\left(\beta^{-1}f_i\right)c_{\beta,\varepsilon}\left(\beta^{-1}f_j\right)$$

$$+ O_R\left(\frac{1}{\beta}+\frac{\varepsilon^2}{\beta^2}\right), \tag{1.4}$$

where $-\Delta^{\mathcal{E},\beta,\varepsilon} \geq 0$ is the corresponding Bochner Laplacian, $k^{\mathcal{F}} = \pi^* k^F \geq \delta$ and $f_1, \ldots, f_{\mathrm{rk}(F)}$ is an orthonormal basis of $(\mathcal{F}, g^{\mathcal{F}})$. Moreover, the analogue of [11, (2.34)] now takes the form

$$\mathrm{ind}\left(P_{R,\beta,\varepsilon,+}^{\mathcal{E}}\right) = \left\langle \widehat{A}(TM)\mathrm{ch}(E), [M]\right\rangle. \tag{1.5}$$

From (1.1), (1.4), (1.5) and proceed as in [11, §2.2 and §2.3], one gets Theorem 1.1 for the case where TM is spin easily. As in [11, §2.5], the same proof applies to give a geometric proof for the case where F is spin, with an obvious modification of the (twisted) sub-Dirac operators (cf. [11, (2.58)]). □

Now for the proof of Theorem 0.2, one follows [6], [8] and chooses a complex vector bundle E_0 over $S^{\dim M}(1)$ such that

$$\left\langle \mathrm{ch}\left(E_0\right), \left[S^{\dim M}(1)\right]\right\rangle \neq 0. \tag{1.6}$$

From Definition 0.1 and [8, (5.8) of Chap. IV], one sees that for any $\varepsilon > 0$, one can find a compact covering $\widetilde{M} \to M$ and a map $f : \widetilde{M} \to S^{\dim M}(1)$ of non-zero degree such that $E = f^*(E_0)$ verifies the leafwise ε-condition. Thus, if there is g^F with $k^F > 0$ over M, then by Theorem 1.1 and in view of either [11, Theorem 0.1] (in the case where M is spin) or [4, Theorem 0.2] (in the case where F is spin), one has

$$0 = \left\langle \widehat{A}\left(T\widetilde{M}\right)\mathrm{ch}\left(E\right), \left[\widetilde{M}\right]\right\rangle$$
$$= (\mathrm{rk}(E_0))\widehat{A}\left(\widetilde{M}\right) + \left\langle \widehat{A}\left(T\widetilde{M}\right)f^*\left(\mathrm{ch}\left(E_0\right) - \mathrm{rk}\left(E_0\right)\right), \left[\widetilde{M}\right]\right\rangle \tag{1.7}$$
$$= \deg(f)\left\langle \mathrm{ch}\left(E_0\right), S^{\dim M}(1)\right\rangle,$$

where the last equality comes from the definition of $\deg(f)$, as $\mathrm{ch}(E_0) - \mathrm{rk}(E_0)$ is a top class on $S^{\dim M}(1)$. This contradicts with (1.6) and completes the proof of Theorem 0.2 for compact $\widetilde{M}$.

Remark 1.2. Since any torus T^n is compactly enlargeable (cf. [8, p. 303]), the proof above already applies to give an alternate proof of [11, Corollary 0.5] on the nonexistence of any foliation with metric of positive leafwise scalar curvature on T^n.

1.2. The case where $\widetilde{M}$ is noncompact

We will deal with the case where $F = TM$ in detail. We will work with compact manifolds, thus giving a new proof of the Gromov–Lawson theorem [7, Theorem 5.8] in the case where $\widetilde{M}$ is noncompact. With this "compact" approach it is easy to prove the foliation extension as in Section 1.1.

We assume that $\widetilde{M}$ is noncompact. To simplify the notation, from now on we simply denote $\widetilde{M}$ by M, or rather M_ε to emphasize the dependence on ε. The key point is that the geometric data on M now comes from isometric liftings of geometric data on a compact manifold.

Thus for any $\varepsilon > 0$, let $f_\varepsilon : M_\varepsilon \to S^{\dim M}(1)$ be as in Definition 0.1. Let $K_\varepsilon \subset M_\varepsilon$ be a compact subset of M_ε such that $f(M_\varepsilon \setminus K_\varepsilon) = x_0$, where x_0 is a (fixed) point on $S^{\dim M}(1)$.[2] Following [7], we take a compact hypersurface H_ε in $M_\varepsilon \setminus K_\varepsilon$. We denote by M_{H_ε} the compact manifold with boundary H_ε containing K_ε.

Let M'_{H_ε} be another copy of M_{H_ε}. We glue M_{H_ε} and M'_{H_ε} along H_ε to get the double, which we denote by $\widehat{M}_{H_\varepsilon}$. Let $g^{T\widehat{M}_{H_\varepsilon}}$ be a metric on $T\widehat{M}_{H_\varepsilon}$ such that $g^{T\widehat{M}_{H_\varepsilon}}|_{M_{H_\varepsilon}} = g^{TM}|_{M_{H_\varepsilon}}$. The existence of $g^{T\widehat{M}_{H_\varepsilon}}$ is clear.[3] Let $S(T\widehat{M}_{H_\varepsilon})$ denote the corresponding spinor bundle.

We extend $f_\varepsilon : M_{H_\varepsilon} \to S^{\dim M}(1)$ to $f_\varepsilon : \widehat{M}_{H_\varepsilon} \to S^{\dim M}(1)$ by setting $f_\varepsilon(M'_{H_\varepsilon}) = x_0$.

Let (E_0, g^{E_0}) be a Hermitian vector bundle on $S^{\dim M}(1)$ verifying (1.6) and carrying a Hermitian connection ∇^{E_0}. Let $(E_1 = \mathbf{C}^k|_{S^{\dim M}(1)}, g^{E_1}, \nabla^{E_1})$, with $k = \mathrm{rk}(E_0)$, be the canonical Hermitian trivial vector bundle on $S^{\dim M}(1)$. Let $v : \Gamma(E_0) \to \Gamma(E_1)$ be an endomorphism such that $v|_{x_0}$ is an isomorphism. Let $v^* : \Gamma(E_1) \to \Gamma(E_0)$ be the adjoint of v with respect to g^{E_0} and g^{E_1}. Set

$$V = v + v^*. \tag{1.8}$$

Then the self-adjoint endomorphism $V : \Gamma(E_0 \oplus E_1) \to \Gamma(E_0 \oplus E_1)$ is invertible near x_0.

Let $(\xi, g^\xi, \nabla^\xi) = (\xi_0 \oplus \xi_1, g^{\xi_0} \oplus g^{\xi_1}, \nabla^{\xi_0} \oplus \nabla^{\xi_1}) = (f_\varepsilon^* E_0 \oplus f_\varepsilon^* E_1, f_\varepsilon^* g^{E_0} \oplus f_\varepsilon^* g^{E_1}, f_\varepsilon^* \nabla^{E_0} \oplus f_\varepsilon^* \nabla^{E_1})$ be the $\mathbf{Z}_2$-graded Hermitian vector bundle with Hermitian connection over $\widehat{M}_{H_\varepsilon}$ (here for simplicity, we do not make explicit the subscript ε in ξ, ξ_0 and ξ_1). Let $R^\xi = (\nabla^\xi)^2$ be the curvature of ∇^ξ. Set $V_{f_\varepsilon} = f_\varepsilon^* V$. Then

$$\left[\nabla^\xi, V_{f_\varepsilon}\right] = 0 \tag{1.9}$$

on M'_{H_ε}.

Let $D^\xi : \Gamma(S(T\widehat{M}_{H_\varepsilon})\widehat{\otimes}\xi) \to \Gamma(S(T\widehat{M}_{H_\varepsilon})\widehat{\otimes}\xi)$ be the canonically defined (twisted) Dirac operator (cf. [8]). Let

$$D_\pm^\xi : \Gamma((S(T\widehat{M}_{H_\varepsilon})\widehat{\otimes}\xi)_\pm) \to \Gamma((S(T\widehat{M}_{H_\varepsilon})\widehat{\otimes}\xi)_\mp)$$

---

[2] Up to an isometry of $S^{\dim M}(1)$, one can always assume that x_0 is fixed and does not depend on ε.

[3] Here we need not assume that $g^{T\widehat{M}_{H_\varepsilon}}$ is of product structure near M_{H_ε}.

be the obvious restrictions, where

$$(S(T\widehat{M}_{H_\varepsilon})\widehat{\otimes}\xi)_+ = S_+(T\widehat{M}_{H_\varepsilon}) \otimes \xi_0 \oplus S_-(T\widehat{M}_{H_\varepsilon}) \otimes \xi_1,$$

while

$$(S(T\widehat{M}_{H_\varepsilon})\widehat{\otimes}\xi)_- = S_-(T\widehat{M}_{H_\varepsilon}) \otimes \xi_0 \oplus S_+(T\widehat{M}_{H_\varepsilon}) \otimes \xi_1.$$

By the Atiyah–Singer index theorem [1] (cf. [8]) and [7], one has

$$\begin{aligned}
\operatorname{ind}\left(D_+^\xi\right) &= \left\langle \widehat{A}\left(T\widehat{M}_{H_\varepsilon}\right)(\operatorname{ch}(\xi_0) - \operatorname{ch}(\xi_1)), \left[\widehat{M}_{H_\varepsilon}\right]\right\rangle \\
&= (\deg(f_\varepsilon))\left\langle \operatorname{ch}(E_0), \left[S^{\dim M}(1)\right]\right\rangle,
\end{aligned} \tag{1.10}$$

where the last equality comes from the definition of $\deg(f_\varepsilon)$ (cf. [7]).

Let k^{TM} denote the scalar curvature of g^{TM}. We assume that there is $\delta > 0$ such that $k^{TM} \geq \delta$ over M.

For any $\varepsilon > 0$, let $D_\varepsilon^\xi : \Gamma(S(T\widehat{M}_{H_\varepsilon})\widehat{\otimes}\xi) \to \Gamma(S(T\widehat{M}_{H_\varepsilon})\widehat{\otimes}\xi)$ be the deformed operator defined by

$$D_\varepsilon^\xi = D^\xi + V_{f_\varepsilon}. \tag{1.11}$$

Proposition 1.3. *There is $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$, one has $\ker(D_\varepsilon^\xi) = \{0\}$.*

Proof. Recall that $x_0 \in S^{\dim M}(1)$ is fixed and $V|_{x_0}$ is invertible. Let $U_{x_0} \subset S^{\dim M}(1)$ be a (fixed) sufficiently small open neighborhood of x_0 such that the following inequality holds on U_{x_0},

$$V^2 \geq \delta_1. \tag{1.12}$$

Let $\psi : S^{\dim 1}(1) \to [0,1]$ be a smooth function such that $\psi = 1$ near x_0 and $\operatorname{Supp}(\psi) \subset U_{x_0}$. Then $\varphi_\varepsilon = 1 - f_\varepsilon^*\psi$ is a smooth function on M_ε (and thus on M_{H_ε}), which extends to a smooth function on $\widehat{M}_{H_\varepsilon}$ such that $\varphi_\varepsilon = 0$ on M'_{H_ε}.

Following [3, p. 115], let $\varphi_{\varepsilon,1}, \varphi_{\varepsilon,2} : \widehat{M}_{H_\varepsilon} \to [0,1]$ be defined by

$$\varphi_{\varepsilon,1} = \frac{\varphi_\varepsilon}{(\varphi_\varepsilon^2 + (1-\varphi_\varepsilon)^2)^{\frac{1}{2}}}, \quad \varphi_{\varepsilon,2} = \frac{1 - \varphi_\varepsilon}{(\varphi_\varepsilon^2 + (1-\varphi_\varepsilon)^2)^{\frac{1}{2}}}. \tag{1.13}$$

Then $\varphi_{\varepsilon,1}^2 + \varphi_{\varepsilon,2}^2 = 1$. Thus, for any $s \in \Gamma(S(T\widehat{M}_{H_\varepsilon})\widehat{\otimes}\xi)$, one has

$$\left\|D_\varepsilon^\xi s\right\|^2 = \left\|\varphi_{\varepsilon,1} D_\varepsilon^\xi s\right\|^2 + \left\|\varphi_{\varepsilon,2} D_\varepsilon^\xi s\right\|^2, \tag{1.14}$$

from which one gets

$$\begin{aligned}
\sqrt{2}\left\|D_\varepsilon^\xi s\right\| &\geq \left\|\varphi_{\varepsilon,1} D_\varepsilon^\xi s\right\| + \left\|\varphi_{\varepsilon,2} D_\varepsilon^\xi s\right\| \\
&\geq \left\|D_\varepsilon^\xi(\varphi_{\varepsilon,1} s)\right\| + \left\|D_\varepsilon^\xi(\varphi_{\varepsilon,2} s)\right\| - \left\|c(d\varphi_{\varepsilon,1}) s\right\| - \left\|c(d\varphi_{\varepsilon,2}) s\right\|,
\end{aligned} \tag{1.15}$$

where we identify $d\varphi_{\varepsilon,i}$, $i = 1, 2$, with the gradient of $\varphi_{\varepsilon,i}$.

Let $e_1, \ldots, e_{\dim M}$ be an orthonormal basis of $g^{T\widehat{M}_{H_\varepsilon}}$. Then by (1.11), one has

$$\left(D_\varepsilon^\xi\right)^2 = \left(D^\xi\right)^2 + \sum_{i=1}^{\dim M} c\left(e_i\right) \left[\nabla_{e_i}^\xi, V_{f_\varepsilon}\right] + V_{f_\varepsilon}^2. \tag{1.16}$$

From (1.16), one has for $j = 1, 2$ that

$$\left\|D_\varepsilon^\xi\left(\varphi_{\varepsilon,j}s\right)\right\|^2 = \left\|D^\xi\left(\varphi_{\varepsilon,j}s\right)\right\|^2 + \sum_{i=1}^{\dim M} \left\langle c\left(e_i\right) \left[\nabla_{e_i}^\xi, V_{f_\varepsilon}\right] \varphi_{\varepsilon,j}s, \; \varphi_{\varepsilon,j}s \right\rangle \tag{1.17}$$
$$+ \left\|\varphi_{\varepsilon,j}V_{f_\varepsilon}s\right\|^2.$$

By the Lichnerowicz formula [9] (cf. [8]), one has on M_{H_ε} that

$$\left(D^\xi\right)^2 = -\Delta^\xi + \frac{k^{TM}}{4} + \frac{1}{2} \sum_{i,\,j=1}^{\dim M} c\left(e_i\right) c\left(e_j\right) R^\xi\left(e_i, e_j\right), \tag{1.18}$$

where Δ^ξ is the corresponding Bochner Laplacian and $k^{TM} \geq \delta$ by assumption.

By Definition 0.1 and proceeding as in [8, (5.8) of Chap. IV], one finds on M_{H_ε} that

$$\frac{1}{2} \sum_{i,\,j=1}^{\dim M} c\left(e_i\right) c\left(e_j\right) R^\xi\left(e_i, e_j\right) + \sum_{i=1}^{\dim M} c\left(e_i\right) \left[\nabla_{e_i}^\xi, V_{f_\varepsilon}\right]$$
$$= \frac{1}{2} \sum_{i,\,j=1}^{\dim M} c\left(e_i\right) c\left(e_j\right) f_\varepsilon^*\left(R^{E_0}\left(f_{\varepsilon*}e_i, f_{\varepsilon*}e_j\right)\right) \tag{1.19}$$
$$+ \sum_{i=1}^{\dim M} c\left(e_i\right) f_\varepsilon^*\left(\left[\nabla_{f_{\varepsilon*}e_i}^{E_0 \oplus E_1}, V\right]\right) = O\left(\varepsilon\right).$$

On the other hand, for any $1 \leq i \leq \dim M$, one verifies that

$$e_i\left(\varphi_\varepsilon\right) = -e_i\left(f_\varepsilon^*\psi\right) = -f_\varepsilon^*\left(\left(f_{\varepsilon*}e_i\right)(\psi)\right) = O(\varepsilon). \tag{1.20}$$

From (1.13) and (1.20), one finds that for $i = 1, 2$,

$$\left|c\left(d\varphi_{\varepsilon,i}\right)\right| = O(\varepsilon). \tag{1.21}$$

From (1.9), (1.12), (1.13), (1.15), (1.17)–(1.19) and (1.21), one deduces that there exists $\delta_2 > 0$ such that when $\varepsilon > 0$ is sufficiently small, one has (compare with [11, p. 1062])

$$\left\|D_\varepsilon^\xi s\right\| \geq \delta_2\|s\|, \tag{1.22}$$

which completes the proof of Proposition 1.3. $\qquad\square$

From Proposition 1.3, one finds $\mathrm{ind}(D_+^\xi) = 0$, which contradicts with (1.10) where the right-hand side is non-zero. Thus, there should be no g^{TM} with $k^{TM} > 0$ over M. This completes the proof of Theorem 0.2 for the case of $F = TM$ (which is

the original Gromov–Lawson theorem [7, Theorem 5.8]), without using the relative
index theorem on noncompact manifolds in [7].

Now to prove Theorem 0.2(i), one simply combines the method in Section 1.1
with the doubling and gluing tricks above. The details are easy to fill. Theorem
0.2(ii) follows by modifying the sub-Dirac operator as in [11, §2.5].

The proof of Theorem 0.2 is completed.

Acknowledgment

References

[1] M.F. Atiyah and I.M. Singer, *The index of elliptic operators I.* Ann. of Math. 87
    (1968), 484–530.

[2] M.-T. Benameur and J.L. Heitsch, *Enlargeability, foliations, and positive scalar cur-
    vature.* Invent. Math. 215 (2019), 367–382.

[3] J.-M. Bismut and G. Lebeau, *Complex immersions and Quillen metrics.* Inst. Hautes
    Études Sci. Publ. Math. (1991), no. 74, ii+298 pp.

[4] A. Connes, *Cyclic cohomology and the transverse fundamental class of a foliation.*
    in *Geometric Methods in Operator Algebras.* H. Araki ed., pp. 52–144, Pitman Res.
    Notes in Math. Series, vol. 123, 1986.

[5] M. Gromov, *Positive curvature, macroscopic dimension, spectral gaps and higher
    signatures.* in *Functional Analysis on the Eve of the 21st Century*, Gindikin, Simon
    (ed.) et al. Progress in Math. 132 (1996), 1–213, Birkhäuser, Basel.

[6] M. Gromov and H.B. Lawson, *Spin and scalar curvature in the presence of a funda-
    mental group I.* Ann. of Math. 111 (1980), 209–230.

[7] M. Gromov and H.B. Lawson, *Positive scalar curvature and the Dirac operator on
    complete Riemannian manifolds.* Publ. Math. I.H.E.S. 58 (1983), 295–408.

[8] H.B. Lawson and M.-L. Michelsohn, *Spin Geometry.* Princeton Univ. Press, Prince-
    ton, NJ, 1989.

[9] A. Lichnerowicz, *Spineurs harmoniques.* C. R. Acad. Sci. Paris, Série A, 257 (1963),
    7–9.

[10] R. Schoen and S.T. Yau, *On the structure of manifolds with positive scalar curvature.*
    Manuscripta Math. 28 (1979), 159–183.

[11] W. Zhang, *Positive scalar curvature on foliations.* Ann. of Math. 185 (2017), 1035–
    1068.

Weiping Zhang
Chern Institute of Mathematics & LPMC
Nankai University
Tianjin 300071, P.R. China
e-mail: `weiping@nankai.edu.cn`