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004

REAL EMBEDDINGS AND THE ATIYAH-PATODI-SINGER INDEX THEOREM FOR DIRAC OPERATORS*

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Abstract. We present the details of our embedding proof, which was announced in [DZ1], of the Atiyah-Patodi-Singer index theorem for Dirac operators on manifolds with boundary [APS1].

Introduction. The index theorem of Atiyah, Patodi and Singer [APS1, (4.3)] for Dirac operators on manifolds with boundary has played important roles in various problems in geometry, topology as well as mathematical physics. Not surprisingly then, there are by now quite a number of proofs of this index theorem other than Atiyah, Patodi and Singer's original proof [APS1]. Among these proofs we mention those of Cheeger [C1, 2] (see also Chou [Ch]), Bismut-Cheeger [BC1] and Melrose [M]. One common point underlying all these proofs (including the original one) is that they can all be viewed, in one way or another, as certain extensions to manifolds with boundary of the heat kernel proof of the local index theorem for Dirac operators on closed manifolds (cf. [BeGV]). That is, one starts with a Mckean-Singer type formula and then studies the small time asymptotics of the corresponding heat kernels. In particular, one makes use of the explicit formulas for the heat kernel of the Laplace operators on the cylinder ([APS1], [M]) and/or cone ([BC1], [C1, 2], [Ch]) (being attached the boundary) for the analysis near the boundary. The η-invariant on the boundary, which was first defined in [APS1], appears naturally during the process.

Now recall that Atiyah and Singer [AS] also have a K-theoretic proof of their index theorem for elliptic operators on closed manifolds. In such a proof, one transforms the problem, through direct image constructions in K-theory, to a sphere and then applies the Bott periodicity theorem on the sphere to establish the result.[1] It is thus natural to ask whether the strategy of Atiyah-Singer's K-theoretic ideas can be used to prove the Atiyah-Patodi-Singer index theorem for manifolds with boundary. The purpose of this paper is to present such a proof, of which an announcement of basic ideas has already appeared in [DZ1].

Briefly speaking, we embed the manifold with boundary under consideration into a ball, instead of a sphere, so that it maps the boundary of the original manifold to the boundary sphere of the ball, and reduce the problem to the ball. Now since any vector bundle on the ball is topologically trivial, one obtains the result immediately. This works even when the original manifold has no boundary, giving a proof of the Atiyah-Singer index theorem for Dirac operators. The Bott periodicity theorem is thus not needed.

Observe that in [AS], Atiyah and Singer made heavy use of the techniques of pseudodifferential operators, which is not suitable for treating directly the global elliptic boundary problems. This is the first serious difficulty in extending directly the arguments in [AS] to deal with the Atiyah-Patodi-Singer boundary problems.

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[1] See also the book of Lawson-Michelsohn [LM] for a comprehensive treatment of this approach.

On the other hand, Bismut and Lebeau developed in [BL] a general and direct localization procedure which applies to a wide range of localization problems involving Dirac type operators. For example, it has lead to a direct analytic treatment of the index theorem for Dirac operators on closed manifolds along the lines of [AS] (cf. [Z, Remark 2.6]), as well as a localization formula for η-invariants of Dirac operators [BZ] which may be viewed as an odd dimensional analogue of the main result in [BL]. It is these techniques and results that will be used in the present paper, giving an embedding proof of the Atiyah-Patodi-Singer index theorem for *Dirac* operators on manifolds with boundary [APS1].

In the proof described in [DZ1], we also used in an essential way Cheeger's cone method [C1, 2]. The reason being, in order to apply Bismut-Lebeau's method [BL], we need to transfer the Atiyah-Patodi-Singer boundary problem to an elliptic problem on certain manifolds with cone-like singularity. Now, in the present paper, we will show that how one can avoid the analysis on the cone at all. This is done by considering the Atiyah-Patodi-Singer type boundary value problem for certain non-differetial operators arising naturally from the analysis in [BL]. In this way, one no longer encounters the heat kernel analysis on cylinders and/or cones which are essential for the other proofs of the Atiyah-Patodi-Singer index theorem. We regard this as a major technical simplification with respect to [DZ1].

In a separate paper [DZ3], we will further extend the main result of this paper to the case of families. In particular, we will give a new proof of the family index theorem of Bismut-Cheeger [BC1, 2] and Melrose-Piazza [MP] along the lines of this paper.

This paper is organized as follows. In Section 1, we prove an important variation formula for the indices of the Atiyah-Patodi-Singer boundary value problems for Dirac operators on manifolds with boundary. In Section 2, we state a localization formula of Riemann-Roch type for the indices of the Atiyah-Patodi-Singer boundary value problems for Dirac operators on manifolds with boundary. In Section 3, we prove the Riemann-Roch property stated in Section 2. In Section 4, by combining the results in Sections 1, 2 with those of Bismut-Zhang [BZ], we complete our proof of the Atiyah-Patodi-Singer index theorem for Dirac operators on manifolds with boundary. There is also an appendix in which we prove a harmonic oscillator property for certain Dirac operators on flat spaces, which plays an essential role in the main text.

1. Dirac type operators on manifolds with boundary: index and its variations. In this section, we recall the definition of the Atiyah-Patodi-Singer boundary value problems [APS1] for Dirac type operators on *Spin* manifolds with boundary. We also prove an important variation formula for the indices of these boundary value problems.

Let X be a compact oriented even dimensional spin manifold with boundary ∂X. We assume that X has been equipped with a fixed spin structure. Then ∂X carries the canonically induced orientation and spin structure.

Let g^{TX} be a metric on TX. Let $g^{T\partial X}$ be its restriction on $T\partial X$. We assume that g^{TX} is of product structure near the boundary ∂X. That is, there is an open neighborhood $U_\alpha = [0, \alpha) \times \partial X$ of ∂X in X with $\alpha > 0$ such that one has the orthogonal splitting on U_α,

$$g^{TX}\big|_{U_\alpha} = dr^2 \oplus \pi_\alpha^* g^{T\partial X}, \tag{1.1}$$

where $\pi_\alpha : [0, \alpha) \times \partial X \to \partial X$ is the obvious projection onto the second factor.

Let ξ be a Hermitian vector bundle over X with Hermitian metric g^ξ. Let ∇^ξ be a Hermitian connection on ξ with respect to g^ξ. We make the assumption that over the open neighborhood U_α of ∂X, one has

$$g^\xi\big|_{U_\alpha} = \pi_\alpha^*\left(g^\xi|_{\partial X}\right), \quad \nabla^\xi\big|_{U_\alpha} = \pi_\alpha^*\left(\nabla^\xi|_{\partial X}\right). \tag{1.2}$$

By taking $\alpha > 0$ sufficiently small, one can always find g^{TX}, g^ξ and ∇^ξ verifying (1.1) and (1.2).

Let $S(TX) = S_+(TX) \oplus S_-(TX)$ be the $\mathbf{Z}_2$-graded Hermitian vector bundle of spinors associated to (TX, g^{TX}). Let $\nabla^{S(TX)}$ be the Hermitian connection on $S(TX)$ canonically induced from the Levi-Civita connection ∇^{TX} of g^{TX}. Then $\nabla^{S(TX)}$ preserves the $\mathbf{Z}_2$-splitting $S(TX) = S_+(TX) \oplus S_-(TX)$. We denote by $\nabla^{S_\pm(TX)}$ the restriction of $\nabla^{S(TX)}$ on $S_\pm(TX)$. Let $\nabla^{S(TX)\otimes\xi}$ (resp. $\nabla^{S_\pm(TX)\otimes\xi}$) be the Hermitian connection on $S(TX) \otimes \xi$ (resp. $S_\pm(TX) \otimes \xi$) obtained from the tensor product of $\nabla^{S(TX)}$ (resp. $\nabla^{S_\pm(TX)}$) and ∇^ξ.

For any $e \in TX$, let $c(e)$ be the Clifford action of e on $S(TX)$. Then $c(e)$ extends to an action on $S(TX) \otimes \xi$ by acting as identity on ξ. We still denote this extended action by $c(e)$.

Let $e_1, \cdots, e_{\dim X}$ be an oriented (local) orthonormal base of TX. We can then define the (total) twisted Dirac operator with coefficient bundle ξ as follows (cf. [BeGV] and [LM]),

$$D^\xi = \sum_{i=1}^{\dim X} c(e_i)\nabla_{e_i}^{S(TX)\otimes\xi} : \Gamma(S(TX) \otimes \xi) \to \Gamma(S(TX) \otimes \xi). \tag{1.3}$$

Let $D_\pm^\xi$ be the restriction of D^ξ on $\Gamma(S_\pm(TX) \otimes \xi)$. Then D_-^ξ is the formal adjoint of D_+^ξ.

Definition 1.1. *By a Dirac type operator on $\Gamma(S(TX) \otimes \xi)$, we mean a first order differential operator $D : \Gamma(S(TX) \otimes \xi) \to \Gamma(S(TX) \otimes \xi)$ such that $D - D^\xi$ is an odd self-adjoint element of zeroth order, and that for $\alpha > 0$ sufficiently small, the following identity holds on U_α,*

$$D = c\left(\frac{\partial}{\partial r}\right)\left(\frac{\partial}{\partial r} + B\right), \tag{1.4}$$

with B independent of r and its restriction on $\Gamma(S(TX)\otimes\xi)|_{\partial X}$ formally self-adjoint. We will also call the restriction D_+ (resp. D_-) of D to $\Gamma(S_+(TX) \otimes \xi)$ (resp. $\Gamma(S_-(TX) \otimes \xi)$) a Dirac type operator.

When there is no confusion, we will also use B to denote its restriction on $(S(TX)\otimes\xi)|_{\partial X}$. Clearly, B preserves the $\mathbf{Z}_2$-grading of $(S(TX)\otimes\xi)|_{\partial X} = (S_+(TX)\otimes\xi)|_{\partial X} \oplus (S_-(TX)\otimes\xi)|_{\partial X}$. We denote by $B_\pm$ the restriction of B on $(S_\pm(TX)\otimes\xi)|_{\partial X}$.

Now consider the formally self-adjoint first order differential operator B_+, which is clearly elliptic, acting on sections of $(S_+(TX) \otimes \xi)|_{\partial X}$. Then the L^2-completion of $(S_+(TX) \otimes \xi)|_{\partial X}$ admits an orthogonal decomposition

$$L^2\left((S_+(TX) \otimes \xi)|_{\partial X}\right) = \bigoplus_{\lambda\in\mathrm{Spec}(B_+)} E_\lambda, \tag{1.5}$$

where E_λ is the eigenspace of λ.

For any $a \in \mathbf{R}$, let $L^2_{\geq a}((S_+(TX)\otimes\xi)|_{\partial X})$ denote the direct sum of the eigenspaces E_λ associated to the eigenvalues $\lambda \geq a$. Let $P_{+,\geq a}$ denote the orthogonal projection from $L^2((S_+(TX)\otimes\xi)|_{\partial X})$ to $L^2_{\geq a}((S_+(TX)\otimes\xi)|_{\partial X})$. We call the particular projection $P_{+,\geq 0}$ the Atiyah-Patodi-Singer projection associated to B_+, to emphasize its role in [APS1].

Following [APS1], one can then impose the boundary value problem

$$(D_+, P_{+,\geq a}) : \{u : u \in \Gamma(S_+(TX)\otimes\xi),\ P_{+,\geq a}u|_{\partial X} = 0\} \to \Gamma(S_-(TX)\otimes\xi), \tag{1.6}$$

which is Fredholm by [APS1]. In particular, we call the boundary problem $(D_+, P_{+,\geq 0})$ the Atiyah-Patodi-Singer boundary problem associated to D_+. We denote by $\operatorname{ind}(D_+, P_{+,\geq a})$ the index of the associated Fredholm operator.

Now let $D_+(s)$, $0 \leq s \leq 1$, be a smooth family of Dirac type operators with the induced boundary operators $B_+(s)$. We can now state the main result of this section, which has been announced in [DZ1, Theorem 1.1], as follows.

Theorem 1.2. *The following identity holds,*

$$\operatorname{ind}(D_+(1), P_{+,\geq 0}(1)) - \operatorname{ind}(D_+(0), P_{+,\geq 0}(0)) = -\operatorname{sf}\{B_+(s), 0 \leq s \leq 1\}, \tag{1.7}$$

where sf *is the notation for the spectral flow of Atiyah-Patodi-Singer [APS2].*

Proof. Take any $0 \leq s_0 \leq 1$. Let $2a_0$ be the minimal absolute value of the nonzero eigenvalues of $B_+(s_0)$. Then there exsists $\varepsilon_0 > 0$ such that for any $s \in [s_0 - \varepsilon_0, s_0 + \varepsilon_0] \cap [0,1]$, a_0 is not an eigenvalue of $B_+(s)$. Then for any $s \in [s_0 - \varepsilon_0, s_0 + \varepsilon_0] \cap [0,1]$, $(D_+(s), P_{+,\geq a_0}(s))$ defines a continuous family of Fredholm operators. Therefore,

$$\operatorname{ind}(D_+(s), P_{+,\geq a_0}(s)) = \operatorname{ind}(D_+(s_0), P_{+,\geq a_0}(s_0)). \tag{1.8}$$

On the other hand, by the classical Agranovič-Dynin type formula (cf. [BoW, Chap. 21]) and the definition of spectral flow [APS2], one verifies easily that

$$\operatorname{ind}(D_+(s), P_{+,\geq a_0}(s)) - \operatorname{ind}(D_+(s), P_{+,\geq 0}(s)) = -\operatorname{sf}\{B_+(s) + ua_0, 0 \leq u \leq 1\},$$

$$\begin{aligned} &\operatorname{ind}(D_+(s_0), P_{+,\geq a_0}(s_0)) - \operatorname{ind}(D_+(s_0), P_{+,\geq 0}(s_0)) \\ &= -\operatorname{sf}\{B_+(s_0) + ua_0, 0 \leq u \leq 1\}. \end{aligned} \tag{1.9}$$

Formula (1.7) follows easily from (1.8), (1.9) and the additivity (using twice) of the spectral flow [APS2]. □

Remark 1.3. For a similar variation formula for η-invariants on odd dimensional manifolds with boundary, see Dai-Freed [DF].

Remark 1.4. For an extension of Theorem 1.2 to the case of families, see [DZ2].

2. A Riemann-Roch theorem under embedding for Dirac operators on manifolds with boundary. In this section, we state a Riemann-Roch type formula for indices of Dirac type operators on manifolds with boundary. This formula will be proved in the next section and will play a key role in our proof of the Atiyah-Patodi-Singer index theorem in Section 4.

This section is organized as follows. In a), we describe the basic geometric data. In b), we state the main result of this section, whose proof will be given in the next section.

a). The geometric construction of direct images under embedding between manifolds with bounday. Let Y be another even dimensional oriented compact spin manifold with boundary ∂Y. Moreover, there is an embedding $i : Y \hookrightarrow X$ such that $\partial Y \subset \partial X$, and that Y intersects transversally with ∂X.

Let $g^{T(\partial Y)}$ be the metric on $T(\partial Y)$ induced from $g^{T(\partial X)}$. Set

$$U'_\alpha = U_\alpha \cap Y. \tag{2.1}$$

We can and we will assume that α is small enough so that U'_α is also a tubular neighborhood of ∂Y. Then U'_α carries a metric $g^{TU'_\alpha}$ naturally induced from g^{TU_α}.

Let $\pi : N \to Y$ be the normal bundle of Y in X. Then $N_{\partial Y} = N|_{\partial Y}$ is the normal bundle to ∂Y in ∂X.

Clearly, $\dim N = \dim X - \dim Y$ is even. Furthermore, since TX, TY are oriented and spin, N is also oriented and spin.

Let g^{TX} be a metric on TX such that its restriction on U_α is g^{TU_α}. Let g^{TY} be the restriction of g^{TX} on Y. For simplicity, we can and we will assume that the embedding $i : (Y, g^{TY}) \hookrightarrow (X, g^{TX})$ is totally geodesic. We identify N with the orthogonal completement of TY in $(TX)|_Y$. Let g^N be the metric on N restricted from $g^{(TX)|_Y}$. Let P^{TY} (resp. P^N) be the orthogonal projection from $(TX)|_Y$ to TY (resp. N) with respect to $g^{(TX)|_Y}$. Then $P^{TY} i^* \nabla^{TX} P^{TY}$, where ∇^{TX} is the Levi-Civita connection of g^{TX}, is the Levi-Civita connection ∇^{TY} of g^{TY} and one has the orthogonal splitting

$$i^* \nabla^{TX} = \nabla^{TY} \oplus \nabla^N, \tag{2.2}$$

where $\nabla^N = P^N i^* \nabla^{TX} P^N$ is the induced Euclidean connection on N.

Let $S(TX) = S_+(TX) \oplus S_-(TX)$ (resp. $S(TY) = S_+(TY) \oplus S_-(TY)$, $S(N) = S_+(N) \oplus S_-(N)$) be the $\mathbf{Z}_2$-graded Hermitian vector bundle of (TX, g^{TX}) (resp. (TY, g^{TY}), (N, g^N)) spinors. Then one has

$$S(TX)|_Y = S(TY) \hat{\otimes} S(N). \tag{2.3}$$

The connections ∇^{TX}, ∇^{TY}, ∇^N lift to unitary connections on $\nabla^{S(TX)}$, $\nabla^{S(TY)}$, $\nabla^{S(N)}$, $\nabla^{S^*(N)}$ on $S(TX)$, $S(TY)$, $S(N)$, $S^*(N)$ respectively, preserving the corresponding $\mathbf{Z}_2$-gradings.

Let $\pi_\alpha : U_\alpha = [0, \alpha) \times \partial X \to \partial X$ (resp. $\pi'_\alpha : U_\alpha = [0, \alpha) \times \partial Y \to \partial Y$) denote the projection from U_α (resp. U'_α) to the boundary of X (resp. Y).

Let $\xi = \xi_+ \oplus \xi_-$ be a $\mathbf{Z}_2$-graded complex vector bundle over X such that $\xi|_{U_\alpha} = \pi^*_\alpha(\xi|_{\partial X})$. Let g^ξ be a Hermitian metric on ξ such that such that $g^\xi|_{U_\alpha} = \pi^*_\alpha(g^\xi|_{\partial X})$ and that ξ_+ and ξ_- are orthogonal to each other with respect to g^ξ.

Let $V \in \Gamma(\mathrm{End}^{\mathrm{odd}}(\xi))$ be a self-adjoint element such that

$$V|_{U_\alpha} = \pi^*_\alpha (V|_{\partial X}). \tag{2.4}$$

We assume that V is invertible on $X \setminus Y$, and that on Y, $\ker V$ has locally constant nonzero dimension, so that $\ker V$ is a nonzero smooth $\mathbf{Z}_2$-graded vector subbundle of $\xi|_Y$. Let $g^{\ker V}$ be the metric on $\ker V$ induced by the metric $g^\xi|_Y$. Let $P^{\ker V}$ be the orthogonal projection from $\xi|_Y$ on $\ker V$.

If $y \in Y$, $U \in T_y X$, let $\partial_U V(y)$ be the derivative of V with repsect to U in any given smooth trivialization of ξ near $y \in X$. One then verifies that $P^{\ker V} \partial_U V(y) P^{\ker V}$ does not depend on the trivialization, and only depends on the image Z of $U \in T_y X$

in N_y. From now on, we will write $\dot{\partial}_Z V(y)$ instead of $P^{\ker V}\partial_U V(y)P^{\ker V}$. Then one verifies easily that $\dot{\partial}_Z V(y)$ is a self-adjoint element of $\mathrm{End}^{\mathrm{odd}}(\ker V_y)$.

If $Z \in N$, let $\tilde{c}(Z) \in \mathrm{End}(S^*(N))$ be the transpose of $c(Z)$ acting on $S(N)$. Let $\tau^{N*} \in \mathrm{End}(S^*(N))$ be the transpose of τ^N defining the $\mathbf{Z}_2$-grading of $S(N) = S_+(N) \oplus S_-(N)$.

Let μ be a complex vector bundle over Y such that $\mu|_{U'_\alpha} = \pi'^*_\alpha(\mu|_{\partial Y})$, equipped with a Hermitian metric g^μ such that $g^\mu|_{U'_\alpha} = \pi'^*_\alpha(g^\mu|_{\partial Y})$. We equip $S^*(N) \otimes \mu$ the tensor product metric $g^{S^*(N)\otimes\mu}$. Also, we extend an endomorphism of $S^*(N)$ to that of $S^*(N) \otimes \mu$ by acting as identity on μ. We now make the fundamental assumption that over the total space of N, we have the identification

$$\left(\pi^* \ker V, \pi^* g^{\ker V}, \dot{\partial}_Z V(y)\right) = \left(\pi^* \left(S^*(N) \otimes \mu\right), \pi^* \left(g^{S^*(N)\otimes\mu}\right), \tilde{c}(Z)\tau^{N*}\right). \tag{2.5}$$

Let ∇^μ be a Hermitian connection on μ which is of product nature near the boundary. Let $\nabla^{S^*(N)\otimes\mu}$ be the Hermitian connection on $S^*(N) \otimes \mu$ obtained from the tensor product of $\nabla^{S^*(N)}$ and ∇^μ.

Let $\nabla^\xi = \nabla^{\xi_+} \oplus \nabla^{\xi_-}$ be a unitary connection on $\xi = \xi_+ \oplus \xi_-$, which preserves the $\mathbf{Z}_2$-grading of ξ and is of product nature near the boundary. Let $\nabla^{\ker V}$ be the unitary connection on $\ker V$ given by

$$\nabla^{\ker V} = P^{\ker V}\nabla^\xi|_Y P^{\ker V}. \tag{2.6}$$

We then make the assumption that under the identification (2.5), we also have the identification of connections

$$\nabla^{\ker V} = \nabla^{S^*(N)\otimes\mu}. \tag{2.7}$$

One easily verifies that there always exists a connection ∇^ξ such that (2.7) holds.

Remark 2.1. By using a well-known construction of Atiyah-Hirzebruch [AH], one verifies easily that given metrics g^μ and g^N on μ and N, there exist $\xi = \xi_+ \oplus \xi_-$, $g^\xi = g^{\xi_+} \oplus g^{\xi_-}$ and V taken as before, such that (2.5) holds (Compare with [BZ, Remark 1.1]). In particular, $\xi_+ - \xi_-$ is a representative of the direct image $i_!\mu \in K(X)$ of $\mu \in K(Y)$ (cf. [LM]).

Remark 2.2. As an easy but important observation, we note that the restriction of the identifications (2.5), (2.7) on the boundary takes forms of exactly the same nature (Compare with [BZ, Sect. 2b)]). In what follows, whenever such an identification on the boundary will be considered, we will simply use a subscript and/or superscript '∂' to indicate the restriction, when there will be no confusion from the context.

b). A Riemann-Roch theorem under emdedding for Dirac type operators on manifolds with boundary. We continue the discussions in a).

Let $D^\xi = D^{\xi_+} + D^{\xi_-}$, D^μ be the Dirac operators defined as in (1.3). We consider a Dirac type operator D_X acting on $\Gamma(S(TX)\hat{\otimes}\xi)$ such that $E_X = D_X - D^\xi$ is an odd endomorphism of $S(TX)\hat{\otimes}\xi$.

From (2.3), (2.5) and (A.1), one finds

$$S(TX)|_Y \hat{\otimes}(\ker V) = S(TY)\hat{\otimes}S(N)\hat{\otimes}S^*(N) \otimes \mu = (S(TY) \otimes \mu)\hat{\otimes} \wedge^* (N^*), \tag{2.8}$$

where $\wedge^*(N^*)$ is the exterior algebra bundle of N^* over Y. Let p be the orthogonal projection from $S(TX)|_Y \otimes(\ker V)$ to $S(TY) \otimes \mu$ which maps as zero on each $S(TY) \otimes \mu \otimes \wedge^i(N^*)$, $i \geq 1$.

Let $E_Y \in \mathrm{End}^{\mathrm{odd}}(S(TY) \otimes \mu)$ be defined by

$$E_Y = p(E_X)|_Y p. \tag{2.9}$$

Let D_Y be the Dirac type operator

$$D_Y = D^{\mu} + E_Y. \tag{2.10}$$

Let B_X (resp. B_Y) be the induced boundary operator from D_X (resp. D_Y) in the sense of (1.4).

The following assumption is essential for this section.

Assumption 2.3. *The operator B_Y has no zero eigenvalue.*

For any $T \in \mathbf{R}$, let $D_T : \Gamma(S(TX)\hat{\otimes}\xi) \to \Gamma(S(TX)\hat{\otimes}\xi)$ be the operator defined by

$$D_T = D_X + TV, \tag{2.11}$$

where $V \in \mathrm{End}(\xi)$ extends as an action on $S(TX)\hat{\otimes}\xi$ by $1\hat{\otimes}V$, etc. Then by (1.4), its induced boundary operator B_T is given by

$$B_T = B_X - Tc\left(\frac{\partial}{\partial r}\right)V|_{\partial X}. \tag{2.12}$$

Let $D_{T,+}$ be the restriction of D_T on $\Gamma(S_+(TX)\otimes\xi_+ \oplus S_-(TX)\otimes\xi_-)$. Let $B_{T,+}$ the associated boundary operator and $P_{T,+,\geq 0}$ the Atiyah-Patodi-Singer projection associated to $B_{T,+}$.

We can now state the main result of this section as follows, whose proof will be given in the next section.

Theorem 2.4. *Under the Assumption 2.3, there exists $T_0 > 0$ such that for any $T \geq T_0$,*

$$\mathrm{ind}(D_{T,+}, P_{T,+,\geq 0}) = \mathrm{ind}(D_{Y,+}, P_{Y,+,\geq 0}). \tag{2.13}$$

3. Proof of Theorem 2.4. The purpose of this section is to prove the Riemann-Roch property, Theorem 2.4, for the index of boundary value problems. The proof we described in [DZ1, Sect. 2] relies on Cheeger's cone method. Here, we will give a more direct proof without passing to manifolds with cone-like singularity. We thus avoid the heat kernel analysis on cylinders and/or cones completely.

The methods and techniques developed by Bismut and Lebeau [BL, Sects. 8, 9] will play an essential role in this section. In fact, what we will do may be thought of as extensions of the Bismut-Lebeau method to manifolds with boundary.

This section is organized as follows. In a), we construct a natural embedding from the space of sections over Y into the space of sections over X. In b), we decompose the total Dirac operator on X to a sum of four operators according to this embedding and introduce a suitable deformation of the Dirac type operators as well as their associated boundary operators. In c), we prove the elliptic estimates for the deformed operators on the boundary. In d), we prove the Fredholm property of the Atiyah-Patodi-Singer type boundary problem for the deformed operators introduced in b). In e), we complete the proof of Theorem 2.4.

Throughout the rest of the paper, we will make the same assumptions and use the same notation as in Section 2.

a). An embedding mapping sections over Y (resp. ∂Y) to sections over X (resp. ∂X). For any $\gamma \geq 0$, let E^γ (resp. $\mathrm{E}^\gamma_{\partial X}$, F^γ, $\mathrm{F}^\gamma_{\partial Y}$) be the set of sections of $S(TX)\hat{\otimes}\xi$ over X (resp. $(S(TX)\hat{\otimes}\xi)|_{\partial X}$ over ∂X, $S(TY)\otimes\mu$ over Y, $(S(TY)\otimes\mu)|_{\partial Y}$ over ∂Y) which lie in the γ^{th} Sobolev space.

Following [BL, Sect. 8g)], for any $y \in Y$, $Z \in N_y$, let $t \in \mathbf{R} \mapsto x_t = \exp^X_y(tZ) \in X$ be the geodesic in X with $x_0 = y$, $\frac{dx_t}{dt}|_{t=0} = Z$. For $\varepsilon > 0$, set $\mathcal{B}_\varepsilon = \{Z \in N : |Z| < \varepsilon\}$. Since X, Y are compact, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the map $(y, Z) \in N \mapsto \exp^X_y(Z) \in X$ is a diffeomorphism from $\mathcal{B}_\varepsilon$ onto a tubular neighborhood $\mathcal{U}_\varepsilon$ of Y in X. From now on, we identify $\mathcal{B}_\varepsilon$ with $\mathcal{U}_\varepsilon$ and use the notation (y, Z) instead of $\exp^X_y(Z)$. In particular, we identify $y \in Y$ with $(y, 0) \in N$.

Let dv_N be the volume form of the fibers in N. Then $dv_Y(y)dv_N(Z)$ is a natural volume form on the total space of N. Let $k(y, Z)$ be the smooth positive function on $\mathcal{B}_{\varepsilon_0}$ defined by

$$dv_X(y, Z) = k(y, Z)dv_Y(y)dv_N(Z). \tag{3.1}$$

The function k has a positive lower bound on $\mathcal{B}_{\varepsilon_0/2}$. Also, $k(y, 0) = 1$.

Now for any $x = (y, Z) \in U_{\varepsilon_0}$, we identify $(S(TX) \otimes \xi)_x$ with $(S(TX) \otimes \xi)_y$ by parallel transport with respect to $\nabla^{S(TX)\otimes\xi}$ along the geodesic $t \mapsto (y, tZ)$. Clearly, this identification preserves the $\mathbf{Z}_2$-grading of $S(TX)\hat{\otimes}\xi$.

Take $\varepsilon \in (0, \varepsilon_0/2]$. Let $\rho : \mathbf{R} \to [0, 1]$ be a smooth function such that $\rho(a) = 1$ if $a \leq 1/2$, while $\rho(a) = 0$ if $a \geq 1$. For $Z \in N$, set $\rho_\varepsilon(Z) = \rho(|Z|/\varepsilon)$.

For $T > 0$, $y \in Y$, set

$$\alpha_T(y) = \int_{N_y} \exp\left(-T|Z|^2\right) \rho^2_\varepsilon(Z)dv_N(Z). \tag{3.2}$$

Definition 3.1. *For any $T > 0$, $\mu \geq 0$, let $J_T : \mathrm{F}^\mu \to \mathrm{E}^\mu$ be defined by*

$$J_T : s \mapsto k^{-1/2}\alpha_T^{-1/2}\rho_\varepsilon(Z)\exp\left(-\frac{T|Z|^2}{2}\right)s. \tag{3.3}$$

One verifies easily that J_T is well-defined. In particular, it induces an isometric embedding $J_T : \mathrm{F}^0 \to \mathrm{E}^0$.

Furthermore, one verifies that (3.3) also induces for any $T > 0$, $\mu \geq 0$, an embedding

$$J_{T,\partial} : \mathrm{F}^\gamma_{\partial Y} \to \mathrm{E}^\gamma_{\partial X}, \tag{3.4}$$

and that $J_{T,\partial} : \mathrm{F}^0_{\partial Y} \to \mathrm{E}^0_{\partial X}$ is an isometric embedding.

b). A decomposition of Dirac type operators under consideration and the associated deformation. For any $T > 0$, let E^0_T (resp. $\mathrm{E}^0_{T,\partial X}$) denote the image of F^0 (resp. $\mathrm{F}^0_{\partial Y}$) under J_T (resp. $J_{T,\partial}$). Let $\mathrm{E}^{0,\perp}_T$ (resp. $\mathrm{E}^{0,\perp}_{T,\partial X}$) be the orthogonal completement of E^0_T (resp. $\mathrm{E}^0_{T,\partial X}$) in E^0 (resp. $\mathrm{E}^0_{\partial X}$). Let p_T, $p^\perp_T$ (resp. $p_{T,\partial X}$, $p^\perp_{T,\partial X}$) be the orthogonal projections from E^0 (resp. $\mathrm{E}^0_{\partial X}$) to E^0_T, $\mathrm{E}^{0,\perp}_T$ (resp. $\mathrm{E}^0_{T,\partial X}$, $\mathrm{E}^{0,\perp}_{T,\partial X}$) respectively.

Recall that the Dirac operators D_T, B_T have been defined in (2.11), (2.12). We now decompose D_T, B_T to

$$D_T = \sum_{i=1}^{4} D_{T,i}\,, \quad B_T = \sum_{i=1}^{4} B_{T,i} \tag{3.5}$$

respectively, where

$$\begin{aligned} D_{T,1} &= p_T D_T p_T, \quad D_{T,2} = p_T D_T p_T^{\perp}, \\ D_{T,3} &= p_T^{\perp} D_T p_T, \quad D_{T,4} = p_T^{\perp} D_T p_T^{\perp}. \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} B_{T,1} &= p_{T,\partial X} B_T p_{T,\partial X}, \quad B_{T,2} = p_{T,\partial X} B_T p_{T,\partial X}^{\perp}, \\ B_{T,3} &= p_{T,\partial X}^{\perp} B_T p_{T,\partial X}, \quad B_{T,4} = p_{T,\partial X}^{\perp} B_T p_{T,\partial X}^{\perp}. \end{aligned} \tag{3.7}$$

We now introduce a deformation of D_T (resp. B_T) according to the decomposition (3.6) (resp. (3.7)).

Definition 3.2. *For any $T > 0$, $u \in [0,1]$, set*

$$D_T(u) = D_{T,1} + D_{T,4} + u\left(D_{T,2} + D_{T,3}\right), \tag{3.8}$$

$$B_T(u) = B_{T,1} + B_{T,4} + u\left(B_{T,2} + B_{T,3}\right). \tag{3.8$'$}$$

One verifies easily that $B_T(u)$ is the boundary operator associated to $D_T(u)$ in the sense of (1.4).

c). Elliptic estimates for $B_T(u)$. The purpose of this subsection is to show that the operators $B_T(u)$ verify the elliptic estimates satisfied by the usual elliptic differential operators, when T is large enough.

In fact, by the geometric assumptions in Section 2a), when restricted to the boundary (see in particular Remark 2.2), as well as Theorem A.3, one can proceed exactly as in [BL, Sect. 8, 9] and [BZ] to show that the following estimates for $B_{T,i}$, $1 \le i \le 4$, hold.

Recall that the construction of J_T depends on a parameter $\varepsilon > 0$.

Proposition 3.3. *There exist $\varepsilon > 0$ such that (a). as $T \to +\infty$,*

$$J_{T,\partial}^{-1} B_{T,1} J_{T,\partial} = B_Y + O\left(\frac{1}{\sqrt{T}}\right) : \Gamma\left((S(TY)\otimes\mu)|_Y\right) \longrightarrow \Gamma\left((S(TY)\otimes\mu)|_Y\right); \tag{3.9}$$

(b). there exist $C_1 > 0$, $C_2 > 0$ and $T_0 > 0$ such that for any $T \ge T_0$, any $s \in \mathrm{E}^{1,\perp}_{T,\partial X} = \mathrm{E}^{0,\perp}_{T,\partial X} \cap \mathrm{E}^1_{\partial X}$, $s' \in \mathrm{E}^1_{T,\partial X} = \mathrm{E}^0_{T,\partial X} \cap \mathrm{E}^1_{\partial X}$, then

$$\begin{aligned} \left\|B_{T,2}s\right\|_0 &\le C_1\left(\frac{\|s\|_1}{\sqrt{T}} + \|s\|_0\right), \\ \left\|B_{T,3}s'\right\|_0 &\le C_1\left(\frac{\|s'\|_1}{\sqrt{T}} + \|s'\|_0\right) \end{aligned} \tag{3.10}$$

and

$$\left\|B_{T,4}s\right\|_0 \ge C_2\left(\|s\|_1 + \sqrt{T}\|s\|_0\right). \tag{3.11}$$

From here, one obtains the following estimates for $B_T(u)$, which says that $B_T(u)$ is a 'small' perturbation of B_T, when T is very large. Thus, it can be regarded as an elliptic estimate for $B_T(u)$.

PROPOSITION 3.4. *There exist $C > 0$ and $T_0 > 0$ such that for any $u \in [0,1]$, $T \geq T_0$ and $s \in \mathrm{E}^1_{\partial X}$, the following inequality holds,*

$$\|B_T s - B_T(u)s\|_0 \leq C\left(\frac{\|B_T s\|_0}{\sqrt{T}} + \|s\|_0\right). \tag{3.12}$$

Proof. By the definitions of B_T and $B_T(u)$, one has

$$B_T s - B_T(u)s = (1-u)(B_{T,2}s + B_{T,3}s). \tag{3.13}$$

From (3.10) and (3.13), one gets that for $u \in [0,1]$, $T \geq T_0$ with $T_0 > 0$ be as in Proposition 3.3, one has

$$\|B_T s - B_T(u)s\|_0 \leq \sqrt{2}C_1\left(\frac{\|s\|_1}{\sqrt{T}} + \|s\|_0\right). \tag{3.14}$$

Now one verifies easily that the super commutator $[B_X, c(\frac{\partial}{\partial r})V|_{\partial X}]$ is of zeroth order. Thus one deduces from (2.12) and the standard estimates for elliptic operators that there exist positive constants A, C_3, C_4 such that

$$\|B_T s\|_0^2 \geq \|B_X s\|_0^2 - TA\|s\|_0^2 \geq C_3\|s\|_1^2 - C_4\|s\|_0^2 - TA\|s\|_0^2. \tag{3.15}$$

It follows then that there exist constants $C_5 > 0$, $C_6 > 0$ such that

$$\|B_T s\|_0 \geq C_5\|s\|_1 - C_6\sqrt{T}\|s\|_0. \tag{3.16}$$

From (3.14) and (3.16), one gets (3.12). □

Since for any $T > 0$, B_T is a self-adjoint elliptic differential operator, Proposition 3.4 and the standard elliptic method enable one to deduce that when $T \geq \max\{T_0, 4C^2\}$, each $B_T(u)$, for $u \in [0,1]$, is self-adjoint and has discrete eigenvalues with finite multiplicity. Let $P_T(u)$ denote the Atiyah-Patodi-Singer projection associated to $B_T(u)$. In the next subsection, we will show that the boundary valued problems $(D_T(u), P_T(u))$, $u \in [0,1]$, are elliptic when T is large enough.

d). The Fredholm property of the boundary problems $(D_T(u), P_T(u))$. We continue the discussion in the previous subsection. In particular, we assume that $T \geq \max\{T_0, 4C^2\}$ so that each $B_T(u)$, $u \in [0,1]$, is formally self-adjoint with discrete eigenvalues of finite multiplicity.

Set, for any $T \geq \max\{T_0, 4C^2\}$ and $u \in [0,1]$,

$$\mathbf{E}^1_T(u) = \left\{s \in \mathrm{E}^1 : P_T(u)\left(s|_{\partial X}\right) = 0\right\}. \tag{3.17}$$

Let

$$D_{T,APS}(u) : \mathbf{E}^1_T(u) \longrightarrow \mathrm{E}^0 \tag{3.18}$$

be the uniquely determined extension of $D_T(u)$.

The main result of this subsection can be stated as follows.

PROPOSITION 3.5. *There exists $T_1 > 0$ such that for any $u \in [0,1]$ and $T \geq T_1$, $D_{T,APS}(u)$ is a Fredholm operator.*

Proof. By standard elliptic methods (cf. [BoW, Chap. 20]), in order to get Proposition 3.5, it suffices to prove the following result.

PROPOSITION 3.6. *There exist $T_1 > 0$, $C_7 > 0$, $C_8 > 0$ such that for any $u \in [0,1]$, $T \geq T_1$ and $s \in \mathbf{E}_T^1(u)$, one has*

$$\|D_T(u)s\|_0 \geq C_7\|s\|_1 - C_8\sqrt{T}\|s\|_0. \tag{3.19}$$

The rest of this subsection is devoted to a proof of Proposition 3.6.

We decompose X into two parts, the interior and the boundary region:

$$X = \left(X \setminus U_{\alpha/3}\right) \cup U_{2\alpha/3}. \tag{3.20}$$

Our proof of Proposition 3.6 consists of three steps, corresponding to the interior, the boundary region and the transition region.

Step 1. The case where s is supported in $X \setminus U_{\alpha/3}$:
Since $\alpha/3 > 0$, using the geometric assumptions in Section 2a), formulas (2.9), (2.10), Theorem A.3 and proceeding as in [BL, Sects. 8, 9] one obtains the following estimates.

LEMMA 3.7. *There exists $\varepsilon > 0$ such that (a). as $T \to +\infty$,*

$$J_T^{-1}D_{T,1}J_T = D_Y + O\left(\frac{1}{\sqrt{T}}\right) : \Gamma\left(S(TY) \otimes \mu\right) \longrightarrow \Gamma\left(S(TY) \otimes \mu\right); \tag{3.21}$$

(b). there exist $C_9 > 0$, $C_{10} > 0$ and $T_2 > 0$ such that for any $T \geq T_2$, any $s \in \mathrm{E}_T^{1,\perp} = \mathrm{E}_T^{0,\perp} \cap \mathrm{E}^1$, $s' \in \mathrm{E}_T^1 = \mathrm{E}_T^0 \cap \mathrm{E}^1$ with $\mathrm{Supp}(|s| + |s'|) \subset X \setminus U_{\alpha/3}$,

$$\|D_{T,2}s\|_0 \leq C_9\left(\frac{\|s\|_1}{\sqrt{T}} + \|s\|_0\right),$$

$$\|D_{T,3}s'\|_0 \leq C_9\left(\frac{\|s'\|_1}{\sqrt{T}} + \|s'\|_0\right) \tag{3.22}$$

and

$$\|D_{T,4}s\|_0 \geq C_{10}\left(\|s\|_1 + \sqrt{T}\|s\|_0\right). \tag{3.23}$$

Now, from (3.21), together with the standard elliptic estimates for D_Y on $Y \setminus U'_{\alpha/3}$ as well as an obvious analogue of [BL, (9.7)], we deduce that there exist constants $C_{11} > 0$, $C_{12} > 0$ such that when T is large enough,

$$\|D_{T,1}p_Ts\|_0 \geq C_{11}\|p_Ts\|_1 - C_{12}\left(1 + \sqrt{T}\right)\|p_Ts\|_0. \tag{3.24}$$

Thus, using (3.8) and (3.22)-(3.24), one deduces that for T large enough and $u \in [0,1]$,

$$\begin{aligned}\|D_T(u)s\|_0 \geq {} & \left(C_{11} - \frac{C_9}{\sqrt{T}}\right)\|p_Ts\|_1 - \left(C_9 + C_{12} + C_{12}\sqrt{T}\right)\|p_Ts\|_0 \\ & + \left(C_{10} - \frac{C_9}{\sqrt{T}}\right)\left\|p_T^\perp s\right\|_1 + \left(C_{10}\sqrt{T} - C_9\right)\left\|p_T^\perp s\right\|_0.\end{aligned} \tag{3.25}$$

Estimate (3.19) follows as a consequence.

Step 2. The case where s is supported in $U_{2\alpha/3}$:

The key observation in this case is that since all the geometric data are of product nature on U_α, one can use separation of variables to split the analysis into those along the $\frac{\partial}{\partial r}$ direction and those along the cross section $\{r\}\times\partial X$'s with $0\le r\le 2\alpha/3$ on which the analysis is the same as on ∂X. In particular, by (1.4), (2.11) and (2.12) one can write on $U_{2\alpha/3}$ that

$$D_T = c\left(\frac{\partial}{\partial r}\right)\left(\frac{\partial}{\partial r}+B_T\right). \tag{3.26}$$

Furthermore, by the definition of the embedding J_T as well as its restriction on ∂X, and thus on each $\{r\}\times\partial X$, $0\le r\le 2\alpha/3$, also, one deduces from (3.26) the following formula on $U_{2\alpha/3}$,

$$D_T(u) = c\left(\frac{\partial}{\partial r}\right)\left(\frac{\partial}{\partial r}+B_T(u)\right),\quad u\in[0,1]. \tag{3.27}$$

One also verifies easily that $B_T(u)$ anti-commutes with $c(\frac{\partial}{\partial r})$. Thus from (3.27) one gets

$$(D_T(u))^2 = -\frac{\partial^2}{\partial r^2}+(B_T(u))^2. \tag{3.28}$$

From (3.27), (3.28) and Green's formula (cf. [BoW, Chap. 3]), one deduces easily that for any $s\in\Gamma(S(TX)\otimes\xi)$ which is supported in $U_{2\alpha/3}$,

$$\|D_T(u)s\|_0^2 = \int_{[0,\frac{2\alpha}{3}]}\langle B_T(u)s, B_T(u)s\rangle_{\{r\}\times\partial X}\,dr + \left\|\frac{\partial s}{\partial r}\right\|_0^2 - \langle s, B_T(u)s\rangle_{\partial X}. \tag{3.29}$$

Now if s also verifies the boundary condition under consideration, that is,

$$P_T(u)\,(s|_{\partial X}) = 0, \tag{3.30}$$

then one finds

$$\langle s, B_T(u)s\rangle_{\partial X}\le 0. \tag{3.31}$$

On the other hand, it is clear that one can apply the analysis in Section 3c) to each $\{r\}\times\partial X$. Thus by (3.12), (3.16) one deduces that there exist constants $C_{13}>0$, $C_{14}>0$ such that when T is large enough,

$$\begin{aligned}\|B_T(u)s\|_{\{r\}\times\partial X,0} &\ge \left(1-\frac{C}{\sqrt{T}}\right)\|B_Ts\|_{\{r\}\times\partial X,0} - C\|s\|_{\{r\}\times\partial X,0}\\ &\ge C_{13}\|s\|_{\{r\}\times\partial X,1} - C_{14}\sqrt{T}\|s\|_{\{r\}\times\partial X,0}.\end{aligned} \tag{3.32}$$

From (3.29), (3.31) and (3.32), one deduces (3.19) easily.

Step 3. The general case:

Now by the results in Steps 1 and 2, one can apply the gluing argument in [BL, pp. 115-117] to complete the proof of Proposition 3.6. □

The proof of Proposition 3.5 is thus also completed. □

e). Proof of Theorem 2.4. We assume that $T \geq T_1$ with T_1 determined by Proposition 3.5. We will first show that when T is large enough, the family of Fredholm operators $D_{T,APS}(u)$, $0 \leq u \leq 1$, constructed in Proposition 3.5 is a continuous family. For this, one establishes the following result.

PROPOSITION 3.8. *There exists $T_2 > 0$ such that for any $T \geq T_2$, $u \in [0,1]$, the operator $B_T(u)$ is invertible.*

Proof. Recall from Assumption 2.3 that B_Y is invertible. Let $c > 0$ be such that

$$\mathrm{Spec}(B_Y) \cap [-2c, 2c] = \emptyset. \tag{3.33}$$

Proposition 3.8 follows from

LEMMA 3.9. *There exists $T_2 > 0$ such that for any $T \geq T_2$, $u \in [0,1]$ and $s \in \mathrm{E}^1_{\partial X}$, then*

$$\|B_T(u)s\|_0 \geq \frac{3c}{2}\|s\|_0. \tag{3.34}$$

Proof. We proceed similarly as in the proof of [TZ, Lemma 4.7]. Write s as $s = s' + s''$ with $s' \in \mathrm{E}^1_{T,\partial X}$ and $s'' \in \mathrm{E}^{1,\perp}_{T,\partial X}$. Then one has

$$\|B_T(u)s\|_0^2 = \|B_{T,1}s' + uB_{T,2}s''\|_0^2 + \|uB_{T,3}s' + B_{T,4}s''\|_0^2, \tag{3.35}$$

from which it follows that for any sufficiently small $\nu > 0$, one has

$$\begin{aligned} \|B_T(u)s\|_0 &\geq \frac{7}{8}\|B_{T,1}s' + uB_{T,2}s''\|_0 + \nu\|uB_{T,3}s' + B_{T,4}s''\|_0 \\ &\geq \frac{7}{8}\|B_{T,1}s'\|_0 - \frac{7}{8}\|B_{T,2}s''\|_0 + \nu\|B_{T,4}s''\|_0 - \nu\|B_{T,3}s'\|_0. \end{aligned} \tag{3.36}$$

In view of (3.33), one sees easily that

$$\left\|J_T B_Y J_T^{-1} s'\right\|_0 \geq 2c\,\|s'\|_0. \tag{3.37}$$

From (3.37) and Proposition 3.3a), one deduces that there exists $C_{15} > 0$ such that when T is sufficiently large, one has

$$\frac{7}{8}\|B_{T,1}s'\|_0 \geq \frac{3c}{2}\|s'\|_0 + \frac{1}{8}\left\|J_T B_Y J_T^{-1}s'\right\|_0 - \frac{C_{15}}{\sqrt{T}}\left(\left\|J_T B_Y J_T^{-1}s'\right\|_0 + \|s'\|_0\right). \tag{3.38}$$

From (3.37), (3.38) one finds that when T is sufficiently large,

$$\frac{7}{8}\|B_{T,1}s'\|_0 \geq \frac{3c}{2}\|s'\|_0 + \frac{1}{16}\left\|J_T B_Y J_T^{-1}s'\right\|_0. \tag{3.39}$$

On the other hand, by standard elliptic estimates as well as an obvious analogue of [BL, (9.7)], there exists constant $C_{16} > 0$ such that

$$\|s'\|_1 \leq C_{16}\left(\left\|J_T B_Y J_T^{-1}s'\right\|_0 + \sqrt{T}\|s'\|_0\right). \tag{3.40}$$

By (3.36)-(3.40) and Proposition 3.3b), one deduces that when T is sufficiently large,

$$
\begin{aligned}
(3.41)\quad \|B_T(u)s\|_0 \geq \frac{3c}{2}\|s'\|_0 + \frac{1}{32}\left\|J_T B_Y J_T^{-1}s'\right\|_0 + \frac{c}{16}\|s'\|_0 - \frac{7C_1}{8}\left(\frac{\|s''\|_1}{\sqrt{T}} + \|s''\|_0\right) \\
+ \nu C_2\left(\|s''\|_1 + \sqrt{T}\|s''\|_0\right) - \nu C_1\left(\frac{C_{16}\left\|J_T B_Y J_T^{-1}s'\right\|_0}{\sqrt{T}} + (C_{16}+1)\|s'\|_0\right) \\
\geq \frac{3c}{2}\|s'+s''\|_0 + \left(\frac{1}{32} - \frac{\nu C_1 C_{16}}{\sqrt{T}}\right)\left\|J_T B_Y J_T^{-1}s'\right\|_0 + \left(\frac{c}{16} - \nu C_1(C_{16}+1)\right)\|s'\|_0 \\
+ \left(\eta C_2 - \frac{7C_1}{8\sqrt{T}}\right)\|s''\|_1 + \left(\nu C_2\sqrt{T} - \frac{7C_1}{8} - \frac{3c}{2}\right)\|s''\|_0.
\end{aligned}
$$

Now if we choose $\nu > 0$ so that one also has

$$
(3.42)\qquad \frac{c}{16} - \nu C_1(C_{16}+1) \geq 0,
$$

then from (3.41) one deduces easily that when T is sufficiently large, (3.34) holds. □

The proof of Proposition 3.8 is completed. □

From Proposition 3.5 and Proposition 3.8, we see that when T is large enough, we have a continuous family of Fredholm operators $\{D_{T,APS}(u)\}_{0\leq u\leq 1}$. Furthermore, by Proposition 3.8 and Green's formula, the operators $D_{T,APS}(u)$, $0 \leq u \leq 1$, are self-adjoint.

Now let τ_X (resp. τ_Y) be the $\mathbf{Z}_2$-grading operator of $S(TX)\hat{\otimes}\xi$ (resp. $S(TY)\otimes\mu$). One verifies directly that

$$
(3.43)\qquad J_T\tau_Y = \tau_X J_T,
$$

that is, J_T preserves the $\mathbf{Z}_2$-gradings of $S(TY)\otimes\mu$ and $S(TX)\hat{\otimes}\xi$.

From the above discussions as well as the homotopy invariance of the index of Fredholm operators, one gets easily that

$$
(3.44)\qquad \operatorname{ind}(D_{T,+}, P_{T,+,\geq 0}) = \operatorname{Tr}\left[\tau_X|_{\ker(D_{T,APS}(1))}\right] = \operatorname{Tr}\left[\tau_X|_{\ker(D_{T,APS}(0))}\right].
$$

Now let $P_{T,1}$ (resp. $P_{T,4}$) be the Atiyah-Patodi-Singer projection associated to $B_{T,1}$ (resp. $B_{T,4}$) acting on $\mathrm{E}^0_{T,\partial X}$ (resp. $\mathrm{E}^{0,\perp}_{T,\partial X}$). Then by using Proposition 3.3 and proceed as in Section 3d), one sees easily that the boundary problems $(D_{T,1}, P_{T,1})$ and $(D_{T,4}, P_{T,4})$ are Fredholm. Furthermore, by (3.11), (3.23) and (3.29), one deduces that when T is large enough,

$$
(3.45)\qquad \ker(D_{T,4}, P_{T,4}) = 0.
$$

On the other hand, for T large enough and $u \in [0,1]$, set

$$
(3.46)\quad D_Y(u) = uD_Y + (1-u)J_T^{-1}D_{T,1}J_T, \quad B_Y(u) = uB_Y + (1-u)J_{T,\partial}^{-1}B_{T,1}J_{T,\partial}.
$$

From (3.9) one can proceed as in (3.37)-(3.39) to see that when T is large enough, $B_Y(u)$ is invertible for every $u \in [0,1]$.

Let $P_Y(u)$ be the Atiyha-Patodi-Singer projection associated to $B_Y(u)$. By (3.9), (3.21) and the above discussion one sees that when T is large enough, $(D_Y(u), P_Y(u))$,

$u \in [0,1]$, form a continuous family of formally self-adjoint Fredholm boundary problems. Thus by the homotopy invariance of the index of Fredholm operators, one gets

$$\mathrm{Tr}\left[\tau_Y|_{\ker(D_Y(0),P_Y(0))}\right] = \mathrm{Tr}\left[\tau_Y|_{\ker(D_Y(1),P_Y(1))}\right] = \mathrm{ind}(D_{Y,+},P_{Y,+,\geq 0}). \tag{3.47}$$

From (3.43)-(3.45) and (3.47) one finds

$$\begin{aligned} \mathrm{ind}\,(D_{T,+},P_{T,+,\geq 0}) &= \mathrm{Tr}\left[\tau_X|_{\ker(D_{T,1},P_{T,1})}\right] + \mathrm{Tr}\left[\tau_X|_{\ker(D_{T,4},P_{T,4})}\right] \\ &= \mathrm{Tr}\left[\tau_Y|_{\ker(D_Y(0),P_Y(0))}\right] = \mathrm{ind}(D_{Y,+},P_{Y,+,\geq 0}), \end{aligned} \tag{3.48}$$

which is exactly (2.13).

The proof of Theorem 2.4 is completed. □

4. The Atiyah-Patodi-Singer index theorem for Dirac operators. In this section, we combine Theorems 1.2 and 2.4 with the results in [BZ] to complete our embedding proof of the Atiyah-Patodi-Singer index theorem [APS1, (4.3)] for Dirac operators.

This section is organized as follows. In a), we use Theorem 2.4 to refine the main result in [BZ] so that the mod **Z** term in [BZ, Theorem 2.2] can now be made specific in our situation. In b), we apply the results proved in a) to the case where X is a ball to obtain the Atiyah-Patodi-Singer index theorem for Dirac operators on Y.

a). Real embeddings and η-invariants. Following [BZ, (1.27)], under the geometric assumptions in Section 2a), let γ^X be the Chern-Simons current on X defined by

$$\gamma^X = \int_0^{+\infty} \mathrm{Tr}_s\left[V \exp\left(-\left(\nabla^\xi + T^{1/2}V\right)^2\right)\right]\frac{dT}{2T^{1/2}}. \tag{4.1}$$

Also, if D is a formally self-adjoint Dirac type operator on a closed odd dimensional spin manifold, we define the reduced η-invariant to be

$$\bar\eta(D) = \frac{\dim(\ker D) + \eta(D)}{2}, \tag{4.2}$$

where $\eta(D)$ is the η-invariant of D in the sense of Atiyah-Patodi-Singer [APS1].

Let $D^{\xi\pm}_{+,APS}$ (resp. $D^{\mu}_{+,APS}$) denote the Atiyah-Patodi-Singer boundary problems associated to $D^{\xi\pm}_{+}$ (resp. D^{μ}_{+}). Let $B^{\xi\pm}_{+}$ (resp. B^{μ}_{+}) be the induced Dirac operators on ∂X (resp. ∂Y) associated to $D^{\xi\pm}_{+}$ (resp. D^{μ}_{+}).

Let R^{TX}, R^{TY} denote the curvature of ∇^{TX}, ∇^{TY} respectively.

We can now state the main result of this subsection, which has been announced in [DZ1, Theorem 2.3], as follows.

Theorem 4.1. *The following identity holds,*

$$\begin{aligned} &\mathrm{ind}\left(D^{\xi+}_{+,APS}\right) + \bar\eta\left(B^{\xi+}_{+}\right) - \mathrm{ind}\left(D^{\xi-}_{+,APS}\right) - \bar\eta\left(B^{\xi-}_{+}\right) \\ &= \mathrm{ind}\left(D^{\mu}_{+,APS}\right) + \bar\eta\left(B^{\mu}_{+}\right) + \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{\dim X}{2}}\int_{\partial X}\det{}^{1/2}\left(\frac{R^{TX}/2}{\sinh\left(R^{TX}/2\right)}\right)\gamma^X. \end{aligned} \tag{4.3}$$

Remark 4.2. A weaker mod $\mathbf{Z}$ version of Theorem 4.1 has been previously proved in [BZ, Theorem 2.2].

The rest of this subsection is devoted to a proof of Theorem 4.1 by making precise the mod $\mathbf{Z}$ contribution in [BZ].

For any $T \geq 0$, let $D^{\xi}_{T,+,APS}$ be the Dirac type operator

$$D^{\xi}_{+} + TV : \Gamma\left(\left(S(TX)\hat{\otimes}\xi\right)_{+}\right) \longrightarrow \Gamma\left(\left(S(TX)\hat{\otimes}\xi\right)_{-}\right) \tag{4.4}$$

verifying the Atiyah-Patodi-Singer boundary condition [APS1]. Let $B^{\xi}_{T,+}$ be the associated boundary operator on ∂X in the sense of (1.4).

We start with the following result which was announced in [DZ1, Prop. 2.1].

Proposition 4.3. *The quantity* $\operatorname{ind}(D^{\xi}_{T,+,APS}) + \bar{\eta}(B^{\xi}_{T,+})$ *does not depend on* $T \geq 0$.

Proof. From Theorem 1.2 and a direct counting argument in using the definition of the reduced η-invariant, one sees easily that $\operatorname{ind}(D^{\xi}_{T,+,APS}) + \bar{\eta}(B^{\xi}_{T,+})$ depends smoothly on T. Proposition 4.3 then follows from the local variation formula [BC3, Theorem 2.7] of Bismut and Cheeger. □

Remark 4.4. To be more precise, in [BC3, Theorem 2.7], Bismut and Cheeger considered the operators of the form $B^{\xi}_{+}\tau^{\xi} + TV$ acting on $\Gamma((S_{+}(TX) \otimes \xi)|_{Y})$, where τ^{ξ} is the $\mathbf{Z}_2$-grading operator of ξ. However, one sees easily that the map $U : \Gamma((S_{+}(TX) \otimes \xi)|_{Y}) \rightarrow \Gamma((S_{+}(TX) \otimes \xi_{+})|_{Y} \oplus (S_{-}(TX) \otimes \xi_{-})|_{Y})$ defines by $U : u \otimes (v_{+} + v_{-}) \mapsto u \otimes v_{+} - c(\frac{\partial}{\partial t}) u \otimes v_{-}$ is unitary and verifies that $U(B^{\xi}_{+}\tau^{\xi} + TV)U^{-1} = B^{\xi}_{T,+}$. This makes it clear that one can apply the results in [BC3] and [BZ] to the present situation.

We can now proceed as in [BZ]. The key observation is that the geometric assumptions in [BZ, Sect. 1b)] correspond almost exactly to the geometric assumptions on ∂X in the current situation, with the minor diffference that we here use $-c(\frac{\partial}{\partial r})\tilde{c}(Z)\tau^{N*}$ to replace $\sqrt{-1}\tilde{c}(Z)$ in [BZ, (1.10)]. As a result, we will use here Theorem A.3 in the Appendix to replace [BZ, Theorem 4.5] in obtaining the analogues of the analytic results of [BZ, Theorems 3.7-3.12].

We now examine the arguments in [BZ, Sect. 3e)]. In order to get the required result, we must find out where the mod $\mathbf{Z}$ terms in [BZ, Sect. 3e)] arise, and replace them by the exact formulas.

In fact, one finds that this integer term is given by

$$-\sharp\left\{\lambda \in \operatorname{Spec}\left(B^{\xi}_{T_0,+}\right) : -a_0 \leq \lambda < 0\right\} \tag{4.5}$$

with $a_0 > 0$ such that B^{μ}_{+} has no non-zero eigenvalues in $[-2a_0, 2a_0]$. This term must be added in the right hand side of the analogue of [BZ, (3.30)].

Secondly, one uses Proposition 4.3 instead of a direct analogue of [BZ, (3.44)].

With the help of these two observations, by proceeding as in [BZ, Sect. 3e)], one finally finds, in our situation, the following refinement of a direct analogue of [BZ, (3.65)] when $T_0 > 0$ is large enough:

$$\begin{aligned} -\sharp\left\{\lambda \in \operatorname{Spec}\left(B^{\xi}_{T_0,+}\right) : -a_0 \leq \lambda < 0\right\} + \operatorname{ind}\left(D^{\xi}_{T_0,+,APS}\right) - \operatorname{ind}\left(D^{\xi}_{0,+,APS}\right) \\ - \bar{\eta}\left(B^{\xi}_{0,+}\right) + \bar{\eta}\left(B^{\mu}_{+}\right) + \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{\dim X}{2}} \int_{\partial X} \det{}^{1/2}\left(\frac{R^{TX}/2}{\sinh\left(R^{TX}/2\right)}\right)\gamma^{X} = 0. \end{aligned} \tag{4.6}$$

We now prove two lemmas which together with (4.6) will give (4.3). The first lemma follows easily from the definitions of the operators under consideration, the definition of the reduced η-invariant as well as the classical Agranovič-Dynin type formula (cf. [BoW, Chap. 21]).

LEMMA 4.5. *The following identity holds,*

(4.7)
$$\operatorname{ind}\left(D^{\xi}_{0,+,APS}\right)+\bar\eta\left(B^{\xi}_{0,+}\right)=\operatorname{ind}\left(D^{\xi+}_{+,APS}\right)+\bar\eta\left(B^{\xi+}_{+}\right)-\operatorname{ind}\left(D^{\xi-}_{+,APS}\right)-\bar\eta\left(B^{\xi-}_{+}\right).$$

An equivalent form of the next lemma has been announced in [DZ1, Theorem 2.2].

LEMMA 4.6. *The following identity holds when $T_0>0$ is sufficiently large,*

$$-\sharp\left\{\lambda\in\operatorname{Spec}\left(B^{\xi}_{T_0,+}\right):-a_0\le\lambda<0\right\}+\operatorname{ind}\left(D^{\xi}_{T_0,+,APS}\right)=\operatorname{ind}\left(D^{\mu}_{+,APS}\right). \tag{4.8}$$

Proof. Let $f:X\to\mathbf{R}$ be a smooth function such that $f\equiv 1$ on $U_{\alpha/3}$ and $f\equiv 0$ outside of $U_{2\alpha/3}$.

Let $D_{T_0,-a_0,+}$ be the Dirac type operator defined by

$$D_{T_0,-a_0,+}=D^{\xi}_{T_0,+}-a_0fc\left(\frac{\partial}{\partial r}\right):\Gamma\left(\left(S(TX)\hat\otimes\xi\right)_+\right)\longrightarrow\Gamma\left(\left(S(TX)\hat\otimes\xi\right)_-\right). \tag{4.9}$$

Let $D_{Y,-a_0,+}$ be the Dirac type operator defined by

$$D_{Y,-a_0,+}=D^{\mu}_{+}-a_0fc\left(\frac{\partial}{\partial r}\right):\Gamma\left(S_+(TY)\otimes\mu\right)\longrightarrow\Gamma\left(S(TY)_-\otimes\mu\right). \tag{4.10}$$

Let $D_{T_0,-a_0,+,APS}$, $D_{Y,-a_0,+,APS}$ be the associated operators verifying the Atiyah-Patodi-Singer boundary condition [APS1].

Since $-a_0$ is not an eigenvalue of B^{μ}_{+}, one sees that the Assumption 2.3 is verified by the boundary operator associated to $D_{Y,-a_0,+}$. Thus one can apply Theorem 2.4 to get that when T_0 is sufficiently large, one has

$$\operatorname{ind}\left(D_{T_0,-a_0,+,APS}\right)=\operatorname{ind}\left(D_{Y,-a_0,+,APS}\right). \tag{4.11}$$

Now by using Theorem 1.2 and the definition of the spectral flow [APS2], one verifies easily that

(4.12)
$$\operatorname{ind}\left(D_{T_0,-a_0,+,APS}\right)=\operatorname{ind}\left(D^{\xi}_{T_0,+,APS}\right)-\sharp\left\{\lambda\in\operatorname{Spec}\left(B^{\xi}_{T_0,+}\right):-a_0\le\lambda<0\right\}$$

and that

$$\operatorname{ind}\left(D_{Y,-a_0,+,APS}\right)=\operatorname{ind}\left(D^{\mu}_{+,APS}\right). \tag{4.13}$$

From (4.11)-(4.13), one gets (4.8). □

From (4.6)-(4.8), one gets (4.3). The proof of Theorem 4.1 is now completed. □

b). A proof of the Atiyah-Patodi-Singer index theorem. We first state an easy consequence of a direct analogue in our situation of [BZ, Theorem 1.4].

LEMMA 4.7. *The following identity holds,*

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{\dim X}{2}}\int_X \det{}^{1/2}\left(\frac{R^{TX}/2}{\sinh\left(R^{TX}/2\right)}\right)\mathrm{Tr}_s\left[\exp\left(-\left(\nabla^{\xi}\right)^2\right)\right]$$
$$-\left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{\dim Y}{2}}\int_Y \det{}^{1/2}\left(\frac{R^{TY}/2}{\sinh\left(R^{TY}/2\right)}\right)\mathrm{Tr}\left[\exp\left(-\left(\nabla^{\mu}\right)^2\right)\right]$$
$$=\left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{\dim X}{2}}\int_{\partial X} \det{}^{1/2}\left(\frac{R^{TX}/2}{\sinh\left(R^{TX}/2\right)}\right)\gamma^X. \tag{4.14}$$

Finally, we are in a position to give a proof of the Atiyah-Patodi-Singer Thorem. Namely, we apply these results to the case where $X = D^{2n}$, the $2n$ dimensional ball with n sufficiently large. That any compact manifold with boundary can be embedded into a large ball in such a fashion is an elementary result from differential topology.

Since D^{2n} is contractable, both $\xi_\pm$ are topologically trivial over D^{2n}. Thus one can deform the metric g^{ξ_+} to g^{ξ_-} by $g(u) = (1-u)g^{\xi_+} + ug^{\xi_-}$, $0 \le u \le 1$. One thus obtains easily a smooth deformation of twisted Dirac operators moving from D^{ξ_+} to D^{ξ_-}. By using Theorem 1.2 as well as the standard local variation formula of the η invariants (cf. [APS2] and [BF, Sect. 2]), one gets easily the following identity

$$\mathrm{ind}\left(D^{\xi_+}_{+,APS}\right)+\bar{\eta}\left(B^{\xi_+}_{+}\right)-\mathrm{ind}\left(D^{\xi_-}_{+,APS}\right)-\bar{\eta}\left(B^{\xi_-}_{+}\right)$$
$$=\left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{\dim X}{2}}\int_X \det{}^{1/2}\left(\frac{R^{TX}/2}{\sinh\left(R^{TX}/2\right)}\right)\mathrm{Tr}_s\left[\exp\left(-\left(\nabla^{\xi}\right)^2\right)\right]. \tag{4.15}$$

From (4.3), (4.14) and (4.15), one finds

$$\mathrm{ind}\left(D^{\mu}_{+,APS}\right)$$
$$=\left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{\dim Y}{2}}\int_Y \det{}^{1/2}\left(\frac{R^{TY}/2}{\sinh\left(R^{TY}/2\right)}\right)\mathrm{Tr}\left[\exp\left(-\left(\nabla^{\mu}\right)^2\right)\right]-\bar{\eta}\left(B^{\mu}_{+}\right), \tag{4.16}$$

which is exactly the Atiyah-Patodi-singer index theorem [APS1, (4.3)] for $D^{\mu}_{+,APS}$.

This completes our embedding proof of the Atiyah-Patodi-Singer index theorem [APS1, (4.3)] for Dirac operators on manifolds with boundary. □

Appendix A. Dirac operators and harmonic oscillators. Let E be a real oriented Euclidean vector space of even dimension. Let $S(E) = S_+(E) \oplus S_-(E)$ be the $\mathbf{Z}_2$-graded Hermitian vector space of E-spinors.

If $e \in E$, let $e^* \in E^*$ corresponds to e by the scalar product. Let $c(e)$ denote the Clifford action of e on $S(E)$. Let $\tilde{c}(e)$ denote the corresponding Clifford action of e on $S^*(E) = S^*_+(E) \oplus S^*_-(E)$.

Let τ be the $\mathbf{Z}_2$-grading operator of $S(E)$, that is, $\tau|_{S_\pm(E)} = \pm \mathrm{id}_{S_\pm(E)}$. Let τ^* be the transpose of τ. Then $\sigma = \tau \otimes \tau^*$ is the $\mathbf{Z}_2$-grading operator on $\wedge(E^*)$.

Recall the identification of the $\mathbf{Z}_2$-graded vector spaces,

$$\wedge(E^*) \simeq S(E)\hat{\otimes}S^*(E). \tag{A.1}$$

For any $e \in E$, let $c(e)$ (resp. $\tilde{c}(e)$) acts on $\wedge(E^*)$ as $c(e)\hat{\otimes}1$ (resp. $1\hat{\otimes}\tilde{c}(e)$). Then, under the identification (A.1), $c(e)$, $\tilde{c}(e)$ acts on $\wedge(E^*)$ as

$$\begin{aligned} c(e) &= e^* \wedge - i_e, \\ \tilde{c}(e)\tau^* &= e^* \wedge + i_e \end{aligned} \tag{A.2}$$

respectively (Compare with [BZ, (4.5)]).

Let $e_1, \ldots, e_{\dim E}$ be an oriented orthonormal base of E. Let $e_1^*, \ldots, e_{\dim E}^*$ be the dual base of E^*.

Let $\Gamma(\wedge(E^*))$ be the vector space of smooth sections of $\wedge(E^*)$ over E.

Definition A.1. *Let $D^{\wedge(E^*)}$ be the operator acting on $\Gamma(\wedge(E^*))$,*

$$D^{\wedge(E^*)} = \sum_{i=1}^{\dim E} c(e_i)\nabla_{e_i}, \tag{A.3}$$

where ∇ is the canonical flat connection acting on $\Gamma(\wedge(E^*))$.

Let Z be the generic point of E. Then $\tilde{c}(Z)\tau^*$ acts on $\Gamma(\wedge(E^*))$.

Proposition A.2. *For any $T \in \mathbf{R}$, the following identity holds,*

$$\left(D^{\wedge(E^*)} + T\tilde{c}(Z)\tau^*\right)^2 = -\sum_{i=1}^{\dim E} \nabla_{e_i}^2 + T^2|Z|^2 + T\left(\dim E - 2\sum_{i=1}^{\dim E} i_{e_i} e_i^* \wedge\right). \tag{A.4}$$

Proof. From (A.2), one gets (A.4) by a direct calculation. □

Now one verifies easily that the lowest eigenvalue of $-\sum_{i=1}^{\dim E} i_{e_i} e_i^* \wedge$ is $-\dim E$ with the corresponding eigenspace being one dimensional and spanned by 1. From this and from the standard property of the harmonic oscillator, one gets

Theorem A.3. *The kernel of the operator $D^{\wedge(E^*)} + T\tilde{c}(Z)\tau^*$ is one demensional and is spanned by*

$$\beta = \exp\left(-\frac{T|Z|^2}{2}\right). \tag{A.5}$$

Furthermore, there exists $C > 0$ such that $D^{\wedge(E^*)} + T\tilde{c}(Z)\tau^*$ has no nonzero eigenvalue in $[-C\sqrt{T}, C\sqrt{T}]$.

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