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Aspects analytiques de la théorie de l'indice

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Abstract. This thesis consists of several topics centered around index theory. These topics include the study of the Ray-Singer analytic torsion, the Atiyah-Patodi-Singer η invariant, spectral flow and its families generalization, the Guillemin-Sternberg geometric quantization conjecture and its various extensions, the index theorems for Dirac operators on even and odd dimensional manifolds with boundary, the analytic espects of the Kervaire semi-characteritic, elliptic genus, Rohklin congruences and its higher dimensional generalizations, Bergman kernels in geometric quantization, etc.

Résumé. Cette thèse contient des résultats relatifs à la théorie de l'indice. On considère en particulier: la torsion analytique de Ray-Singer, l'invariant η d'Atiyah-Patodi-Singer, le flot spectral et sa généralisation en famille, la conjecture de la quantification géométrique de Guillemin-Sternberg et ses diverses extensions, le théorème de l'indice de l'opérateur de Dirac pour les variétés à bord en dimension paire ou impaire, les aspects analytiques de la semi-caractéristique de Kervaire, le genre elliptique, les congruences de Rohklin et leur généralisation en dimension supérieure, le noyau de Bergman en quantification géométrique, etc.

Contents

Pour la commodité du lecteur, des notes aux Comptes-Rendus ont été ajoutées au texte de l'habilitation.

PRÉSENTATION DES TRAVAUX:

ANALYTIC ASPECTS OF INDEX THEORY

1. Earlier works

These are the papers I wrote before coming to France to pursue my Ph. D. study.

In [1], we gave a slight generalization of the Bott residue formula on complex manifolds, where the zeroes of the holomorphic vector field are allowed to be non-degenerate complex submanifolds (instead of points as in Bott's original formula). The proof is inspired by Bismut's Gaussian proof of the famous Duistermaat-Heckman and Berline-Vergne localization formulas.

In the joint work with Lafferty and Yanlin Yu [4] and [59], we gave a direct geometric proof of the Atiyah-Bott-Segal-Singer equivariant index theorem for Dirac operators. Comparing with Bismut's proof using probability as well as Berline-Vernge's proof using frame bundles, our proof is closer in spirit to the heat kernel proofs of the local index theorem for Dirac operators due to Getzler and Yanlin Yu respectively.

In paper [2], we proved a regularity result for the equivariant η-function of Dirac operators by using the method in [4]. Our result generalizes the corresponding result in the non-equivariant case due to Bismut and Freed.

In [10], we gave an alternate proof without using probability of the local index theorem due to Bismut for Dirac operators associated with certain non torsion-free connections.

There is another paper [58] in which we made an effort to understand Bismut's local index theorem for a family of Dirac operators.

2. Reidemeister torsion and Ray-Singer analytic torsion

Reidemeister torsion is a classical concept in topology associated to orthogonal representations of the fundamental group of a CW complex.

Inspired by the Atiyah-Singer index theory, Ray and Singer studied in 1970 an analogue of the Reidemeister torsion for the de Rham complex on a smooth manifold. They called this analogue the analytic torsion and discovered that it has a lot of similar properties like that of Reidemeister torsion. They further made the conjecture that their analytic torsion (now widely called the Ray-Singer torsion) equals to the Reidemeister torsion.

The above Ray-Singer conjecture was proved in 1978, independently by Cheeger and Müller. The Cheeger-Müller theorem holds for orthogonal representations of the fundamental group. Later, Müller extended it to the case of unimoduler representations.

The proofs of Cheeger and Müller use many ideas in topology and reduce the problem to that on spheres.

In [3] and [6], jointly with Bismut, we extended the Cheeger-Müller theorem to the case of arbitrary representations of the fundamental group. Moreover, the method we used is purely analytic and is quite different from what used by Cheeger and Müller. It uses the Witten deformation for the de Rham complex by a Morse function and relies on the analytic techniques developed in the long paper of Bismut-Lebeau on complex immersions and Quillen metrics. An adaptation of Helffer-Sjöstrand's rigorous proof of Witten's proposal of deriving the Thom-Smale complex through the Witten deformation also plays an important role in this proof.

In [12], again jointly with Bismut, we further generalized the results in [3] and [6] to the case where the manifold under consideration admits a finite group action. The results obtained generalize the earlier results in this direction obtained by Lott-Rothenberg and Lück. Moreover, this paper also contains an alternate treatment of the relationship between the Thom-Smale complex and the Witten complex, simpler than that of Helffer and Sjöstrand.

In [46], we generalized the main result in [3], [6] to the case of infinite Galois covering of a smooth manifold. Here one deals with the L^2-torsions, introduced first by Lott and Mathai, which are defined with the help of the von Neumann trace associated to the covering group. Our main result is stated for extended cohomologies and thus generalizes the L^2 generalization, due to Burghelea et al, of the Cheeger-Müller theorem, in that we do not assume the so-called determinant class condition. It also extends the corresponding result of Braverman-Carey-Farber-Mathai to the case of general representations. An important anomaly for L^2-analytic torsions was established in [46] during the process of the proof.

In the joint paper [61] with Xiaonan Ma, we proved an anomaly formula for L^2-analytic torsions on manifolds with boundary. It generalizes the anomaly formula in [46] as well as the anomaly formula for the usual analytic torsion on manifolds with boundary established by Brünning and Ma.

Paper [54], which is a joint work with Guangxiang Su, deals with the complex analytic torsion introduced by Burghelea and Haller. In it, we solved a conjecture of Burghelea and Haller, identifying the complex analytic torsion with the corresponding complex Reidemeister torsion. The statement and proof in [54] are parallel to what in [6]. The key difference is that in dealing with complex torsions, one deals with the non-self-adjoint Laplacians instead of the self-adjoint ones in [6]. However, we showed that the main method in [6] still applies in the current situation, with necessary modifications. Burghelea and Haller proved their conjecture, up to sign, around the same time as ours. Their proof is different from ours.

In the joint papers [49], [63] with Weiping Li, we introduced what we call the L^2-Alexander-Conway invariant for knots. These invariants can be interpreted as the twisted L^2-Reidemeister torsion of the knot complement. A surprising rigidity in the twisted $U(1)$-representation case was established.

3. The adiabatic limits of η-invariants and Rokhlin congruences

It is well-known that the η-invariant, introduced by Atiyah-Patodi-Singer in their study of the index theorem for manifolds with boundary, appears in many places in geometry, topology as well as mathematical physics. For example, Witten proposed that the adiabatic limit of the η-invariant associated to Dirac operators on a fibered manifold over a circle is closely related to the anomaly in physics (This conjecture of Witten was proved independently by Bismut-Freed and Cheeger). Later on, Bismut-Cheeger and Xianzhe Dai studied systematically the adiabatic limits of η-invariants of Dirac operators on general fibered manifolds.

In [5] and [11], we applied the above results of Bismut-Cheeger and Dai to circle bundles and obtained an explicit expression of the adiabatic limit of η-invariants for Dirac operators on circle bundles (This result was also obtained independently in an unpublished work of Dai). We then applied this computation to establish a higher dimensional Rokhlin type congruence and studied various extensions.

We state a special case of the general result in [11] as follows.

Let K be an $8k+4$ dimensional compact oriented Spin^c-manifold. Let c be a closed two form on K such that the corresponding de Rham class $[c] \in H^2(K)$ verifies that $[c] \equiv w_2(K) \bmod 2$, where $w_2(K)$ is the second Stiefel-Whitney class of K.

Let B be an oriented $8k+2$ dimensional closed submanifold of K such that $[B] \subset H_{8k+2}(K)$ is dual to $[c]$. Then $K \setminus B$ is a spin manifold, and every spin structure on $K \setminus B$ induces naturally a spin structure on B. Moreover, all the induced spin structures on B lie in the same spin cobordism class.

Let $\widehat{\mathcal{A}}(B) \in \mathbf{Z}_2$ be the Atiyah-Milnor-Singer invariant determined by this spin cobordism class on B.

A classical theorem due to Atiyah-Hirzebruch states that $\langle \widehat{A}(TK)\exp(\frac{c}{2}), [K]\rangle$ is an integer. The following result in [11] refines this result.

Theorem 3.1. *The following identity holds,*

$$\left\langle \widehat{A}(TK)\exp\left(\frac{c}{2}\right), [K]\right\rangle \equiv \widehat{\mathcal{A}}(B) \mod 2\mathbf{Z}. \tag{3.1}$$

Corollary 3.2. (Atiyah-Hirzebruch) *If K is an $8k+4$ dimensional compact oriented spin manifold, then $\langle \widehat{A}(TK), [K]\rangle$ is an even number.*

Corollary 3.3. (Rokhlin) *If K is as in Theorem 3.1 and* $\dim K = 4$*, then*

$$\frac{\operatorname{sign}(B\cdot B) - \operatorname{sign}(K)}{8} \equiv \widehat{\mathcal{A}}(B) \mod 2\mathbf{Z}. \tag{3.2}$$

Both Corollaries 3.2 and 3.3 generalize the following classical Rokhlin divisibility theorem: the signature of a smooth closed oriented spin four dimensional manifold is divisible by 16.

Indeed the result established in [11] is still valid for the case where B is non-orientable. In this case, the obtained result generalizes the congruences of Guillou-Marin and Kirby-Taylor in four dimension to higher dimensions.

It might be worth to mention that the congruences of Rokhlin (3.2), Guillou-Marin and Kirby-Taylor have important applications in real algebraic geometry (related to Hilbert's 17th problem) and low dimensional topology.

As an application of the Rokhlin type congruence (3.1), we computed in [18] the Atiyah-Milnor-Singer invariant of spin complex hypersurfaces of dimension $4k+1$ ($k \geq 1$). Combining this computation with a well-known result of Stolz, we determined in [18] when a spin complex hypersurfaces of dimensional $4k+1$ ($k \geq 1$) admits a Riemannian metric with positive scalar curvature.

In another paper [15], which is joint with Dai, the computation of the adiabatic limit of the Dirac operator on circle bundles is applied to get an analytic computation of a cobordism invariant of Kreck-Stolz.

Paper [9] contains a K-theoretic proof of Theorem 3.1, as well as a generalization of it to the case of arbitrary twisted bundles.

On the other hand, we proposed in [11] that there should be an intrinsic relation between the KO-characteristic class and the Hirzebruch L-class. This problem was later solved by Kefeng Liu who generalized the 12 dimensional "miraculous cancellation" formula due to Alvarez-Gaumé and Witten to arbitrary $8k+4$ dimensional manifolds.

In the joint paper [13] with Liu, we applied Liu's above result to give an analytic interpretation of the Finashin invariant, by using ideas in elliptic genus.

In joint papers [41] and [44] with Fei Han, we further extended Liu's result to the case where an extra twisted complex line bundle shows up. Combining with earlier results in [11] and [13], we get direct geometric proofs of the higher dimensional Rokhlin type congruences due to Ochanine and Finashin.

4. Real embedding, η-invariant and index theory on manifolds with boundary

In the joint paper with Bismut [7], we applied the analytic techniques developed by Bismut-Lebeau to study the η-invariant of Atiyah-Patodi-Singer.

To be more precise, let $i : Y \to X$ be an embedding between two closed oriented spin odd dimensional Riemannian manifolds. Let N be the normal bundle to Y in X carrying the induced metric. Then for any Hermitian vector bundle μ on Y carrying a Hermitian connection, we can give a geometric construction of the direct image $i_!\mu \in \widetilde{K}(X)$ of Atiyah-Hirzebruch. The main result of [7] states that when mod $\mathbf{Z}$, the η-invariant of D^μ, the Dirac operator twisted by μ on Y, can be expressed by the η-invariant of $D^{i_!\mu}$, the Dirac operator twisted by $i_!\mu$ on X, plus some extra purely geometric terms.

The proof in [7] relies heavily on the techniques of Bismut-Lebeau developed for a problem in complex geometry. In view of the more flexible nature of η-invariants (with respect to the holomorphic analytic torsion in the paper of Bismut-Lebeau), it is nature to ask whether one can prove the above embedding formula for η-invariants in more geometric ways.

In [45], we made a first step in this direction. We started with a simple observation that if X is taken to be a sphere, then with the help of the Bott periodicity, one gets a purely geometric formula of the analytically defined invariant $\overline{\eta}(D^\mu)$ mod $\mathbf{Z}$. Then by using the Freed-Melrose mod k index theorem as well as the original Atiyah-Patodi-Singer index theorem, we showed that for any μ on Y, there always exists an embedding of Y to a higher dimensional sphere for which the embedding formula in [7] can be proved without using the Bismut-Lebeau techniques.

The geometric proof of the remaining general case was later carried out in the joint paper [55] with Huitao Feng and Guangbo Xu. A notable feature in [55] is that a Riemann-Roch type theorem for the Chern-Simons currents constructed in [7], under successive embeddings, is established.

The methods and results developed in [7] have some further applications:

In [8], we used them to give a new proof of the Atiyah-Singer mod 2 index theorem for Dirac operators.

In [16], we used them to give a new proof of the mod k index theorem of Freed-Melrose for Dirac operators, and obtained some mod 2 refinement in the real case.

In [14] and [34], which were joint with Dai, we applied the method in [7] to give a new proof of the Atiyah-Patodi-Singer index theorem for Dirac operators on manifolds with boundary. There are two more features in this works with respect to what in [7]. First, since the formula in [7] is only stated in a mod $\mathbf{Z}$ version, we need to be more careful in dealing with the missing "mod $\mathbf{Z}$" terms by examine the spectral flows appeared in the context. Secondly, we need to establish a Riemann-Roch type theorem under embeddings for the indices of Dirac operators on manifolds with boundary (thus carrying the Atiyah-Patodi-Singer boundary conditions).

In [42], we applied the techniques in [34] to prove a generalization of the famous Atiyah-Hirzebruch vanishing theorem for circle actions to the case of $\mathbf{Z}/k$-manifolds.

5. Families index theory for Dirac operators

We describe some joint works with Xianzhe Dai on families index theory in this section.

In [20], we established a splitting formula for the families index of Dirac operators. It solves a question of Bismut-Cheeger and generalizes the corresponding splitting theorem of Atiyah-Patodi-Singer to the case of families.

In [17], [24], we generalized the concept of spectral flow due to Atiyah-Patodi-Singer to the family case and introduced what we call the higher spectral flow.

As is well-known, the concept of spectral flow was introduced by Atiyah-Patodi-Singer in their study of the index theory on manifolds with boundary. It appeared later in many places in mathematics as well as in mathematical physics, and became an important invariant for elliptic operators.

Bismut-Cheeger and Melrose-Piazza studied the families generalizations of the Atiyah-Patodi-Singer index theorem, and obtained the families index theorem in the sense of taking Chern characters.

It is nature to ask whether the concept of spectral flow can also be extended to the family case and this is what we did in [17], [24].

We first obtained a new formula for the spectral flow, so that the original (discrete) definition by counting eigenvalues now admits a continuous version. With this continuous version at hand one then gets a natural generalization of the concept of spectral flow to the family case. Several basic properties of the higher spectral flow just defined were studied in [17], [27], showing that it is the "right" generalization of the classical concept of Atiyah-Patodi-Singer. Moreover, we used our spectral flow to give a heat kernel proof of the families index theorem for Toeplitz operators. The later can be viewed as an odd dimensional analogue of the Bismut local families index theorem for Dirac operators.

Our concept of higher spectral flow was later generalized by Fangbing Wu to the non-commutative case, which in turn plays roles in a series of papers by Leichtnam-Piazza on the Atiyah-Patodi-Singer type index theorem on covering spaces.

6. Geometric quantization conjecture and holomorphic Morse inequalities

6.1. **Analytic approach of the geometric quantization conjecture.** The famous Guillemin-Sternberg geometric quantization conjecture in symplectic geometry, in roughly speaking, says that "geometric quantization commutes with symplectic reduction".

To be more precise, let (M,ω) be a compact symplectic manifold. Let J be an almost complex structure on TM such that

$$g^{TM}(v,w)=\omega(v,Jw)$$

defines a Riemmannian metric on TM.

Let G be a compact connected Lie group. Let $\mathbf{g}$ be the Lie algebra of G. Assume G acts on (M,ω) in a Hamiltonian way, and preserves J. Then there exists a G-equivariant **moment map**

$$\mu : M \to \mathbf{g}^*$$

such that for any $V\in\mathbf{g}$, one has

$$i_{V_M}\omega = d\langle\mu, V\rangle,$$

where $V_M\in\Gamma(TM)$ denotes the vector field on M generated by $V\in\mathbf{g}$. Clearly, G preserves $\mu^{-1}(0)$.

Definition 6.1. The Marsden-Weinstein symplectic reduction space M_G is defined to be

$$M_G=\mu^{-1}(0)/G.$$

Basic assumption: $0\in\mathbf{g}^*$ is a regular value of the moment map $\mu : M\to\mathbf{g}^*$.

Then $\mu^{-1}(0)$ is a closed manifold. For simplicity, we also assume that G acts on $\mu^{-1}(0)$ freely, then M_G is a closed manifold and carries an induced symplectic form ω_G. Moreover, J induces an almost complex structure J_G on TM_G such that $\omega_G(v,J_Gw)$ determines a Riemannian metric g^{TM_G} on TM_G.

Remark 6.2. If (M,ω,J) is Kähler, then (M_G,ω_G,J_G) is also Kähler.

Now, let L be an Hermitian line bundle over M carrying an Hermitian connection ∇^L such that

$$\frac{\sqrt{-1}}{2\pi}\left(\nabla^L\right)^2=\omega.$$

When such an L exists, we call (M,ω) **pre-quantizable**, and call L the pre-quantized line bundle. We assume the existence of L now.

We make the assumption that the Hamiltonian G action lifts to an action on L, which preserves the Hermitian metric and Hermitian connection on L. Then L descends to a pre-quantized line bundle L_G over M_G carrying a canonically induced Hermitian metric and Hermitian connection ∇^{L_G}.

Remark 6.3. When (M,ω,J) is Kähler and L is a holomorphic line bundle over M, then L_G is also holomorphic over M_G.

Let D^L be the Spin^c-Dirac operator (twisted by L) acting on $\Gamma(\Lambda^{(0,*)}(T^*M)\otimes L)$. Then it commutes with the induced G-action on $\Gamma(\Lambda^{(0,*)}(T^*M)\otimes L)$. Thus, G preserves $\ker D^L_{\pm}$, which are the restrictions of D^L on $\Gamma(\Lambda^{(0,\frac{\text{even}}{\text{odd}})}(T^*M)\otimes L)$ respectively.

Let $(\ker D^L_{\pm})^G$ denote the G-invariant part in $\ker D^L_{\pm}$.

Define the *reduction* of the quantization space

$$Q(L)=\left(\ker D^L_+\right)-\left(\ker D^L_-\right)$$

of L to be

$$Q(L)^G=\left(\ker D^L_+\right)^G-\left(\ker D^L_-\right)^G.$$

Let

$$Q\left(L_G\right)=\left(\ker D^{L_G}_+\right)-\left(\ker D^{L_G}_-\right).$$

be the quantization space of L_G on M_G.

The Guillemin-Sternberg conjecture:

$$\dim Q(L)^G=\dim Q\left(L_G\right). \tag{$*$}$$

Remark 6.4. Guillemin-Sternberg first proved in 1982 that when (M,ω,J) is Kähler and L is holomorphic,

$$\dim H^{(0,0)}(M,L)^G=\dim H^{(0,0)}\left(M_G,L_G\right)$$

and proposed (*) as a conjecture.

When G is abelian, (*) was first proved by Guillemin (1995) in a special case, and later in general by Meinrenken (JAMS 1996) and Vergne (DMJ 1996) independently. The remaining non-abelian case was proved by Meinrenken (Adv. in Math. 1998) by using the technique of *symplectic cut* of Lerman.

There are also approaches of Duistermaart-Guillemin-Meinrenken-Wu (for circle actions) and Jeffrey-Kirwan (for non-abelian group actions with certain extra conditions).

Remark 6.5. All of the above proofs use the Atiyah-Bott-Segal-Singer equivariant index theorem in an essential way: first relate $\dim Q(L)^G$ to quantities on the fixed point set of the G-action, and then try to relate the later to quantities on the symplectic quotient (through symplectic cut or through the Jeffrey-Kirwan-Witten non-abelian localization formulas).

Natural question. Whether there is an approach relating $\dim Q(L)^G$ *directly* to $\dim Q(L_G)$?

In the joint papers [21] and [23] with Youliang Tian, we gave a direct analytic proof of the Guillemin-Sternberg conjecture, answering the above natural question.

The main idea is to introduce a deformation of the Spin^c Dirac operator D^L by using the Hamiltonian vector field associated to the norm square of the moment map μ.

More precisely, we equip with $\mathbf{g}^*$ an Ad_G-invariant metric and set $\mathcal{H} = |\mu|^2$. Let $X^{\mathcal{H}}$ be the associated Hamiltonian vector field, i.e.,

$$i_{X^{\mathcal{H}}}\omega = d\mathcal{H}.$$

For any $T \in \mathbf{R}$, we introduce the following deformation of D^L,

$$D_T^L = D^L + \frac{\sqrt{-1}T}{2} c\left(X^{\mathcal{H}}\right) : \Gamma\left(\Lambda^{0,*}(T^*M)\otimes L\right) \to \Gamma\left(\Lambda^{0,*}(T^*M)\otimes L\right).$$

Remark 6.6. If (M, ω, J) is Kähler and L is holomorphic, then one has

$$D_T^L = \sqrt{2}\left(e^{\frac{-T\mathcal{H}}{2}}\overline{\partial}^L e^{\frac{T\mathcal{H}}{2}} + e^{\frac{T\mathcal{H}}{2}}\left(\overline{\partial}^L\right)^* e^{\frac{-T\mathcal{H}}{2}}\right).$$

This might be thought of as an analogue of the Witten deformation in Morse theory, but now in a holomorphic non-abelian context.

By using this deformation, one can then apply the analytic localization technique of Bismut-Lebeau to complete the proof of the Guillemin-Sternberg conjecture. In particular, a direct relation between $\dim Q(L)^G$ and $\dim Q(L_G)$ is obtained.

Several immediate extensions of the geometric quantization formula were also obtained in [23] by using our analytic method. First of all, we showed that the line bundle L can be replaced by any Hermitian vector bundle verifying certain "positivity" condition. We also showed that when $\mu^{-1}(0)$ is not empty, then

$$\mathrm{Todd}(M) = \mathrm{Todd}(M_G),$$

a result also obtained independently by Meinrenken and Sjamaar. On the other hand, when (M, ω, J) is Kähler, we showed that the quantization formula can be refined to a series of Morse type inequalities.

The analytic method developed in [21] and [23] allows us to obtain further generalizations of the geometric quantization formula. Here we list a few of them.

1). In the joint paper [25] with Tian, we generalized the holomorphic Morse inequalities in [23] to the case of singular reductions;

2). In the joint paper [26] with Tian, we generalized the results of Meinrenken and Vergne, in the case where G is abelian, to a series of weighted quantization formulas;

3). In the joint paper [28] with Tian, we extended the quantization formula to the case of symplectic manifolds with boundary. As a consequence, we obtained an analytic version of the residue formula of Guillemin-Kalkman-Martin;

4). In [29], we showed that in the Kähler case, the holomorphic Morse inequalities in [23] are indeed equalities, our results were stated for singular reductions and were thus stronger than similar results proposed earlier by Braverman;

5). In [30], we generalized the quantization formula to the family case;

6). In the joint paper [38] with Huitao Feng and Wenchuan Hu, we further generalized the results in [30] to the case of manifolds with boundary. The result obtained is a common generalization of the main results in [28] and [30];

7). More recently, in the joint paper [56] with Mathai, we generalized the Guillemin-Sternberg geometric quantization conjecture to the case where both M and G are allowed to be non-compact. This will be discussed in more details in Subsection 6.4.

6.2. **Equivariant holomorphic Morse inequalities in the sense of Witten.** It is now well-known that Witten's analytic proof of (real) Morse inequalities has had wide range influence in mathematics. In a paper appeared shortly after Witten's paper on his proof of (real) Morse inequalities, Witten wrote another paper concerning holomorphic circle actions on Kähler manifolds and proposed what he called the homomorphic More inequalities. As usual, Witten only outlined his proof based on physics ideas and he only stated precisely his inequalities in the case where the fixed points of the circle action are isolated.

In 1996, Mathai and Wu gave a first rigorous proof of Witten's equivariant holomorphic Morse inequalities, in the case where the fixed points are isolated, by using heat kernel method.

In the joint paper [22] with Siye Wu, by using the localization techniques developed by Bismut-Lebeau, we proved the generalized version of the Witten holomorphic Morse inequalities where the fixed point set need not be discrete. Relations with the Guillemin-Sternberg geometric quantization conjecture were also discussed.

6.3. **An application of the quantization formula on manifolds with boundary.** In the joint paper [28] with Tian about the quantization formula on manifolds with boundary, as an application, we proved a universal geometric quantization formula which holds for the case where the symplectic reduction might be singular, while the total symplectic manifold is closed. The contribution near the symplectic reduction is expressed by an (analytic) invariant index of Atiyah-Patodi-Singer type. Here, we will show this APS type index can actually be interpreted as an invariant index of certain transversally elliptic operator in the sense of Atiyah, Paradan and Vergne. Thus it acquires a natural topological interpretation.

We will first recall the above mentioned analytic result from [28].

Following the notation in Section 6.1, it is clear that there exists $\delta > 0$ small enough such that $\mathcal{H}^{-1}((0,\delta])$ does not contain any critical point of $\mathcal{H} = |\mu|^2$. Thus for any $c \in (0,\delta]$, a regular value of $\mathcal{H}$, $\mathcal{H}^{-1}(c)$ is a G-invariant hypersurface of M, cutting M into two parts $M = M_+^c \cup M_-^c$ with $M_+^c = \mathcal{H}^{-1}([0,c])$ and the common boundary $M_+^c \cap M_-^c = \mathcal{H}^{-1}(c)$.

Now for any $T \in \mathbf{R}$, let $D^L_{M_+^c,T}$ denote the restriction of D^L_T on M_+^c. Let $D^L_{M_+^c,T,\pm,APS}$ be the corresponding elliptic operators verifying the Atiyah-Patodi-Singer boundary condition (here since the product nature no longer holds near boundary, one need to modify the induced boundary operator a little bit, see [28] for more details).

It is clear that both $D^L_{M_+^c,T,\pm,APS}$ are G-equivariant. Moreover, since there is no critical point of $\mathcal{H}$ on $\partial M_+^c = \mathcal{H}^{-1}(c)$, one verifies easily that when $T \geq 0$ is large enough, $D^L_{M_+^c,T,-,APS}$ is the formal adjoint of $D^L_{M_+^c,T,+,APS}$, thus one can define the quantization space

$$Q_{APS,T}\left(M_+^c, L|_{M_+^c}\right) = \left(\operatorname{Ker} D^L_{M_+^c,T,+,APS}\right) - \left(\operatorname{Ker} D^L_{M_+^c,T,-,APS}\right).$$

We can now state the universal quantization formula proved in [28, Theorem 6.1] as follows.

Theorem 6.7. *There exists $T_0 \geq 0$ such that for any $T \geq T_0$, the following identity holds,*

$$\dim Q(M,L)^G = \dim Q_{APS,T}\left(M_+^c, L|_{M_+^c}\right)^G.$$

From the above theorem, one sees that $\dim Q_{APS,T}\left(M_+^c, L|_{M_+^c}\right)^G$ does not depend on T when $T \geq 0$ is large enough. Indeed, if $0 \in \mathbf{g}^*$ is a regular value of μ, one can further identify $\dim Q_{APS,T}\left(M_+^c, L|_{M_+^c}\right)^G$, when $T \geq 0$ is large enough, with $\dim Q(L_G)$, thus getting the original Guillemin-Sternberg conjecture.

Now in general, $0 \in \mathbf{g}^*$ might not be a regular value of μ. However, by an observation due to Paradan and Vergne, since $X^{\mathcal{H}}$ is a G-invariant vector field on M_+^c tangent to the orbits of the G-action and $\mu^{-1}(0)$ is the only zero set of $X^{\mathcal{H}}$ in M_+^c, one has a well-defined transversally elliptic Dirac operator associated to $X^{\mathcal{H}}$ and the prequantized line bundle L, denoted by $c_{X^{\mathcal{H}},L}$. And its index can be interpreted as a distribution on G. We denote its G-invariant index by $\mathrm{ind}_G(c_{X^{\mathcal{H}},L})$.

The following proposition identifies $\mathrm{ind}_G(c_{X^{\mathcal{H}},L})$ and $\dim Q_{APS,T}\left(M_+^c, L|_{M_+^c}\right)^G$ for large enough $T \geq 0$.

Proposition 6.8. *There exists $T_0 \geq 0$ such that for any $T \geq T_0$, the following identity holds,*

$$\mathrm{ind}_G\left(c_{X^{\mathcal{H}},L}\right) = \dim Q_{APS,T}\left(M_+^c, L|_{M_+^c}\right)^G .$$

Proof. One first deforms in a G-invariant way the metrics and connections near the boundary of M_+^c to the situation of product nature. Then extend these geometric data to the complete manifold obtained by attaching a cylinder to the boundary.

By using a standard argument going back to Atiyah-Patodi-Singer, one sees that when $T \geq 0$ is large enough, one can interpret $\dim Q_{APS,T}(M_+^c, L|_{M_+^c})^G$ by the corresponding G-invariant L^2-index of the extended operator on the complete cylindrical manifold.

On the other hand, a result due to Braverman shows that the above G-invariant L^2-index equals exactly to $\mathrm{ind}_G(c_{X^{\mathcal{H}},L})$.

By combining the above two observations, one completes the proof of the Proposition. Q.E.D.

By combining Theorem 6.7 and Proposition 6.8, one gets analytically the following result, which should be contained in Paradan's paper on his proof of the Guillemin-Sternberg conjecture.

Proposition 6.9. *The following identity holds,*

$$\dim Q(M, L)^G = \mathrm{ind}_G\left(c_{X^{\mathcal{H}},L}\right) .$$

Note that the right hand side now admits a purely topological interpretation.

6.4. **Quantization formula for proper actions.** In this subsection, we describe a recent joint work with Mathai [56], where a generalization of the Guillemin-Sternberg conjecture to the case of non-compact groups and manifolds is established. This essentially solves a conjecture of Hochs and Landsman.

So from now on we assume that G and M are non-compact. The quantization formula we proved is for the case where G acts on M properly with compact quotient M/G. Note that in this case $M_G = \mu^{-1}(0)/G$ is still compact. Thus, if we still assume that everything is G-invariant[1] and that $0 \in \mathbf{g}^*$ is a regular value of the moment map μ with G acting on $\mu^{-1}(0)$ freely, then the quantity $\dim Q(L_G)$ is still well-defined.

The first difficulty one encounters in stating a possible quantization formula in this situation is that since here M is non-compact, $\operatorname{Ker} D^L_\pm$ need not be finite dimensional, so tautologically $\dim(\operatorname{Ker} D^L_\pm)^G$ might not be well-defined.

Nevertheless, Hochs and Landsman, by making use of the KK-theory, proposed to define an integer which in the case where both G and M are compact, coincides with $\dim Q(L)^G$. Then they made the conjecture that in the case where G is unimodular, the integer they defined equals to $\dim Q(L_G)$. Moreover, they proved that their conjecture holds in the case where G admits a normal discrete subgroup Γ such that G/Γ is compact.

In the joint work with Mathai [56], we first defined, in a direct analytic way, what we call a G-invariant index (denoted by $\operatorname{ind}_G(D^L_+)$) associated to D^L_+, and then show that for this index, the following generalization of the Guillemin-Sternberg conjecture holds.

Theorem 6.10. *In the general case where G is merely assumed to be locally compact, there exists $p_0 > 0$ such that for any integer $p \geq p_0$,*

$$\operatorname{ind}_G\left(D^{L^p}_+\right) = \operatorname{ind}\left(D^{L^p_G}_+\right). \tag{6.1}$$

Moreover, if $\mathbf{g}^*$ admits an Ad_G-invariant metric, then one can take $p = 1$ in (6.1).

In particular, in the case considered by Hochs and Landsman where G admits a normal discrete subgroup Γ such that G/Γ is compact, G admits an Ad_G-invariant metric so that one can take $p = 1$ in (6.1).

For completeness we now briefly recall our definition of $\operatorname{ind}_G(D^L_+)$.

Since M/G is compact, one sees easily that there exists a compact subset Y of M such that $G(Y) = M$.

[1]This can always be achieved with the help of a cut-off function which is a smooth nonnegative function c on M such that c has compact support and $\int_G c(gx)^2 dg = 1$ for any $x \in M$.

Let U, U' be two open subsets of M such that $Y \subset U$ and that the closures $\overline{U}$ and $\overline{U'}$ are both compact in M, and that $\overline{U} \subset U'$. The existence of U, U' is clear.

Then one can construct a smooth function $f : M \to [0,1]$ such that $f|_U = 1$ and $\mathrm{Supp}(f) \subset U'$.[2]

We now consider the space $\Gamma(\Lambda^{(0,*)}(T^*M) \otimes L)^G$, the subspace of G-invariant sections of $\Gamma(\Lambda^{(0,*)}(T^*M) \otimes L)$.

By using the property that $G(Y) = M$, it is easy to see that there exists a positive constant $C > 0$ such that for any $s \in \Gamma(\Lambda^{(0,*)}(T^*M) \otimes L)^G$,

$$\|s\|_{U,0} \leq \|fs\|_0 \leq \|s\|_{U',0} \leq C\|s\|_{U,0}, \tag{6.2}$$

where for $V = U$ or U',

$$\|s\|^2_{V,0} = \int_V \langle s(x), s'(x)\rangle \, dx. \tag{6.3}$$

Let $\mathbf{H}^0_f(M,L)^G$ be the completion of the space $\{fs : s \in \Gamma(\Lambda^{(0,*)}(T^*M) \otimes L)^G\}$ under the standard L^2-norm $\|\cdot\|_0$. Let $\mathbf{H}^1_f(M,L)^G$ be the completion of $\{fs : s \in \Gamma(\Lambda^{(0,*)}(T^*M) \otimes L)^G\}$ under a (fixed) G-invariant first Sobolev norm $\|\cdot\|_1$.

Let P_f be the orthogonal projection from $L^2(M,L)$ to its subspace $\mathbf{H}^0_f(M,L)^G$.

It is clear that $P_f D^L$ maps an element of $\mathbf{H}^1_f(M,L)^G$ into $\mathbf{H}^0_f(M,L)^G$

Proposition 6.11. *The induced operator*

$$P_f D^L : \mathbf{H}^1_f(M,L)^G \to \mathbf{H}^0_f(M,L)^G \tag{6.4}$$

is a Fredholm operator.

Proof. For any $s \in \Gamma(\Lambda^{(0,*)}(T^*M) \otimes L)^G$, one has

$$D^L(fs) = fD^L s + c(df)s, \tag{6.5}$$

where we identify the one form df with its metric dual $(df)^*$.

It is clear that $P_f(fD^L s) = fD^L s$, while in view of (6.2),

$$\left\|P_f(c(df)s)\right\|_0 \leq \|c(df)s\|_0 \leq C_1\|fs\|_0 \tag{6.6}$$

for some constant $C_1 > 0$.

From (6.2), (6.5) and (6.6), one verifies easily that

$$\left\|P_f D^L(fs)\right\|_0 \geq C_2\|fs\|_1 - C_3\|fs\|_0, \tag{6.7}$$

for some constants C_2, $C_3 > 0$.

Since f is of compact support, from the Gärding type inequality (6.7) one completes the proof of Proposition 6.11. Q.E.D.

[2]With this function, one can construct the cut-off function c by $c(x) = \frac{f(x)}{(\int_G f(gx)^2 dg)^{1/2}}$ for any $x \in M$.

Remark 6.12. Besides the Fredholm property in Proposition 6.11, the following self-adjoint property also holds: for any $s,\ s' \in \Gamma(\Lambda^{(0,*)}(T^*M)\otimes L)^G$, one has

$$\left\langle P_f D^L(fs), fs'\right\rangle = \left\langle fs, P_f D^L\left(fs'\right)\right\rangle . \tag{6.8}$$

Remark 6.13. If $(\widetilde{U},\ \widetilde{U}',\ \widetilde{f})$ is another triple of open subsets and the cut-off function as above, then by taking the deformation $f_t = (1-t)f + t\widetilde{f}$, one gets easily a continuous family of Fredholm operators $P_{f_t}D^L$.

Now let $D^L_{\pm} : \Gamma(\Lambda^{(0,\frac{\text{even}}{\text{odd}})}(T^*M)\otimes L) \to \Gamma(\Lambda^{(0,\frac{\text{odd}}{\text{even}})}(T^*M)\otimes L)$ be the restriction of D^L on $\Gamma(\Lambda^{(0,\frac{\text{even}}{\text{odd}})}(T^*M)\otimes L)$.

Let $\mathbf{H}^i_{f,\pm}(M,L)^G$ $(i=0,\ 1)$ be the subspaces of $\mathbf{H}^i_f(M,L)^G$ obtained by completing the $\frac{\text{even}}{\text{odd}}$ forms.

By Proposition 6.11 and (6.8), the induced operator $P_f D^L_+ : \mathbf{H}^1_{f,+}(M,L)^G \to \mathbf{H}^0_{f,-}(M,L)^G$ is Fredholm. Moreover its index, $\mathrm{ind}(P_f D^L_+)$, does not depend on the choice of f, in view of Remark 6.13. Similarly, it is also easy to see that this index does not depend on the choices of G-invariant metrics and connections involved.

Definition 6.14. We call $\mathrm{ind}(P_f D^L_+)$ defined above the G-invariant index associated to D^L_+ and denote it by $\mathrm{ind}_G(D^L_+)$.

In particular, one can show that in the case where both G and M are compact, one has $\mathrm{ind}(P_f D^L_+) = \dim Q(L)^G$. Thus, tautologically $\mathrm{ind}(P_f D^L_+)$ should be the right generalization of $\dim Q(L)^G$ in the non-compact case.

Now since both G and M are non-compact, one can not apply the Atiyah-Bott-Segal-Singer equivariant index theorem to compute $\mathrm{ind}(P_f D^L_+)$. Instead, we will adapt the analytic approach developed in [23] to the current situation.

However, since now G is non-compact, there need not exist Ad_G-invariant metrics on $\mathbf{g}^*$. Thus the function $\mathcal{H} = |\mu|^2$ as well as the Hamiltonian vector field $X^{\mathcal{H}}$ need not be G-invariant, and consequently, the deformed operator D^L_T defined in Subsection 6.1 need not be G-equivariant.

In order to get a G-equivariant deformation of D^L, we set

$$X^{\mathcal{H}}_G = \int_G c(gx)^2 X^{\mathcal{H}}_g dg, \tag{6.9}$$

where $X^{\mathcal{H}}_g$ denotes the pullback of $X^{\mathcal{H}}$ by $g \in G$.

It is clear that $X^{\mathcal{H}}_G$ is G-invariant, and we can define the following deformed operator of D^L,

$$D^L_T = D^L + \frac{\sqrt{-1}T}{2} c\left(X^{\mathcal{H}}_G\right), \tag{6.10}$$

which is G-equivariant.

However, the appearance of the cut-off function c in the definition of $X_G^{\mathcal{H}}$ makes it difficult for the pointwise localization argument in [23, Section 2] to proceed here. This is why we need to replace L by L^p for positive integer p.

More precisely, for any open neighborhood W of $\mu^{-1}(0)$ in M, we can show that the following analogue of [23, Theorem 2.1] holds.

Proposition 6.15. *There exists $p_0 \geq 1$ such that for any integer $p \geq p_0$, there exist $C > 0$, $b > 0$ verifying the following property: for any $T \geq 1$ and $s \in \Omega^{0,*}(M, L^p)^G$ with $\mathrm{Supp}(s) \cap \overline{U'} \subset \overline{U'} \setminus W$, one has*

$$\left\| P_f D_T^{L^p}(fs) \right\|_0^2 \geq C \left(\|fs\|_1^2 + (T-b)\|fs\|_0^2 \right). \tag{6.11}$$

Moreover, if $\mathbf{g}^*$ admits an Ad_G-invariant metric, then one can take $p_0 = 1$.

This key localization property allows us to reduce the proof of Theorem 6.10 to sufficiently small neighborhoods of $\mu^{-1}(0)$, on which one can apply the known analytic technique (which goes back to Bismut-Lebeau) to complete the proof of Theorem 6.10.

Remark 6.16. In the Appendix in [56], Bunke showed that our G-invariant index $\mathrm{ind}_G(D_+^L)$ actually admits a KK-theoretic interpretation, which implies that our Theorem 6.10 indeed resolves the Hochs-Landsman conjecture essentially.

Remark 6.17. Indeed, one can show that both $(\mathrm{Ker}\, D_\pm^L)^G$ are of finite dimension. Moreover, in the case where G is unimodular, we showed in [56] that

$$\mathrm{ind}_G\left(D^L\right) = \dim\left(\mathrm{Ker}\, D_+^L\right)^G - \dim\left(\mathrm{Ker}\, D_-^L\right)^G, \tag{6.12}$$

which further justifies that Theorem 6.10 is a "right" extension of the Guillemin-Sternberg conjecture in the non-compact case.

7. Vector fields on manifolds, Poincaré-Hopf formula and the Kervaire semi-characteristic number

The famous Poincaré-Hopf index formula for vector fields states that for any vector field with isolated zeros on a closed manifold, one can define an integer at each zero, called the index of the vector field at this zero, such that the Euler characteristic number of the manifold equals to the sum of these indices.

The Poincaré-Hopf theorem is interesting mainly for even dimensional manifolds, as a classical result of Hopf says that on any odd dimensional manifold their always exists a vector field without any zero.

In his paper on the analytic proof of Morse inequalities, Witten also proposed an analytic proof of the Poincaré-Hopf formula. But his proof holds only for the case where the isolated zeros are non-degenerate.

In [60], we gave an analytic proof of the Poincaré-Hopf formula which holds without the non-degenerate assumption. It is based on an old idea due to Atiyah in 1970's and makes use of the η-invariant.

Another natural question relating to the Poincaré-Hopf formula is that whether there is an odd dimensional analogue of Euler characteristic as well as a corresponding Poincaré-Hopf type formula for them. A natural candidate is the Kervaire semi-characteristic defined in what follows.

Let M be a $4q+1$ dimensional closed oriented manifold. Then its Kervaire semi-characteristic is defined by

$$k(M) \equiv \sum_{i=0}^{2q} \dim H^{2i}(M;\mathbf{R}) \mod 2\mathbf{Z}, \tag{7.1}$$

where $H^{2i}(M;\mathbf{R})$ is the $2i$-th cohomology of M.

An important property of $k(M)$ is that it can be expressed through the mod 2 index of skew-adjoint elliptic operators in the sense of Atiyah-Singer. On the other hand, it is not multiplicative under coverings. Thus it is more subtle in some sense than the Euler characteristic.

In 1969, Atiyah proved his famous vanishing result for $k(M)$: if there exist two nowhere linearly dependent vector fields on M, then $k(M) = 0$.

Inspired by this vanishing result, Atiyah and Dupont proved a counting formula for $k(M)$ which is of Poincaré-Hopf type: let V_1, V_2 be two vector fields on M, and V_1 and V_2 are linearly dependent only at a finite number of points on M (we call these points the singular points of $\{V_1, V_2\}$), then for any singular point x one can define a mod 2 index $\mathrm{ind}_{V_1,V_2}(x) \in \mathbf{Z}_2$. The formula of Atiyah-Dupont then states

$$k(M) \equiv \sum \mathrm{ind}_{V_1,V_2}(x) \mod 2\mathbf{Z}. \tag{7.2}$$

However, the condition in formula (7.2) needs that V_1 and V_2 are linearly dependent only at a finite number of points, while this condition does not hold for all manifolds (indeed, it requires that the $4q$-th Stiefel-Whitney class of M vanishes). Thus a natural question is whether there is a generic counting formula for $k(M)$ which holds for all $4q+1$ oriented manifolds.

In [32], we proposed a positive answer to the above question. Our starting point is the Hopf theorem we mentioned: on any $4q+1$ closed oriented manifold M there exists at least one nowhere zero vector field V. Let E be the normal bundle to V in TM, then the generic self-intersection of E is a one dimensional manifold consisting of a union of disjoint circles. For any such circle F, one can define canonically a real line bundle $o_F(V)$. The counting formula established in

[32] can now be stated as follows,

$$k(M) \equiv \#\{F : o_F(V) \text{ is orientable on } F\} \mod 2\mathbf{Z}. \tag{7.3}$$

Clearly, (7.3) holds for any $4q+1$ manifolds, thus it can be viewed as a universal counting formula for $k(M)$. Moreover, a difference between (7.3) and the Atiyah-Dupont formula (7.2) is that here we count the number of circles instead of points.

The proof we gave in [32] for (7.3) is purely analytic. It is based on an analytic interpretation of $k(M)$ given in [27].

To be more precise, let g^{TM} be a Riemannian metric on TM. Let $d : \Omega^*(M) \to \Omega^*(M)$ be the exterior differential operator and $d^* : \Omega^*(M) \to \Omega^*(M)$ the formal adjoint of d with respect to the inner product on $\Omega^*(M)$ naturally induced from g^{TM}. Let D_V be defined by

$$D_V = \widehat{c}(V)\,(d+d^*) - (d+d^*)\,\widehat{c}(V) : \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{even}}(M). \tag{7.4}$$

Then one verifies that D_V is skew-adjoint and elliptic. Moreover, the following formula is proved in [27],

$$k(M) = \mathrm{ind}_2\,(D_V)\,, \tag{7.5}$$

where ind_2 is the mod 2 index in the sense of Atiyah and Singer.

Now let E be the normal bundle to V in TM such that V is perpendicular to E with respect to g^{TM}. Let X be a transversal section of E. Then the zero set of X, $\mathrm{zero}(X)$, represents the self-intersection of E.

For any $T \in \mathbf{R}$, the following deformation of D_V is introduced in [32],

$$D_{V,T} = \widehat{c}(V)\,(d+d^*+T\widehat{c}(X)) - (d+d^*+T\widehat{c}(X))\,\widehat{c}(V), \tag{7.6}$$

which may be thought of as a Witten type deformation for D_V. Then one still has

$$k(M) = \mathrm{ind}_2\,(D_{V,T}) \tag{7.7}$$

for any $T \geq 0$.

One then lets $T \to +\infty$ to get (7.3) by applying the analytic localization techniques developed by Bismut-Lebeau. The line bundle $o_F(V)$ shows up naturally in the process.

A more topological proof of (7.3), as well as a more topological interpretation of $o_F(V)$, was later given by Zizhou Tang.

In [31], we extended the main result in [32] to the twisted bundle case. We call the obtained result the mod 2 index theorem for twisted Signature operators. It can be thought of as an odd dimensional analogue of the classical Hirzebruch Signature theorem. It also generalizes a result of Farber and Turaev.

On the other hand, besides giving an analytic interpretation of $k(M)$, [27] also contains an odd dimensional analogue of the Gauss-Bonnet-Chern formula.

In another paper [39], which is joint with Tang, we extended the Atiyah-Dupont theory for vector fields on tangent bundles to the case of vector bundles. As an application, we get the following non-existence theorem of almost complex structures.

Theorem 7.1. *(*Tang-Zhang [39]*) Let $\overline{\mathbf{C}P}^{2k}$ denote the "negative" complex projective space which carries the opposite orientation with respect to the standard complex projective space $\mathbf{C}P^{2k}$. Then for any $k > 1$, $\overline{\mathbf{C}P}^{2k}$ admits no almost complex structure.*

8. Elliptic genus: vanishing and rigidity results in K-theory

A famous vanishing theorem of Atiyah-Hirzebruch states that if a closed oriented spin manifold M admits a nontrivial circle action, then $\widehat{A}(M) = 0$. This vanishing result was applied by Witten formally on the loop space $\mathcal{L}M$, and obtained what we now call the Witten rigidity theorem. Here, the $\widehat{A}$-genus of $\mathcal{L}M$ was interpreted as the elliptic genus on M. This is the way Witten derived his rigidity result for elliptic genus. Natually, this proof is still a physics proof.

The first rigorous proof of the Witten rigidity was given by Taubes by analytic method. Later, Bott and Taubes simplified Taubes' proof by using the Lefschetz fixed point theorem. An even simpler proof was in turn found by Kefeng Liu who used effectively the properties of modular forms involved. Moreover, Liu also obtained along his proof several vanishing results concerning elliptic genus.

From the point of view of index theory, a natural question is whether one can generalize the Witten rigidity to the family case. The positive answer to this question appeared in 1999, when Liu and Xiaonan Ma proved such a generalization by extending Liu's method accordingly.

The key feature of the theorem of Liu-Ma is that they proved the rigidity of the Chern character of the index bundle under consideration. Thus, the natural question still remained whether the index bundle itself admits certain rigidity property.

In the joint papers [33], [40] with Liu and Ma, we solved the above problem by establishing the rigidity theorem for the index bundle associated to elliptic genus in the K-theoretic sense. Our proof generalizes in principle the original proof of Taubes, in which we applied the techniques developed in [22] to simplify many treatment. In particular, the K-theoretic generalization to families of the Atiyah-Bott-Segal-Singer equivariant index theorem was established at the very beginning of the proof.

In [35], a generalization of the above K-theoretic rigidity to Spin^c-manifolds is given.

9. Vanishing theorems on foliations

In the joint paper [36] with Kefeng Liu, we gave a geometric interpretation of the Bott connection of foliation by using the idea of adiabatic limits. Our original motivation is to give a more geometric proof of the following vanishing theorem of Connes.

Theorem 9.1. *If M is an oriented closed manifold. Let $F \subset TM$ be an oriented spin integrable subbundle of TM. We assume that there is a metric on F such that it induces pointwise positive scalar curvature on any leaf induced by F. Then $\widehat{A}(M) = 0$.*

When $F = TM$, this is the classical theorem of Lichnerowicz.

The original proof of Connes is highly noncommutative. In [36], we applied our geometric interpretation of the Bott connection to give a geometric proof of the Connes vanishing theorem for what we called almost Riemannian foliation which contains Riemannian foliations as special cases. The general case seems still lacks a purely geometric proof.

In another paper [37], jointly with Liu and Ma, we proved certain generalizations of the vanishing theorem of Liu for elliptic genus to the case of foliations. As a special case, we proved the following generalization of the famous Atiyah-Hirzebruch vanishing theorem to foliations.

Theorem 9.2. *If M is an oriented closed manifold. Let $F \subset TM$ be an oriented spin integrable subbundle of TM. We assume that F admits a nontrivial circle action. Then $\widehat{A}(M) = 0$.*

10. Sub-signature operators and applications

In [19], for any oriented subbundle E of the tangent bundle TM of an oriented closed manifold M, we constructed an elliptic operator D_E, such that

$$\operatorname{ind} D_E = \left\langle \widehat{L}(E) e(TM/E), [M] \right\rangle . \tag{10.1}$$

In particular, by taking E to be zero or TM, we get the Gauss-Bonnet-Chern theorem and the Hirzebruch Signature theorem respectively.

The key point of the above construction is that it does not use any spin condition. Moreover, the local index computation still applies to D_E.

It turns out that the main idea involved in the above construction has certain applications, for examples, in [27], [31], [32], [36], [37].

On another direction of application, we applied D_E to fibered manifolds and extended the Atiyah-Patodi-Singer invariant associated with unitary flat vector bundles to the non-unitary case, and established a Riemann-Roch type theorem, under fibration, for these invariants. The main technique here is the computation of the adiabatic limit of the η-invariants of the involved operators.

In the joint paper [52] with Xiaonan Ma, we applied these techniques to give an alternate proof of Bismut-Lott's Riemann-Roch formula for flat vector bundles. In doing so, we introduced the following deformation for flat vector bundles.

Let (F, ∇^F) be a complex flat vector bundle over a manifold M. Let g^F be a Hermitian metric on F. Set

$$\omega\left(F, g^F\right) = \left(g^F\right)^{-1} \nabla^F g^F. \tag{10.2}$$

Then

$$\nabla^{F,u} = \nabla^F + \frac{1}{2}\omega\left(F, g^F\right) \tag{10.3}$$

is a unitary connection on F.

For any $a \in \mathbf{R}$, we introduced in [52] the following deformation

$$\nabla^{F,u}(a) = \nabla^{F,u} + \frac{\sqrt{-1}a}{2}\omega\left(F, g^F\right). \tag{10.4}$$

Then we get a family of unitary connections on F. By studying the η-invariants of Dirac operators associated to these connections, we can get a new proof of the Bismut-Lott theorem. Moreover, an extension of the Bismut-Lott theorem from the original $\mathbf{R}$ version to the $\mathbf{C}/\mathbf{Q}$ version is also obtained in this framework.

In two other papers [48], [65], also jointly with Ma, we applied the above deformation to get new properties of η-invariants associated to non-unitary connections on twisted bundles, by identifying the real and imaginary parts respectively. In particular, we showed analytically that one can define a holomorphic function on the representation space of the fundamental group of a smooth manifold which has the Ray-Singer analytic torsion as its absolute value.

In a recent joint paper [57] with Dai, we showed how the Bismut-Lott real analytic torsion form can show up by considering the adiabatic limit of the Bismut-Freed connection associated to certain families of sub-signature operators.

11. An index theorem for Toeplitz operators on manifolds with boundary

The classical Toeplitz operator on circle, as well as the corresponding index theorem due to Gohberg-Krein, has a natural extension to odd dimensional manifolds, described as follows.

Let M be an odd dimensional closed oriented spin manifold. Let g^{TM} be a metric on TM. Let $D : \Gamma(S(TM)) \to \Gamma(S(TM))$ be the Dirac operator acting on the spinor bundle $S(TM)$. Then D is elliptic and formally self-adjoint.

Let

$$L^2_{\geq 0}(S(TM)) = \bigoplus_{\lambda \geq 0} E_\lambda$$

be the direct sum of the eigenspaces associated to nonnegative eigenvalues of D. Let $P_{\geq 0}$ be the orthogonal projection from $L^2(S(TM))$ to $L^2_{\geq 0}(S(TM))$.

Let $\mathbf{C}^N|_M$ be the trivial vector bundle on M carrying the trivial metric and connection.

Let $g \in \mathrm{Aut}(\mathbf{C}^N|_M)$ be an automorphism of $\mathbf{C}^N|_M$.

The associated Toeplitz operator T_g can be defined as follows,

$$T_g = \left(P_{\geq 0} \otimes \mathrm{Id}_{\mathbf{C}^N|_M}\right) \mathrm{Id}_{S(TM)} \otimes g \left(P_{\geq 0} \otimes \mathrm{Id}_{\mathbf{C}^N|_M}\right) : \\ L^2_{\geq 0}(S(TM)) \otimes \mathbf{C}^N|_M \to L^2_{\geq 0}(S(TM)) \otimes \mathbf{C}^N|_M. \tag{11.1}$$

One can prove that T_g is a Fredholm operator. Its index is computed by Baum-Douglas as an application of the Atiyah-Singer index theorem for elliptic pseudo-differential operators.

Index Theorem for T_g. *The following identity holds,*

$$\operatorname{ind} T_g = \left\langle \widehat{A}(TM)\mathrm{ch}(g), [M] \right\rangle, \tag{11.2}$$

where $\mathrm{ch}(g)$ *is the odd Chern character associated to* g*.*

There is also a heat kernel proof of the above index theorem through η-invariants. In such a proof, one first identifies $\operatorname{ind} T_g$ with certain spectral flow, then computes this spectral flow by evaluating the variation of η-invariants.

In the joint paper [50] with Xianzhe Dai, we established a generalization of the above index theorem to the case of manifolds with boundary. Our result can be thought of as an odd dimensional analogue of the Atiyah-Patodi-Singer index theorem for Dirac operators on manifolds with boundary.

In the process, we also constructed an η-invariant associated to g on even dimensional manifolds, which is of independent interests.

We now briefly summarize our main result in the following subsections.

11.1. **Toeplitz operators on manifolds with boundary.** Let M be an odd dimensional oriented spin manifold with boundary ∂M. We assume that M carries a fixed spin structure. Then ∂M carries the canonically induced orientation and spin structure. Let g^{TM} be a Riemannian metric on TM such that it is of product structure near the boundary ∂M. That is, there is a tubular neighborhood, which, without loss of generality, can be taken to be $[0,1)\times\partial M\subset M$ with $\partial M=\{0\}\times\partial M$ such that

$$g^{TM}\big|_{[0,1)\times\partial M}=dx^2\oplus g^{T\partial M},$$

where $x\in[0,1)$ is the geodesic distance to ∂M and $g^{T\partial M}$ is the restriction of g^{TM} on ∂M. Let ∇^{TM} be the Levi-Civita connection of g^{TM}. Let $S(TM)$ be the Hermitian bundle of spinors associated to (M,g^{TM}). Then ∇^{TM} extends naturally to a Hermitian connection $\nabla^{S(TM)}$ on $S(TM)$.

Let E be a Hermitian vector bundle over M. Let ∇^E be a Hermitian connection on E. We assume that the Hermitian metric g^E on E and connection ∇^E are of product structure over $[0,1)\times\partial M$. That is, if we denote $\pi:[0,1)\times\partial M\to\partial M$ the natural projection, then

$$g^E\big|_{[0,1)\times\partial M}=\pi^*\left(g^E\big|_{\partial M}\right),\quad \nabla^E\big|_{[0,1)\times\partial M}=\pi^*\left(\nabla^E\big|_{\partial M}\right).$$

For any $X\in TM$, we extend the Clifford action $c(X)$ of X on $S(TM)$ to an action on $S(TM)\otimes E$ by acting as identity on E, and still denote this extended action by $c(X)$. Let $\nabla^{S(TM)\otimes E}$ be the tensor product connection on $S(TM)\otimes E$ obtained from $\nabla^{S(TM)}$ and ∇^E.

The canonical (twisted) Dirac operator D^E is defined by

$$D^E=\sum_{i=1}^{\dim M}c(e_i)\nabla^{S(TM)\otimes E}_{e_i}:\Gamma(S(TM)\otimes E)\longrightarrow\Gamma(S(TM)\otimes E),$$

where $e_1,\dots,e_{\dim M}$ is an orthonormal basis of TM. One verifies easily that over $[0,1)\times\partial M$, one has

$$D^E=c\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}+\pi^*D^E_{\partial M}\right),$$

where $D^E_{\partial M}:\Gamma((S(TM)\otimes E)|_{\partial M})\to\Gamma((S(TM)\otimes E)|_{\partial M})$ is the induced Dirac operator on ∂M. The later is elliptic and self-adjoint.

We now introduce the APS type boundary conditions for D^E.

Let $L^2_+((S(TM)\otimes E)|_{\partial M})$ be the space of the direct sum of eigenspaces of positive eigenvalues of $D^E_{\partial M}$. Let $P_{\partial M}$ denote the orthogonal projection operator from $L^2((S(TM)\otimes E)|_{\partial M})$ to $L^2_+((S(TM)\otimes E)|_{\partial M})$ (for simplicity we suppress the dependence on E).

As is well known, the APS projection $P_{\partial M}$ is an elliptic global boundary condition for D^E. However, to get self adjoint boundary conditions, we need to modify it by a Lagrangian subspace of $\ker D^E_{\partial M}$, namely, a subspace L of $\ker D^E_{\partial M}$ such that $c(\frac{\partial}{\partial x})L=L^\perp\cap(\ker D^E_{\partial M})$. Since ∂M bounds M, by the cobordism invariance of the index, such Lagrangian subspaces always exist.

The modified APS projection is obtained by adding the projection onto the Lagrangian subspace. Let $P_{\partial M}(L)$ denote the orthogonal projection operator from $L^2((S(TM)\otimes E)|_{\partial M})$ to $L^2_+((S(TM)\otimes E)|_{\partial M})\oplus L$:

$$P_{\partial M}(L) = P_{\partial M} + P_L,$$

where P_L denotes the orthogonal projection from $L^2((S(TM)\otimes E)|_{\partial M})$ to L.

The pair $(D^E, P^E_{\partial M}(L))$ forms a self-adjoint elliptic boundary problem, and $P_{\partial M}(L)$ is called an Atiyah-Patodi-Singer boundary condition associated to L. We will also denote the corresponding elliptic self-adjoint operator by $D^E_{P_{\partial M}(L)}$.

Let $L^{2,+}_{P_{\partial M}(L)}(S(TM)\otimes E))$ be the space of the direct sum of eigenspaces of *non-negative* eigenvalues of $D^E_{P_{\partial M}(L)}$. This can be viewed as an analog of the Hardy space. We denote by $P_{P_{\partial M}(L)}$ the orthogonal projection from $L^2(S(TM)\otimes E)$ to $L^{2,+}_{P_{\partial M}(L)}(S(TM)\otimes E))$.

Let $N>0$ be a positive integer, let $\mathbf{C}^N$ be the trivial complex vector bundle over M of rank N, which carries the trivial Hermitian metric and the trivial Hermitian connection. Then all the above construction can be developed in the same way if one replaces E by $E\otimes \mathbf{C}^N$. And all the operators considered here extend to act on $\mathbf{C}^N$ by identity. If there is no confusion we will also denote them by their original notation.

Now let $g: M\to GL(N,\mathbf{C})$ be a smooth automorphism of $\mathbf{C}^N$. With simple deformation, we can assume that g is unitary. That is, $g: M\to U(N)$. Furthermore, we make the assumption that g is of product structure over $[0,1)\times\partial M$, that is,

$$g|_{[0,1)\times\partial M} = \pi^*\left(g|_{\partial M}\right).$$

Clearly, g extends to an action on $S(TM)\otimes E\otimes \mathbf{C}^N$ by acting as identity on $S(TM)\otimes E$. We still denote this extended action by g.

Since g is unitary, one verifies easily that the operator $gP_{\partial M}(L)g^{-1}$ is again an orthogonal projection on $L^2((S(TM)\otimes E\otimes\mathbf{C}^N)|_{\partial M})$, and that $gP_{\partial M}(L)g^{-1} - P_{\partial M}(L)$ is a pseudodifferential operator of order less than zero. Moreover, the pair $(D^E, gP_{\partial M}(L)g^{-1})$ forms a self-adjoint elliptic boundary problem. We denote its associated elliptic self-adjoint operator by $D^E_{gP_{\partial M}(L)g^{-1}}$. Thus $D^E_{gP_{\partial M}(L)g^{-1}}$ has the boundary condition which is the conjugation by g of the previous APS type condition.

The necessity of using the conjugated boundary condition here is from the fact that, if $s\in L^2(S(TM)\otimes E\otimes\mathbf{C}^N)$ verifies $P_{\partial M}(L)(s|_{\partial M})=0$, then gs verifies $gP_{\partial M}(L)g^{-1}((gs)|_{\partial M})=0$.

Thus, consider also the analog of Hardy space for the conjugated boundary value problem, $L^{2,+}_{gP_{\partial M}(L)g^{-1}}(S(TM)\otimes E\otimes\mathbf{C}^N)$ which is the space of the direct sum of eigenspaces of *nonnegative* eigenvalues of $D^E_{gP_{\partial M}(L)g^{-1}}$. Let $P_{gP_{\partial M}(L)g^{-1}}$ denote the orthogonal projection from $L^2(S(TM)\otimes E\otimes \mathbf{C}^N)$ to $L^{2,+}_{gP_{\partial M}(L)g^{-1}}(S(TM)\otimes E\otimes \mathbf{C}^N)$.

Definition 11.1. The Toeplitz operator $T^E_g(L)$ is defined by

$$T^E_g(L) = P_{gP_{\partial M}(L)g^{-1}}\circ g\circ P_{P_{\partial M}(L)}:$$
$$L^{2,+}_{P_{\partial M}(L)}\left(S(TM)\otimes E\otimes \mathbf{C}^N\right)\to L^{2,+}_{gP_{\partial M}(L)g^{-1}}\left(S(TM)\otimes E\otimes \mathbf{C}^N\right).$$

One verifies that $T^E_g(L)$ is a Fredholm operator.

11.2. **Perturbation.** The analysis of the conjugated elliptic boundary value problem $D^E_{gP_{\partial M}(L)g^{-1}}$ turns out to be surprisingly subtle and difficult. To circumvent this difficulty, we now construct a perturbation of the original problem.

Let $\psi=\psi(x)$ be a cut off function which is identically 1 in the ϵ-tubular neighborhood of ∂M ($\epsilon>0$ sufficiently small) and vanishes outside the 2ϵ-tubular neighborhood of ∂M. Consider the Dirac type operator

$$D^\psi = (1-\psi)D^E + \psi g D^E g^{-1}.$$

The effect of this perturbation is that, near the boundary, the operator D^ψ is actually given by the conjugation of D^E, and therefore, the elliptic boundary problem $(D^\psi, gP_{\partial M}(L)g^{-1})$ is now the conjugation of the APS boundary problem $(D^E, P_{\partial M}(L))$.

All previous consideration applies to $(D^\psi, gP_{\partial M}(L)g^{-1})$ and its associated self-adjoint elliptic operator $D^\psi_{gP_{\partial M}(L)g^{-1}}$. In particular, we have the perturbed Toeplitz operator

$$T^E_{g,\psi}(L) = P^\psi_{gP_{\partial M}(L)g^{-1}}\circ g\circ P_{P_{\partial M}(L)}: \tag{11.3}$$
$$L^{2,+}_{P_{\partial M}(L)}\left(S(TM)\otimes E\otimes \mathbf{C}^N\right)\to L^{2,+,\psi}_{gP_{\partial M}(L)g^{-1}}\left(S(TM)\otimes E\otimes \mathbf{C}^N\right),$$

where $P^\psi_{gP_{\partial M}(L)g^{-1}}$ is the APS projection associated to $D^\psi_{gP_{\partial M}(L)g^{-1}}$.

We will also need to consider the conjugation of D^ψ:

$$D^{\psi,g} = g^{-1}D^\psi g = D^E + (1-\psi)g^{-1}[D^E, g]. \tag{11.4}$$

11.3. **An invariant of η type for even dimensional manifolds.** Given an even dimensional closed spin manifold X, we consider the cylinder $[0,1]\times X$ with the product metric. Let $g: X\to U(N)$ be a map from X into the unitary group which extends trivially to the cylinder. Similarly, $E\to X$ is an Hermitian vector bundle which is also extended trivially to the cylinder. We make the assumption that $\operatorname{ind} D^E_+ = 0$ on X.

Consider the analog of $D^{\psi,g}$ as defined in (11.4), but now on the cylinder $[0,1]\times X$ and denote it by $D^{\psi,g}_{[0,1]}$. We equip it with the boundary condition $P_X(L)$ on one of the boundary component $\{0\}\times X$ and the boundary condition $\mathrm{Id}-g^{-1}P_X(L)g$ on the other boundary component $\{1\}\times X$ (Note that the Lagrangian subspace L exists by our assumption of vanishing index). Then $(D^{\psi,g}_{[0,1]}, P_X(L), \mathrm{Id}-g^{-1}P_X(L)g)$ forms a self-adjoint elliptic boundary problem. For simplicity, we will still denote the corresponding elliptic self-adjoint operator by $D^{\psi,g}_{[0,1]}$.

Let $\eta(D^{\psi,g}_{[0,1]}, s)$ be the η-function of $D^{\psi,g}_{[0,1]}$ which, when $\mathrm{Re}(s)>>0$, is defined by

$$\eta(D^{\psi,g}_{[0,1]}, s)=\sum_{\lambda\neq 0}\frac{\mathrm{sgn}(\lambda)}{|\lambda|^s},$$

where λ runs through the nonzero eigenvalues of $D^{\psi,g}_{[0,1]}$.

One knows that the η-function $\eta(D^{\psi,g}_{[0,1]}, s)$ admits a meromorphic extension to $\mathbf{C}$ with $s=0$ a regular point (and only simple poles). One then defines the η-invariant of $D^{\psi,g}_{[0,1]}$, denoted by $\eta(D^{\psi,g}_{[0,1]})$, to be the value at $s=0$ of $\eta(D^{\psi,g}_{[0,1]}, s)$, and the reduced η-invariant by

$$\overline{\eta}(D^{\psi,g}_{[0,1]})=\frac{\dim\ker D^{\psi,g}_{[0,1]}+\eta(D^{\psi,g}_{[0,1]})}{2}.$$

Definition 11.2. We define an invariant of η type for the complex vector bundle E on the even dimensional manifold X (with vanishing index) and the K^1 representative g by

$$\overline{\eta}(X,g)=\overline{\eta}(D^{\psi,g}_{[0,1]})-\mathrm{sf}\left\{D^{\psi,g}_{[0,1]}(s); 0\le s\le 1\right\}, \tag{11.5}$$

where "sf" is the notation for spectral flow and $D^{\psi,g}_{[0,1]}(s)$ is a path connecting $g^{-1}D^E g$ with $D^{\psi,g}_{[0,1]}$ defined by

$$D^{\psi,g}(s)=D^E+(1-s\psi)g^{-1}[D^E,g]$$

on $[0,1]\times X$, with the boundary condition $P_X(L)$ on $\{0\}\times X$ and the boundary condition $\mathrm{Id}-g^{-1}P_X(L)g$ at $\{1\}\times X$.

We proved in [50, Section 5] that $\overline{\eta}(X,g)$ does not depend on the cut off function ψ and is thus a well-defined analytic invariant.

In our application, we will apply this construction to the cylinder $[0,1]\times\partial M$. i.e., $X=\partial M$ is a boundary.

11.4. **An index theorem for $T_g^E(L)$.** Recall that $g : M \to U(N)$. Thus $g^{-1}dg$ defines a $\Gamma(\mathrm{End}(\mathbf{C}^N))$-valued 1-form on M. Let $\mathrm{ch}(g)$ denote the odd Chern character form of g defined by (cf. [66, Chap. 1])

$$\mathrm{ch}(g) = \sum_{n=0}^{\frac{\dim M-1}{2}} \frac{n!}{(2n+1)!}\mathrm{Tr}\left[\left(g^{-1}dg\right)^{2n+1}\right].$$

Recall also that ∇^{TM} is the Levi-Civita connection associated to the Riemannian metric g^{TM}, and ∇^E is the Hermitian connection on E. Let $R^{TM} = (\nabla^{TM})^2$ (resp. $R^E = (\nabla^E)^2$) be the curvature of ∇^{TM} (resp. ∇^E).

Let $\mathcal{P}_M$ denote the Calderón projection associated to $D^{E\otimes\mathbf{C}^N}$ on M. Then $\mathcal{P}_M$ is an orthogonal projection on $L^2((S(TM)\otimes E\otimes\mathbf{C}^N)|_{\partial M})$, and that $\mathcal{P}_M - P_{\partial M}(L)$ is a pseudodifferential operator of order less than zero.

Let $\tau_\mu(gP_{\partial M}(L)g^{-1}, P_{\partial M}(L), \mathcal{P}_M) \in \mathbf{Z}$ be the Maslov triple index in the sense of Kirk and Lesch.

We can now state our main result as follows.

Theorem 11.3. *The following identity holds,*

(11.6)
$$\begin{aligned}\mathrm{ind}\, T_g^E(L) = -\left(\frac{1}{2\pi\sqrt{-1}}\right)^{(\dim M+1)/2} \int_M \widehat{A}\left(R^{TM}\right)\mathrm{Tr}\left[\exp\left(-R^E\right)\right]\mathrm{ch}(g,d)\\ - \overline{\eta}(\partial M, g) + \tau_\mu\left(gP_{\partial M}(L)g^{-1}, P_{\partial M}(L), \mathcal{P}_M\right).\end{aligned}$$

Remark 11.4. The formula (11.6) is closely related to the so called WZW theory in physics. When $\partial M = S^2$ or a compact Riemann surface and E is trivial, the local term in (11.6) is precisely the Wess-Zumino term, which allows an integer ambiguity, in the WZW theory. Thus, our eta invariant $\overline{\eta}(\partial M, g)$ gives an intrinsic interpretation of the Wess-Zumino term without passing to the bounding 3-manifold. In fact, for $\partial M = S^2$, it can be further reduced to a local term on S^2 by using Bott's periodicity.

The following immediate consequence is of independent interests.

Corollary 11.5. *The number*

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^{(\dim M+1)/2} \int_M \widehat{A}\left(R^{TM}\right)\mathrm{Tr}\left[\exp\left(-R^E\right)\right]\mathrm{ch}(g,d) + \overline{\eta}(\partial M, g)$$

is an integer.

Our proof of Theorem 11.3 given in [50] divides into two steps. In the first step, we proved by using heat kernel method an index theorem for the perturbed Toeplitz operator defined in (11.3). Then in the second step we connected the index of the perturbed Toeplitz operator and that of the (un-perturbed) Toeplitz operator by using spectral flow.

11.5. **A conjectural relation between two cylindrical η-invariants.** We have pointed out that the eta type invariant $\overline{\eta}(X,g)$, which we introduced in Subsection 11.3 using a cut off function, is in fact independent of the cut off function. This leads naturally to the question of whether $\overline{\eta}(X,g)$ can actually be defined directly. We now state a conjecture for this question.

Let $D^{[0,1]}$ be the Dirac operator on $[0,1]\times X$. We equip the boundary condition $gP_X(L)g^{-1}$ at $\{0\}\times X$ and the boundary condition $\mathrm{Id}-P_X(L)$ at $\{1\}\times X$.

Then $(D^{[0,1]}, gP_X(L)g^{-1}, \mathrm{Id}-P_X(L))$ forms a self-adjoint elliptic boundary problem. Let $D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}$ denote the corresponding elliptic self-adjoint operator.

Let $\eta(D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}, s)$ be the η-function of $D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}$. By results due to Grubb and Kirk-Lesch, one knows that the η-function $\eta(D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}, s)$ admits a meromorphic extension to $\mathbf{C}$ with poles of order at most 2. One then defines the η-invariant of $D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}$, denoted by $\eta(D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)})$, to be the constant term in the Laurent expansion of $\eta(D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}, s)$ at $s=0$.

Let $\overline{\eta}(D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)})$ be the associated reduced η-invariant.

Conjecture 11.6. The following identity holds,

$$\overline{\eta}(X,g) = \overline{\eta}\left(D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}\right).$$

If this conjecture would be correct, then the result stated in [62, Theorem 5.2] would also be correct. A previous version of [50] was devoted to a proof of [62, Theorem 5.2], and a referee pointed out a gap in that version. This is why we later introduced a new η-type invariant, which makes the picture clearer.

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Géométrie différentielle/*Differential Geometry*

Métriques de Reidemeister et métriques de Ray-Singer sur le déterminant de la cohomologie d'un fibré plat : une extension d'un résultat de Cheeger et Müller

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Résumé – Soit M une variété compacte, soit F un fibré plat sur M. On munit TM et F de métriques. Soit K une triangulation de M. On considère le fibré en droites réel det $H^{\cdot}$ (M, F), qu'on munit de deux métriques naturelles, la métrique de Reidemeister associée à K et la métrique de Ray-Singer. On établit des formules d'anomalie pour la métrique de Ray-Singer. On montre que les métriques de Reidemeister et de Ray-Singer sont reliées par une formule locale sur M. On généralise ainsi le résultat de Cheeger et Müller relatif à l'égalité des deux métriques dans le cas où la métrique de F est plate, résultat récemment étendu par Müller au cas où F possède une forme volume invariante. Le terme local mesurant le rapport des deux métriques fait précisément intervenir la variation locale de la forme volume sur F.

Reidemeister metrics and Ray-Singer metrics on the determinant of the cohomology of a flat bundle: an extension of a result of Cheeger and Müller

Abstract – *Let* M *be a compact manifold, let* F *be a flat bundle over* M. *We equip* TM *and* F *with smooth metrics. Let* K *be a smooth triangulation of* M. *We consider the real line* det $H^{\cdot}$ (M, F), *which we equip with two natural metrics, the Reidemeister metric and the Ray-Singer metric. We establish anomaly formulas for the Ray-Singer metric. We also show that the Reidemeister metric and the Ray-Singer metric are related by a formula which is local over* M. *We thus extend the result of Cheeger and Müller, who proved the equality of the two metrics in the case where the metric of* F *is flat. This last result was recently extended by Müller to the case where the volume form of* F *is flat. In our local term measuring the ratio of the Ray-Singer metric to the Reidemeister metric, the local variation of the volume form of* F *appears explicitly.*

Abridged English Version – Let M be a compact manifold. Let F be a real flat bundle on M and let F* be its dual. Let $H^{\cdot}$ (M, F) be the cohomology groups of the sheaf of flat sections of F. Let det $H^{\cdot}$ (M, F) be the real line

$$\det H^{\cdot}(M, F) = \bigotimes_{i=0}^{\dim M} (\det H^{i}(M, F))^{(-1)^{i}}.$$

Let g^{TM}, g^{F} be smooth metrics on TM, F. Let K be a smooth triangulation of M, let B be the finite set of barycenters of the simplexes of K, let $(C_{\cdot}(K, F^*), \partial_K)$ be the simplicial complex associated to K, F*, whose homology is canonically isomorphic to $H_{\cdot}(K, F^*)$, and let $(C^{\cdot}(K, F), \partial_K^*)$ be the dual complex. The restriction of g^F to B determines a metric on $C_{\cdot}(K, F^*)$, and so a metric on the line $\det C^{\cdot}(K, F) = \bigotimes_{i=0}^{\dim M} (\det(C^{i}(K, F))^{(-1)^{i}}$. Since

$$\det H^{\cdot}(M, F) \simeq \det C^{\cdot}(K, F),$$

we obtain a metric $\| \ \|^{R, K}_{\det H^{\cdot}(M, F)}$ on the line det $H^{\cdot}$ (M, F), called the Reidemeister metric. As was shown by Reidemeister (*see* [8]), if the metric g^F is flat, the Reidemeister metric is invariant under subdivision of K. This result was recently extended by Müller [10] to the case where the metric induced by g^F on the line det F is flat.

Let $\| \ \|^{RS}_{\det H^{\cdot}(M, F)}$ be the Ray-Singer metric on the line det $H^{\cdot}$ (M, F) associated to the metrics g^{TM}, g^F. This is the product of the standard L_2 metric on det $H^{\cdot}$ (M, F) [obtained

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by identifying $H^{\bullet}(M, F)$ with the harmonic elements in the De Rham complex $\Omega(M, F)$] by the Ray-Singer analytic torsion of the complex $(\Omega(M, F), d^F)$ [12].

In two celebrated papers, Cheeger [5] and Müller [9] proved a conjecture of Ray and Singer [12] saying that when the metric g^F is flat, the Reidemeister and Ray-Singer metrics coincide. This result was recently extended by Müller [10] to the case where the metric on det F is flat.

We here announce the result that in full generality, the ratio of the Ray-Singer metric to the Reidemeister metric is given by an explicit local formula on M. We thus extend the results of Cheeger and Müller to arbitrary flat bundles with metrics.

As an intermediary step, we establish anomaly formulas for the Ray-Singer metric, which are the analogues of the corresponding holomorphic anomaly formulas [1] in the De Rham category. These anomaly formulas are tautologically trivial when M is odd dimensional, and also when the metric g^F is flat.

To establish our main result, we inspire ourselves from prior work by Bismut-Lebeau [3] on a corresponding result in the holomorphic category. We then use the deformation of the De Rham complex suggested by Witten [14] using a Morse function and also the results of Helffer and Sjöstrand [6], who established the asymptotic matrix structure of the d-operator acting on certain finite dimensional vector subcomplexes of the De Rham complex for the trivial flat vector bundle as a parameter T tends to $+\infty$, when the Morse function f verifies the Smale transversality conditions.

Also, we choose a Morse function adapted to the triangulation K by using a result of Poźniak [11], who constructs a Morse function whose Thom complex coincides with the simplicial complex of K.

The idea of using Helffer and Sjöstrand's results [6] to establish the equality of the Reidemeister and Ray-Singer metrics goes back to Tangerman [13] who announced he had made progress in this direction.

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1. Introduction. — Soit M une variété compacte de dimension n. Soit F un fibré plat sur M, soit F^* son dual.

Soit K une triangulation C^∞ de M, soit B l'ensemble fini des barycentres des simplexes de K. Soit $(C_\bullet(K, F^*), \partial_K)$ le complexe simplicial associé à K et F^*, dont l'homologie est égale à $H_\bullet(M, F^*)$. Soit $(C^\bullet(K, F), \partial_K^*)$ le complexe dual, dont la cohomologie s'identifie à $H^\bullet(M, F)$. On pose

$$\det C^\bullet(K, F) = \bigotimes_{i=0}^{n} (\det C^i(K, F))^{(-1)^i},$$

$$\det H^\bullet(M, F) = \bigotimes_{i=0}^{n} (\det H^i(M, F))^{(-1)^i}.$$

Alors les droites $\det C^\bullet(K, F)$ et $\det H^\bullet(M, F)$ sont canoniquement isomorphes.

Pour $x \in B$, soit $\| \ \|_{\det F_x}$ une métrique sur la droite $\det F_x$. Les métriques $\| \ \|_{\det F_x}$ $(x \in B)$ induisent une métrique sur $\det C^\bullet(M, F)$. On appelle métrique de Reidemeister et on note $\| \ \|^{R,K}_{\det H^\bullet(M, F)}$ la métrique correspondante sur la droite $\det H^\bullet(M, F)$.

Dans le prolongement de résultats de Reidemeister [8], Müller [10] a établi récemment que si les métriques $\| \ \|_{\det F_x}$ proviennent d'une métrique plate sur $\det F$, alors la métrique de Reidemeister est invariante par subdivision.

Soit $(\Omega(M, F), d^F)$ le complexe de De Rham des sections C^∞ de $\Lambda(T^*M)\otimes F$. Soient g^{TM}, g^F des métriques sur TM, F. Soit $*$ l'opérateur de Hodge associé à la métrique g^{TM}. On munit $\Omega(M, F)$ du produit scalaire

$$\alpha, \alpha' \in \Omega(M, F) \mapsto \langle \alpha, \alpha' \rangle = \int_M \langle \alpha \wedge *\alpha' \rangle. \tag{1}$$

Soit d^{F*} l'adjoint formel de d^F relativement au produit scalaire (1). Soit

$$\square^F = d^F d^{F*} + d^{F*} d^F$$

le laplacien correspondant.

Par la théorie de Hodge, $H^\bullet(M, F) \simeq \operatorname{Ker} \square^F$. On munit $\operatorname{Ker} \square^F$ de la métrique induite par (1). On désigne par $|\ |_{\det H^\bullet(M, F)}$ la métrique sur la droite $\det H^\bullet(M, F)$ induite par cet isomorphisme.

Soit P la projection orthogonale de $\Omega(M, F)$ sur $\operatorname{Ker} \square^F$. On pose $P^\perp = 1 - P$. Soit N l'opérateur de nombre de $\Omega(M, F)$. On note Tr_s la supertrace. On pose pour $s \in \mathbb{C}$, $\operatorname{Re}(s) > n/2$

$$\theta(s) = -\operatorname{Tr}_s[N(\square^F)^{-s} P^\perp].$$

Définition 1. — On appelle *métrique de Ray-Singer* sur la droite $\det H^\bullet(M, F)$ la métrique

$$\|\ \|^{RS}_{\det H^\bullet(M, F)} = |\ |_{\det H^\bullet(M, F)} \exp\left\{\frac{1}{2}\theta'(0)\right\}.$$

Le terme $\exp\{1/2\,\theta'(0)\}$ est appelé *torsion analytique de Ray-Singer* [12].

Dans des articles célèbres, Cheeger [5] et Müller [9] ont montré que, quand la métrique g^F est plate, les métriques de Reidemeister et de Ray-Singer coïncident, établissant ainsi une conjecture de Ray-Singer [12]. Plus récemment, Müller a étendu le résultat de [5], [9] au cas où la métrique $\|\ \|_{\det F}$ induite par g^F sur $\det F$ est plate.

L'objet de cette Note est d'annoncer une formule reliant les métriques de Reidemeister et de Ray-Singer dans le cas général. Les preuves sont développées dans [4].

2. Formules d'anomalie pour la métrique de Ray-Singer. — Quand M est de dimension impaire, il est bien connu par Ray-Singer [12] que la métrique $\|\ \|^{RS}_{\det H^\bullet(M, F)}$ ne dépend pas des métriques g^{TM} et g^F. Quand M est de dimension paire et quand la métrique g^F est plate, il résulte aussi de [12] que la métrique $\|\ \|^{RS}_{\det H^\bullet(M, F)}$ est triviale, essentiellement à cause de la dualité de Poincaré.

Nous traitons ici le cas général. Soit g^F une métrique sur F. On pose

$$\omega(F, g^F) = (g^F)^{-1} d^F g^F$$
$$\theta(F, g^F) = \operatorname{Tr}[\omega(F, g^F)].$$

Alors $\theta(F, g^F)$ est une 1-forme fermée, dont la classe de cohomologie ne dépend pas de g^F.

Soient maintenant (g^{TM}, g^F) et (g'^{TM}, g'^F) deux couples de métriques sur TM, F. Soient $\|\ \|_{\det F}$ et $\|\ \|'_{\det F}$ les métriques induites par g^F et g'^F sur le fibré en droites $\det F$. Soient $\|\ \|^{RS}_{\det H^\bullet(M, F)}$ et $\|\ \|'^{RS}_{\det H^\bullet(M, F)}$ les métriques de Ray-Singer correspondantes sur la droite $\det H^\bullet(M, F)$.

Soit ∇^{TM} (resp. ∇'^{TM}) la connexion de Levi-Civita sur (TM, g^{TM}) [resp. (TM, g'^{TM})]. Soit $e(TM, \nabla^{TM})$ [resp. $e(TM, \nabla'^{TM})$] la forme différentielle sur M associée à la connexion ∇^{TM} (resp ∇'^{TM}) représentant la classe d'Euler de TM en théorie de Chern-Weil.

Soit $\tilde{e}(\mathrm{TM}, \nabla^{\mathrm{TM}}, \nabla'^{\mathrm{TM}})$ la classe de formes de Chern-Simons de degré $n-1$, définie à un cobord près, telle que

$$d\tilde{e}(\mathrm{TM}, \nabla^{\mathrm{TM}}, \nabla'^{\mathrm{TM}}) = e(\mathrm{TM}, \nabla'^{\mathrm{TM}}) - e(\mathrm{TM}, \nabla^{\mathrm{TM}}).$$

Notons que si n est impair, $e(\mathrm{TM}, \nabla^{\mathrm{TM}}) = e(\mathrm{TM}, \nabla'^{\mathrm{TM}}) = 0$, et que $\tilde{e}(\mathrm{TM}, \nabla^{\mathrm{TM}}, \nabla'^{\mathrm{TM}}) = 0$.

THÉORÈME 1. — *On a l'identité*

$$(2)\quad \mathrm{Log}\left(\frac{\| \quad \|'^{\mathrm{RS},\,2}_{\det H^\bullet(M,\,F)}}{\| \quad \|^{\mathrm{RS},\,2}_{\det H^\bullet(M,\,F)}}\right) = \int_M \mathrm{Log}\left(\frac{\| \quad \|'^{2}_{\det F}}{\| \quad \|^{2}_{\det F}}\right) e(\mathrm{TM}, \nabla^{\mathrm{TM}}) - \int_M \theta(\mathrm{F}, g'^{\mathrm{F}})\, \tilde{e}(\mathrm{TM}, \nabla^{\mathrm{TM}}, \nabla'^{\mathrm{TM}}).$$

3. COMPLEXE SIMPLICIAL ET FONCTION DE MORSE. — Soit K une triangulation C^∞ de M. Pour $0 \leqq i \leqq n$, soit K^i la réunion des simplexes de K de dimension $\leqq i$. Soit B l'ensemble fini des barycentres de M.

Alors par Poźniak [11], on peut construire une fonction de Morse f: $\mathrm{M} \to \mathbb{R}$ et une métrique g^{TM} sur TM ayant les propriétés suivantes :

— Les points critiques de f sont les barycentres de K. De plus si $\sigma \in K^i \setminus K^{i-1}$, l'indice de f au barycentre de σ est égal à i.

— Les cellules descendantes W_f^- associées à ∇f s'identifient aux simplexes de K.

— f vérifie les conditions de transversalité de Smale. Le complexe de Thom associé à f s'identifie au complexe simplicial associé à K.

— Si $x \in \mathrm{B}$, il existe des coordonnées $y = (y^1, \ldots, y^n)$ près de x tel que dans ces coordonnées

$$(3)\quad \begin{cases} f(y) = f(x) - \dfrac{1}{2}\displaystyle\sum_{1}^{\mathrm{ind}\,(x)} |y^i|^2 + \dfrac{1}{2}\sum_{\mathrm{ind}\,(x)+1}^{n} |y^i|^2, \\ g^{\mathrm{TM}} = \displaystyle\sum_{1}^{n} |dy^i|^2. \end{cases}$$

— Les simplexes de K de dimension $\leqq n-1$ sont totalement géodésiques dans M.

4. FONCTION DE MORSE ET FORME D'ANGLE. — Soit π la projection $\mathrm{TM} \to \mathrm{M}$. Soit $\psi(\mathrm{TM}, \nabla^{\mathrm{TM}})$ le courant de degré $n-1$ sur l'espace total de TM construit dans [7], [2], qui vérifie l'équation de courants

$$d\psi(\mathrm{TM}, \nabla^{\mathrm{TM}}) = \pi^* e(\mathrm{TM}, \nabla^{\mathrm{TM}}) - \delta_{\mathrm{M}}.$$

La restriction de ψ aux fibres de TM est la forme d'angle de TM relativement à la métrique g^{TM}.

Soit f: $\mathrm{M} \to \mathbb{R}$ une fonction de Morse, et soit B_f l'ensemble de ses points critiques. Soient ∇f et $\nabla' f$ les gradients de f relativement aux métriques g^{TM} et g'^{TM}.

On utilise les mêmes notations qu'en 2, 3.

THÉORÈME 2. — *On a l'identité*

$$(4)\quad -\int_M \theta(\mathrm{F}, g'^{\mathrm{F}})(\nabla' f)^* \psi(\mathrm{TM}, \nabla'^{\mathrm{TM}}) + \int_M \theta(\mathrm{F}, g^{\mathrm{F}})(\nabla f)^* \psi(\mathrm{TM}, \nabla^{\mathrm{TM}}) = \int_M \mathrm{Log}\left(\frac{\| \quad \|'^{2}_{\det F}}{\| \quad \|^{2}_{\det F}}\right) e(\mathrm{TM}, \nabla^{\mathrm{TM}}) - \int_M \theta(\mathrm{F}, g'^{\mathrm{F}})\, \tilde{e}(\mathrm{TM}, \nabla^{\mathrm{TM}}, \nabla'^{\mathrm{TM}}) - \sum_{x \in \mathrm{B}_f} (-1)^{\mathrm{ind}\,(x)} \mathrm{Log}\left(\frac{\| \quad \|'^{2}_{\det F_x}}{\| \quad \|^{2}_{\det F_x}}\right)$$

Soit $\sigma \in K^n \setminus K^{n-1}$. La 1-forme $\theta(F, g^F)$ possède sur σ une primitive $V_\sigma(F, g^F)$ définie à une constante près. Si $x \in \sigma \cap B$, soit $\alpha(\sigma, x) \in [0, 1]$ l'angle solide sous lequel x voit σ relativement à la métrique g^{T_xM}.

THÉORÈME 3. — *Si f et g^{TM} sont choisies comme en* 3, *on a l'identité*

$$(5) \quad -\int_M \theta(F, g^F)(\nabla f)^* \psi(TM, \nabla^{TM}) = \sum_{\sigma \in K^n \setminus K^{n-1}} \int_\sigma V_\sigma(F, g^F) e(TM, \nabla^{TM}) - \sum_{x \in B} (-1)^{\mathrm{ind}(x)} \sum_{\substack{\sigma \in K^n \setminus K^{n-1} \\ x \in \sigma}} \alpha(\sigma, x) V_\sigma(F, g^F)(x)$$

Remarque 4. — Des théorèmes 2 et 3, on tire en particulier que pour des métriques g^{TM}, g^F arbitraires, $-\int_M \theta(F, g^F)(\nabla f)^* \psi(TM, \nabla^{TM})$ ne dépend pas de la fonction f construite en 3.

5. COMPARAISON DES MÉTRIQUES DE REIDEMEISTER ET DE RAY-SINGER. — Soit K une triangulation C^∞ de M. Soit $f: M \to \mathbb{R}$ une fonction de Morse prise comme en 3 relativement à K.

Soient g^{TM}, g^F des métriques C^∞ sur TM, F. Pour $x \in B$, on munit $\det F_x$ de la métrique induite par g^{F_x}.

Soient $\| \ \|^{R, K}_{\det H^\bullet(M, F)}$ et $\| \ \|^{RS}_{\det H^\bullet(M, F)}$ les métriques de Reidemeister et de Ray-Singer correspondantes sur la droite $\det H^\bullet(M, F)$.

On a alors l'extension suivante du résultat de Cheeger [5] et Müller [9], [10].

THÉORÈME 5. — *On a l'identité*

$$(6) \quad \mathrm{Log}\left(\frac{\| \ \|^{RS, 2}_{\det H^\bullet(M, F)}}{\| \ \|^{R, K, 2}_{\det H^\bullet(M, F)}}\right) = -\int_M \theta(F, g^F)(\nabla f)^* \psi(TM, \nabla^{TM}).$$

Remarque 6. — De (6), on déduit en particulier le comportement de la métrique de Reidemeister par subdivision.

6. PRINCIPE DE LA PREUVE DU THÉORÈME 5. — La preuve du théorème 5 est proche dans son principe de la preuve d'un résultat de Bismut-Lebeau [3] dans une situation holomorphe formellement comparable.

Par les théorèmes 1 et 2, on voit qu'il suffit de montrer (6) pour un seul couple (g^{TM}, g^F) de métriques sur (TM, F).

On choisit g^{TM} adapté à f commme en 3. De plus on suppose que g^F est plate sur un voisinage de B.

Pour $T \geqq 0$, on munit F de la métrique $g^F_T = e^{-2Tf} g^F$. Soit d^{F*}_T l'adjoint de d^F relativement au produit scalaire (1) associé aux métriques (g^{TM}, g^F_T). On pose

$$D_T = d^F + d^{F*}_T.$$

On a un résultat très proche d'un résultat de Bismut-Lebeau [3].

THÉORÈME 6. — *Soit $\alpha_{t, T}$ la 1-forme sur $\mathbb{R}^*_+ \times \mathbb{R}_+$*

$$(7) \quad \alpha_{t, T} = \frac{dt}{2t} \mathrm{Tr}_s[N \exp(-t D^2_T)] - dT\, \mathrm{Tr}_s[f \exp(-t D^2_T)]$$

Alors $\alpha_{t, T}$ est fermée.

Soient ε, A, T_0 avec $0<\varepsilon<1<A<+\infty$, $0<T_0<+\infty$.

Soit Γ le contour orienté de $\mathbb{R}_+^* \times \mathbb{R}_+$ qui borde le rectangle $\Delta=\{(u, T),\ \varepsilon \leqq u \leqq A,\ 0 \leqq T \leqq T_0\}$. Soit Γ_1 le côté $\{(u, T),\ \varepsilon \leqq u \leqq A,\ T=T_0\}$. Les côtés Γ_2, Γ_3, Γ_4 sont obtenus à partir de Γ_1 par orientation de Γ dans le sens direct. On pose $I_j=\int_{\Gamma_j} \alpha$. Du théorème 6, on tire

(8) $$\sum_1^4 I_j=0$$

Comme dans [3], on étudie le comportement des termes I_j ($1 \leqq j \leqq 4$) quand $A \to +\infty$, $T_0 \to +\infty$, $\varepsilon \to 0$ dans cet ordre. (6) résultera alors de (8) et des résultats intermédiaires suivants, qui sont montrés dans [4].

Soit $\chi(F)$ la caractéristique d'Euler de F. On pose

$$\tilde{\chi}'(F)=\operatorname{rg}(F) \sum_{x \in B} (-1)^{\operatorname{ind}(x)} \operatorname{ind}(x),$$

$$\operatorname{Tr}_s^B[f]=\sum_{x \in B} (-1)^{\operatorname{ind}(x)} f(x).$$

Soit $P_T^{]1,+\infty[}$ la projection orthogonale de $\Omega(M, F)$ sur le sous-espace de $\Omega(M, F)$ engendré par des espaces propres de D_T^2 associé à des valeurs propres >1.

Théorème 7. — *Étant donnés* ε, A *avec* $0<\varepsilon<A<+\infty$, *il existe* $C>0$ *tel que si* $t \in [\varepsilon, A]$, $T \geqq 1$, *alors*

$$\| \operatorname{Tr}_s[N \exp(-t D_T^2)]-\tilde{\chi}'(F) \| \leqq \frac{C}{\sqrt{T}}.$$

Pour tout $t>0$, *on a*

$$\lim_{T \to +\infty} \operatorname{Tr}_s[N \exp(-t D_T^2) P_T^{]1,+\infty[}]=0.$$

Il existe $c>0$, $C>0$ *tels que pour tout* $t \geqq 1$, $T \geqq 0$, *alors*

$$\| \operatorname{Tr}_s[N \exp(-t D_T^2) P_T^{]1,+\infty[}] \| \leqq c \exp(-C t).$$

Théorème 8. — *Pour tout* $t>0$, *il existe* $c>0$ *tel que quand* $T \to +\infty$, *on ait*

$$\operatorname{Tr}_s[f \exp(-t D_T^2)]=\operatorname{rg}(F) \operatorname{Tr}_s^B[f]+\left(\frac{n}{4} \chi(F)-\frac{1}{2} \tilde{\chi}'(F)\right) \frac{1}{T}+O(e^{-c T}).$$

On utilise maintenant le formalisme de l'intégrale de Berezin [7]. Soit R^{TM} la courbure de la connexion de Levi-Civita ∇^{TM}. Soit $e_1, \ldots, e_n$ une base orthonormale de TM. On pose

(9) $$\dot{R}^{TM}=\frac{1}{4} \sum \langle e_k, R^{TM}(e_i, e_j) e_l \rangle e^i \wedge e^j \wedge \hat{e}^k \wedge \hat{e}^l.$$

Pour $T \geqq 0$, on pose

$$B_T=\frac{\dot{R}^{TM}}{2}+\sqrt{T} \sum \langle \nabla_{e_i}^{T^*M} df, e_j \rangle e^i \wedge \hat{e}^j+T|df|^2.$$

L'intégrale de Berezin $\int^B$ transforme un polynôme en les variables anticommutantes $e^1, \ldots, e^n, \hat{e}^1, \ldots, \hat{e}^n$ en un polynôme en les variables $e^1, \ldots, e^n$, *i.e.* en une forme différentielle sur M. Une puissance de π convenable est incorporée dans l'intégrale $\int^{\mathrm{B}}$ pour simplifier les formules qui suivent.

On pose

$$\mathrm{L}=\frac{1}{2}\sum_{1}^{n} e^i \wedge \hat{e}^i.$$

THÉORÈME 9. — *Quand* $t \to 0$, *alors*

$$\mathrm{Tr}_s[\mathrm{N}\exp(-t\mathrm{D}_{\mathrm{T}}^2)]=\frac{n}{2}\chi(\mathrm{F})+O(t) \qquad \textit{si}\quad n \textit{ est pair},$$

$$=\mathrm{rg}(\mathrm{F})\int_{\mathrm{M}}\int^{\mathrm{B}} \mathrm{L}\exp\left(-\frac{\dot{\mathrm{R}}^{\mathrm{TM}}}{2}\right)\frac{1}{\sqrt{t}}+O(\sqrt{t}) \qquad \textit{si}\quad n \textit{ est impair}.$$

On pose

$$\mathrm{D}=d^{\mathrm{F}}+d^{\mathrm{F}*},$$
$$\hat{c}(df)=df\wedge + i_{\nabla f}.$$

L'opérateur $\mathrm{D}+\mathrm{T}\,\hat{c}(df)$ a été introduit par Witten [14].

THÉORÈME 10. — *Il existe* $\mathrm{C}>0$ *tel que pour* $0<t\leqq 1$, $0\leqq \mathrm{T}\leqq 1/t$, *on ait*

$$\left|\frac{1}{t^2}\left\{\mathrm{Tr}_s[f\exp(-(t\mathrm{D}+\mathrm{T}\,\hat{c}(df))^2)]-\mathrm{rg}(\mathrm{F})\int_{\mathrm{M}} f\int^{\mathrm{B}}\exp(-\mathrm{B}_{\mathrm{T}^2}) + t\int_{\mathrm{M}}\frac{\theta}{2}(\mathrm{F},g^{\mathrm{F}})\int^{\mathrm{B}}\widehat{df}\exp(-\mathrm{B}_{\mathrm{T}^2})\right\}\right|\leqq \mathrm{C}.$$

THÉORÈME 11. — *Pour tout* $\mathrm{T}>0$, *on a*

$$\lim_{t\to 0}\frac{1}{t^2}\left(\mathrm{Tr}_s\left[f\exp\left(-\left(t\mathrm{D}+\frac{\mathrm{T}}{t}\hat{c}(df)\right)^2\right)\right]-\mathrm{rg}(\mathrm{F})\,\mathrm{Tr}_s^{\mathrm{B}}[f]\right) = \left(\frac{n}{4}\chi(\mathrm{F})-\frac{1}{2}\tilde{\chi}'(\mathrm{F})\right)\frac{1}{\mathrm{T}\tanh(\mathrm{T})}.$$

THÉORÈME 12. — *Il existe* $c>0$, $\mathrm{C}>0$ *tel que pour* $t\in]0,1[$, $\mathrm{T}\geqq 1$, *alors*

$$\left|\frac{1}{t^2}\left(\mathrm{Tr}_s\left[f\exp\left(-\left(t\mathrm{D}+\frac{\mathrm{T}}{t}\hat{c}(df)\right)^2\right)\right]-\mathrm{rg}(\mathrm{F})\,\mathrm{Tr}_s^{\mathrm{B}}[f]-\frac{t^2}{\mathrm{T}}\left(\frac{n}{4}\chi(\mathrm{F})-\frac{1}{2}\tilde{\chi}'(\mathrm{F})\right)\right)\right| \leqq c\exp(-\mathrm{CT}).$$

Soit $\mathrm{D}_{\mathrm{T}}^{]0,1]}$ la restriction de l'opérateur D_{T} à la somme directe des espaces propres de $\mathrm{D}_{\mathrm{T}}^2$ pour des valeurs propres $\lambda\in]0,1]$.

Soit $|\ |_{\det \mathrm{H}^\bullet(\mathrm{M},\mathrm{F}),\mathrm{T}}$ la métrique L_2 sur la droite $\det \mathrm{H}^\bullet(\mathrm{M},\mathrm{F})$ associée aux métriques $(g^{\mathrm{TM}}, g_{\mathrm{T}}^{\mathrm{F}})$.

En utilisant une extension des résultats de Helffer-Sjöstrand [6] à des fibrés plats F munis de métriques arbitraires et le quasi-isomorphisme canonique du complexe de Rham $(\Omega(\mathrm{M},\mathrm{F}), d^{\mathrm{F}})$ au complexe simplicial $(\mathrm{C}^\bullet(\mathrm{K},\mathrm{F}), \partial_{\mathrm{K}}^*)$, on montre dans [4] le résultat clef suivant :

THÉORÈME 13. — *On a l'identité* :

$$\lim_{T \to +\infty} \left\{ \operatorname{Tr}_s [N \operatorname{Log} (D_T^{[0,1],2})] + \operatorname{Log} \left(\frac{|\ |^2_{\det H^\bullet(M,F),T}}{|\ |^2_{\det H^\bullet(M,F)}} \right) + 2 \operatorname{rg}(F) \operatorname{Tr}_s^B [f] T + \left(\frac{n}{2} \chi(F) - \tilde{\chi}'(F) \right) \operatorname{Log} \left(\frac{T}{\pi} \right) \right\} = \operatorname{Log} \left(\frac{\|\ \|^{R,K,2}_{\det H^\bullet(M,F)}}{|\ |^2_{\det H^\bullet(M,F)}} \right).$$

De l'identité (8) et des théorèmes 7-13, on tire dans [4] le théorème 5.

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Géométrie/*Geometry*

Spin^c-manifolds and Rokhlin congruences

Weiping ZHANG

Abstract – We establish a general congruence formula of Rokhlin type for spin^c-manifolds. This result refines the integrality theorems of Atiyah and Hirzebruch [1]. It also extends the previous congruences due to Rokhlin [7], Atiyah-Rees [2], Esnault-Seade-Viehweg [3] and Zhang [10]-[12].

Varietés spin^c et congruences de Rokhlin

Résumé – *Nous établissons une formule de congruence générale du type Rokhlin pour des variétés* spin^c. *Ce résultat précise le théorème de divisibilité d'Atiyah-Hirzebruch* [1]. *Il étend également des congruences dues à Rokhlin* [7], *Atiyah-Rees* [2], *Esnault-Seade-Viehweg* [3] *et Zhang* ([10]-[12]).

Version française abrégée – Soit K une variété compacte spin de dimension $8k+4$. Soit E un fibré vectoriel orienté réel sur K. Soit E_C la complexification de E. Alors, par un théorème classique d'Atiyah-Hirzebruch [1], le nombre caractéristique

$$\langle \hat{A}(TK)\, \mathrm{ch}\,(E_C),\, [K] \rangle$$

est un entier pair.

D'autre part, si K est une variété compacte connexe orientée non-spin de dimension 4, soit B une sous-variété compacte connexe orientable de dimension 2 de K telle que $[B] \in H_2(K, Z_2)$ soit duale à la deuxième classe de Stiefel-Whitney $w_2(K) \in H^2(K, Z_2)$. Rokhlin [7] a établi une formule de congruence pour la signature de K du type

$$\frac{\mathrm{Sign}\,(B.B) - \mathrm{Sign}\,(K)}{8} \equiv \Phi(B) \quad (\mathrm{mod}\ 2Z)$$

où B.B désigne l'auto-intersection de B et $\Phi(B)$ est un invariant de cobordisme spin de B associé à (K, B).

Dans cette Note, nous étendons les résultats d'Atiyah-Hirzebruch et de Rokhlin aux variétés spin^c *de dimension supérieure*.

Comme corollaire, nous donnons une formule intrinsèque pour des indices mod 2 des fibrés vectoriels orientables de dimension 2 sur une variété spin de dimension $8k+2$.

In this Note, we establish an extended Rokhlin type congruence formula for spin^c-manifold. This result generalizes the formulas proved in Zhang ([10]-[12]).

Although it turns out that both of the proofs appearing in [11] and [12] can be used to prove this formula, we here present a third proof whose idea goes back to Atiyah and Hirzebruch [1]. From the topological point of view, this proof seems more close to the heart of the problem, in comparing with the cobordism proof in [12].

1. A CONGRUENCE FORMULA FOR SPIN^c-MANIFOLDS. – Let K be a compact connected oriented spin^c-manifold of dimension $8k+4$. Let ξ be a complex line bundle on K such that the formula $c_1(\xi) \equiv w_2(TK)$ (mod 2) holds. We fix a spin structure on $\xi \oplus TK$.

Let B be an $8k+2$ dimensional compact connected orientable submanifold of K such that $[B] \in H_{8k+2}(K, Z)$ is Poincaré dual to $c = c_1(\xi)$. Then B carries an induced spin structure (*cf.* [5]). We call B a c-characteristic submanifold of K.

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If F is a real vector bundle on B, we denote by $\mathrm{ind}_2(\mathrm{F})$ the associated mod 2 index of Atiyah and Singer (*cf.* [4], [13]).

Let now E be a real vector bundle over K. Note $E_{\mathbf{C}}$ the complexification of E.

Let $i: B \hookrightarrow K$ denote the canonical embedding of B in K.

The main result of this Note can be stated as follows.

THEOREM 1. – *The following identity holds,*

$$\text{(1)} \qquad \left\langle \hat{A}(TK) \exp\left(\frac{c}{2}\right) \mathrm{ch}(E_{\mathbf{C}}), [K] \right\rangle \equiv \mathrm{ind}_2(i^* E) \quad (\mathrm{mod}\ 2\mathbf{Z}).$$

Proof. – We use ideas of Atiyah-Hirzebruch [1] and Atiyah-Rees [2] to prove (1).

Let m, n be two sufficiently large positive integers. Let $f: K \to \mathbf{C}P^{4m+2}$ be a classifying map of ξ, and let $g: K \hookrightarrow S^{8n}$ be an embedding. Let $h = (f, g): K \hookrightarrow \mathbf{C}P^{4m+2} \times S^{8n}$ be the induced embedding.

Let γ be the canonical complex line bundle over $\mathbf{C}P^{4m+2}$. Let

$$\pi: \quad \mathbf{C}P^{4m+2} \times S^{8n} \to \mathbf{C}P^{4m+2}$$

be the projection map. Then one has

$$\text{(2)} \qquad \xi = f^*(\gamma) = h^* \pi^*(\gamma).$$

Let $j: \mathbf{C}P^{4m+1} \hookrightarrow \mathbf{C}P^{4m+2}$ be the canonical embedding. Clearly $M := \mathbf{C}P^{4m+1} \times S^{8n}$ is a $c_1(\pi^* \gamma)$-characteristic submanifold of $\mathbf{C}P^{4m+2} \times S^{8n}$. Furthermore, by perturbing f and g, we get a transversal intersection $B' = K \cap M$, which is another $c_1(\xi)$-characteristic submanifold of K. Note $i': B' \hookrightarrow K$, $j': B' \hookrightarrow M$ the canonical embeddings.

Then by an easy modification of an argument in Ochanine ([5], Section 2.5) and by the spin cobordism invariance of the mod 2 index, one gets

$$\text{(3)} \qquad \mathrm{ind}_2(i'^* E) = \mathrm{ind}_2(i^* E).$$

Also, since $K \cap M$ is a transversal intersection, one has the following identification of KO direct images:

$$\text{(4)} \qquad (j \times \mathrm{id}_{S^{8n}})^* (h_! E) = j'_! (i'^* E).$$

On the other hand, by the Atiyah-Hirzebruch theorem the Riemann-Roch property for spin^c-manifolds as well as for the mod 2 index, one obtains that

$$\text{(5)} \quad \left\langle \hat{A}(TB) \exp\left(\frac{c}{2}\right) \mathrm{ch}(E_{\mathbf{C}}), [K] \right\rangle$$

$$= \left\langle \hat{A}(T(\mathbf{C}P^{4m+2} \times S^{8n})) \exp\left(\frac{\pi^* c_1(\gamma)}{2}\right) \mathrm{ch}((h_! E)_{\mathbf{C}}), [\mathbf{C}P^{4m+2} \times S^{8n}] \right\rangle,$$

and

$$\text{(6)} \qquad \mathrm{ind}_2(i'^* E) = \mathrm{ind}_2(j'_!(i'^* E)).$$

By using (3)-(6), we then reduce (1) to the case where $K = \mathbf{C}P^{4m+2} \times S^{8n}$, $B = \mathbf{C}P^{4m+1} \times S^{8n}$. This in turn, via the Bott periodicity theorem, can be reduced to the case where $K = \mathbf{C}P^{4m+2}$ and $B = \mathbf{C}P^{4m+1}$ for which the validity of (1) has been proved by Atiyah and Rees [3]. □

Remark 2. – Since $KO(\mathbf{C}P^{4m+2})$ has been calculated explicitly (Sanderson [8]), the formula (1) for complex projective spaces can also be verified directly.

Remark 3. – Theorem 1 provides a partial way of calculating the mod 2 index of a real vector bundle over an $8k+2$ dimensional spin manifold, at least when this bundle can be extended through some circle bundle to a spin^c-manifold.

Remark 4. – Special cases of (1) for complex manifolds have been proved in Atiyah-Rees [2] and Esnault-Seade-Viehweg [3].

2. SOME APPLICATIONS. – We state some corollaries of theorem 1.

COROLLARY 5 (Atiyah-Hirzebruch [1]). – *Let* K *be a compact spin manifold of dimension* $8k+4$, E *a real vector bundle over* K, *then* $\langle \hat{A}(TK)\,\mathrm{ch}(E_{\mathbf{C}}), [K]\rangle$ *is an even interger.*

Let (K, B) be a characteristic pair as in Section 1. Let N be the normal bundle to B in K. Note e the Euler class of N.

Let Ξ be an integral power operation on KO.

COROLLARY 6 (Zhang [11]). – *The following identity holds,*

$$(7)\quad \langle \hat{A}(TK)\,\mathrm{ch}(\Xi_{\mathbf{C}}(TK)), [K]\rangle \equiv \mathrm{ind}_2(\Xi(TB\oplus\mathbf{R}^2))$$
$$+\left\langle \hat{A}(TB)\,\frac{\mathrm{ch}(\Xi_{\mathbf{C}}(TB\oplus N))-\cosh(e/2)\,\mathrm{ch}(\Xi_{\mathbf{C}}(TB\oplus\mathbf{R}^2))}{2\sinh(e/2)}, [B]\right\rangle \pmod{2\mathbf{Z}}.$$

Proof. – The formula (7) follows from (1) by setting $E=\Xi(TK\oplus\mathbf{R}^2-\xi)$, where ξ is the complex line bunde over K associated to B. □

The following congruence formula for elliptic genera φ_q (*cf.* [6], [12]) and the Ochanine genus β_q [6] is a direct consequence of corollary 6.

COROLLARY 7 (Zhang [11], [12]). – *The following identity holds,*

$$(8)\quad \langle \varphi_q(TK), [K]\rangle \equiv \beta_q(B)+\left\langle \varphi_q(TB)\,\frac{\tanh(e/2)\,\varphi_q(e)-(e/2)}{e\tanh(e/2)}, [B]\right\rangle \pmod{2\mathbf{Z}[[q]]}.$$

Remark 8. – Recall that in [11], we use an analytic method of calculating the adiabatic limits of η-invariants of Dirac operators on circle bundles to prove corollary 6, while the proof of (8) in [12] is based on a cobordism theoretic method. Both two methods can be modified to prove theorem 1 immediately.

Remark 9. – In [11], we have also proved a congruence formula for the case where B is allowed to be non-orientable. This result has not received a purely topological proof.

By setting $E=\xi$ in (1), we get

COROLLARY 10. – *The following identity holds,*

$$(9)\quad \left\langle \hat{A}(TK)\exp\left(\frac{3c}{2}\right), [K]\right\rangle+\left\langle \hat{A}(TK)\exp\left(\frac{c}{2}\right), [K]\right\rangle \equiv \mathrm{ind}_2(N) \pmod{2\mathbf{Z}}.$$

Example 11. – Let $S^2=\mathbf{C}P^1 \subsetneq \mathbf{C}P^2$ be the canonical embedding, then N is the Hopf bundle H over S^2. By (9), one gets immediately that $\mathrm{ind}_2(H)=1$. Combining with the analytic approach mentioned in remark 8, this provides an explanation of [13], remark 2.4.

Remark 12. – By using theorem 1 for $E=\mathbf{R}$, one sees from (9) that $\mathrm{ind}_2(N)$ can be computed using the Atiyah invariants of some other characteristic submanifolds. This is particularly strange when B is a nonsingular spin complex hypersurface $V^d(4k+1)$ of $\mathbf{C}P^{4k+2}$ $(k>0)$, for which, in view of a result of Stolz [9], the fact that $\mathrm{ind}_2(N)$ is zero or nonzero relies on whether $V^{3d}(4k+1)$ and/or $V^d(4k+1)$ would or would not carry metrics of positive scalar curvature. We refer to [12] for a determination of whether a nonsingular spin complex hypersurface in $\mathbf{C}P^{4k+2}$ can carry a metric of positive scalar curvature.

We conclude this Note with the following generalization of example 11.

Corollary 13. — *Let* B *be a compact connected spin manifold of dimension* $8k+2$, *let* N *be a complex line bundle over* B. *Then the following identity holds,*

$$\text{(10)} \qquad \operatorname{ind}_2(\mathrm{N}) \equiv \langle \hat{\mathrm{A}}(\mathrm{TB})\,\mathrm{ch}(\mathrm{N}), [\mathrm{B}] \rangle \pmod{2\mathbf{Z}}.$$

Proof. — Clearly N can be extended through the circle bundle associated to itself to a spin^c-manifold K, such that (K, B) is a characteristic pair. The formula (10) then follows easily from (9). □

Remark 14. — It might be interesting to note that according to (10), $\operatorname{ind}_2(\mathrm{N})$ does not depend on the spin structure on B.

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Géométrie différentielle/*Differential Geometry*

Symplectic reduction and quantization

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Abstract. We present a direct analytic proof of the Guillemin-Sternberg geometric quantization conjecture [2]. Further extensions are also obtained.

Réduction symplectique et quantification

Résumé. *Nous présentons une preuve analytique d'une conjecture de Guillemin-Sternberg* [2], *ainsi que des extensions de ce résultat.*

Version française abrégée

Soit G un groupe de Lie compact connexe agissant sur une variété symplectique compacte (M, ω) par une action hamiltonienne. Soit (L, ∇^L) un fibré en droites hermitien muni d'une connexion hermitienne, supposé G-équivariant, et tel que $\nabla^{L,2} = \frac{2\pi}{\sqrt{-1}}\omega$. Soit $\mu : M \to g^*$ l'application moment associée. Soit J une structure presque complexe G-invariante sur TM, telle que $g^{TM}(u,v) = \omega(u, Jv)$ est une métrique riemannienne sur TM.

Soit $D^L : \Omega^{0,*}(M, L) \to \Omega^{0,*}(M, L)$ l'opérateur Spin^c de Dirac associé (*voir* [4]). Alors, on a une représentation virtuelle $RR(M, L)$ de G donnée par

$$RR(M, L) = \Omega^{0,\,\mathrm{pair}}(M, L) \cap \ker D^L - \Omega^{0,\,\mathrm{impair}}(M, L) \cap \ker D^L.$$

Supposons que $0 \in g^*$ soit une valeur régulière de μ, et que G agisse librement sur $\mu^{-1}(0)$. On note $M_G = \mu^{-1}(0)/G$ la réduction symplectique de Marsden-Weinstein. Le fibré $L_G = (L|_{\mu^{-1}(0)}/G)$ est un fibré hermitien en droites sur M_G. On obtient ainsi un espace virtuel $RR(M_G, L_G)$.

Conjecture (Guillemin-Sternberg, [2]) *On a*

$$\dim RR(M, L)^G = \dim RR(M_G, L_G).$$

Note présentée par Jean-Michel BISMUT.

Dans cette Note, nous présentons une preuve analytique de cette conjecture et on obtient également des extensions de ce résultat. Ainsi, si (M, ω) est kählérienne, on montre des inégalités de type Morse relative à la partie invariante de la cohomologie de L

In this Note, we present a direct analytic proof of the Guillemin-Sternberg geometric quantization conjecture [2]. Besides deriving an alternative proof of this conjecture in the full nonabelian group action case, our methods also lead to immediate generalizations in various contexts. Details and further applications will appear in [7].

1. The Guillemin-Sternberg conjecture

Let (M, ω) be a closed symplectic manifold such that there is a Hermitian line bundle L over M admitting a Hermitian connection ∇^L with the property that $\nabla^{L,2} = \frac{2\pi}{\sqrt{-1}}\omega$. Let J be an almost complex structure on TM so that $g^{TM}(u,v) = \omega(u, Jv)$ defines a riemannian metric on TM.

With these data, one can construct canonically a Spin^c-Dirac operator (*see* [4, Appendix D])

$$D^L : \Omega^{0,*}(M, L) \to \Omega^{0,*}(M, L), \tag{1.1}$$

which gives rise to the finite dimensional virtual vector space

$$RR(M, L) = \Omega^{0,\mathrm{even}}(M, L) \cap \ker D^L - \Omega^{0,\mathrm{odd}}(M, L) \cap \ker D^L. \tag{1.2}$$

Now suppose that a compact connected Lie group G acts on (M, ω) in a Hamiltonian way, which lifts to L naturally and preserves J, ∇^L, etc. Let $\mu : M \to \mathrm{g}^*$ be the corresponding moment map. We assume that $0 \in \mathrm{g}^*$ is a regular value of μ, and for simplicity, that G acts on $\mu^{-1}(0)$ freely. Then $M_G = \mu^{-1}(0)/G$ is a smooth manifold. On the other hand, ω descends to a symplectic form ω_G on M_G. Thus we get the Marsden-Weinstein symplectic reduction space (M_G, ω_G). The pair (L, ∇^L) also descends to a pair (L_G, ∇^{L_G}) over M_G. Then one defines the corresponding Spin^c-Dirac operator and in particular the virtual vector space $RR(M_G, L_G)$.

Since G preserves everything, it commutes with D^L. Thus $RR(M, L)$ is a virtual representation space of G. Denote by $RR(M, L)^G$ the G-trivial representation component of $RR(M, L)$.

THEOREM 1.1. – $\dim RR(M, L)^G = \dim RR(M_G, L_G)$.

Theorem 1.1 was first proved by Guillemin-Sternberg [2] in the holomorphic category when (M, g^{TM}) is Kähler. They raised it as a conjecture for general symplectic manifolds. When G is abelian, this conjecture was proved by Meinrenken [5] and Vergne ([8], [9]). A proof for the full nonabelian case was given by Meinrenken [6].

2. Quantized Witten deformation and its Laplacian

For any $X \in \Gamma(TM)$ with complexification $X = X_1 + X_2 \in \Gamma(T^{(1,0)}M \oplus T^{(0,1)}M)$, set $c(X) = \sqrt{2}\bar{X}_1^* \wedge - \sqrt{2} i_{X_2}$, where $\bar{X}_1^* \in \Gamma(T^{*(0,1)}M)$ is the metric dual of X_1 (*see* [1], Section 5). Then $c(X)$ extends to an action on $\Omega^{0,*}(M, L)$.

Let g and thus g* be equipped with an Ad G-invariant metric. Let $|\mu|^2$ be the norm square of the moment map. Let $Jd|\mu|^2 \in \Gamma(T^*M) \simeq \Gamma(TM)$ be the 1-form introduced by Witten [10].

DEFINITION 2.1. – For any $T \in \mathbf{R}$, the quantized Witten symplectic deformation operator D_T is the formally self-adjoint first order elliptic differential operator given by

$$D_T = D^L - \frac{\sqrt{-1}T}{2} c(Jd|\mu|^2) : \Omega^{0,*}(M, L) \to \Omega^{0,*}(M, L). \tag{2.1}$$

Remark 2.2. – If J is integrable, so that (M, g^{TM}) is Kähler, one has

$$D_T = \sqrt{2}\left(e^{-T|\mu|^2/2}\,\bar\partial^L\,e^{T|\mu|^2/2} + e^{T|\mu|^2/2}\,(\bar\partial^L)^*\,e^{-T|\mu|^2/2}\right). \tag{2.2}$$

Also, a similar deformation has been used by Vergne [9] on the symbol level.

Let $h_1, \ldots, h_{\dim G}$ be an orthonormal base of $\mathbf{g}^*$. Then μ has the expression $\mu = \sum_{i=1}^{\dim G} \mu_i h_i$, where each μ_i is a real function on M. Let V_i be the killing vector field on M induced by the dual of h_i. Using (2.1) and the Kostant formula [3] for the infinitesimal action of G on L, one obtains the following Bochner type formula.

THEOREM 2.3. – *The following identity holds,*

$$\begin{aligned} D_T^2 = D^{L,2} + \sqrt{-1}T \sum_{i=1}^{\dim G} c(d\mu_i)c(V_i) + 4\pi T|\mu|^2 + T^2\left|\sum_{i=1}^{\dim G}\mu_i V_i\right|^2 \\ + \sqrt{-1}T\sum_{i=1}^{\dim G}\mu_i\left(\frac{1}{2}\sum_{j=1}^{\dim M} c(e_j)c(\nabla_{e_j}V_i) - \operatorname{Tr}[\nabla^{(1,0)}V_i|_{T^{(1,0)}M}] - 2\hat L_{V_i}\right), \end{aligned} \tag{2.3}$$

where $\hat L_{V_i}$ denotes the infinitesimal action of V_i on $\Omega^{0,*}(M, L)$, and $\nabla^{(1,0)}$ denotes the connection on $T^{(1,0)}M$ induced from the Levi-Civita connection ∇ of g^{TM}.

3. Localization to neighbourhoods of $\mu^{-1}(0)$

In this section, we show that the proof of Theorem 1.1 can be localized to arbitrary small neighbourhoods of $\mu^{-1}(0)$. The main difficulty arises from the fact that the nonzero critical point set of $|\mu|^2$ may not be nondegenerate in the sense of Bott. We overcome this difficulty by doing pointwise estimates instead of global estimates used in the standard analytic Morse theory.

Let $\Omega_G^{0,*}(M, L)$ denote the G-invariant part of $\Omega^{0,*}(M, L)$.

THEOREM 3.1. – *For any open neighbourhood U of $\mu^{-1}(0)$, there exist constants $C > 0$, $b > 0$ such that for any $T \geq 1$ and any $s \in \Omega_G^{0,*}(M, L)$ with* $\operatorname{Supp} s \subset M\backslash U$,

$$\|D_T s\|_0^2 \geq C(\|s\|_1^2 + (T - b)\|s\|_0^2). \tag{3.1}$$

We prove Theorem 3.1 in two steps. The first step is to prove the following key pointwise estimate.

PROPOSITION 3.2. – *Let*

$$Q_T = D_T^2 + 2\sqrt{-1}T\sum_{i=1}^{\dim G}\mu_i \hat L_{V_i} \tag{3.2}$$

act on $\Omega^{0,*}(M, L)$. For any $x \in M\backslash U$, there exist an open neighborhood W of x and constants $C_x > 0$, $b_x > 0$ such that if $s \in \Omega^{0,*}(M, L)$ with $\operatorname{Supp} s \subset W$, *then for any $T \geq 1$,*

$$\langle Q_T s, s\rangle \geq C_x(\|s\|_1^2 + (T - b_x)\|s\|_0^2). \tag{3.3}$$

If x is not a critical point of $|\mu|^2$, the proof of (3.3) is trivial. We now assume that x is a *nonzero critical* point of $|\mu|^2$. Then one can find an orthonormal basis $f_1, \ldots, f_{\dim M}$ of $T_x M$ with the corresponding normal coordinates $y_1, \ldots, y_{\dim M}$ such that near x, $|\mu|^2$ can be written as

$$|\mu(y)|^2 = |\mu(x)|^2 + \sum_{j=1}^{\dim M} a_j y_j^2 + O(|y|^3), \tag{3.4}$$

where the constants a_j's may possibly be zero.

From (3.4), one can see directly that at x,

$$\sqrt{-1} \sum_{i=1}^{\dim G} c(d\mu_i)\, c(V_i) + \sqrt{-1}\, T \sum_{i=1}^{\dim G} \mu_i \left(\frac{1}{2} \sum_{j=1}^{\dim M} c(f_j)\, c(\nabla_{f_j} V_i) - \operatorname{Tr}[\nabla^{(1,0)} V_i] \right) \geq - \sum_{j=1}^{\dim M} |a_j|. \tag{3.5}$$

From (3.5), (3.4), (3.2) and (2.3), one gets (3.3).

The second step of the proof of Theorem 3.1 is to glue together the pointwise estimates in Proposition 3.2. The key point is that when restricted to $\Omega_G^{0,*}(M, L)$, one has $\hat{L}_{V_i} = 0$. Thus $D_T^2 = Q_T$ on $\Omega_G^{0,*}(M, L)$. On the other hand, since $M \backslash U$ is compact, finitely many glueing suffice.

4. The analysis near $\mu^{-1}(0)$ and a proof of Theorem 1.1

Theorem 3.1 allows us to reduce the proof of Theorem 1.1 to a sufficiently small open neighbourhood U of $\mu^{-1}(0)$. We take U to be equivariant.

Since $0 \in \mathbf{g}^*$ is a regular value of μ, $\mu^{-1}(0)$ is a nondegenerate critical submanifold of $|\mu|^2$ in the sense of Bott. One can then apply directly here methods and techniques of the paper of Bismut-Lebeau [1, Sections 8, 9] and localize everything at $\mu^{-1}(0)$. As G acts on $\mu^{-1}(0)$ freely, $G \to \mu^{-1}(0) \xrightarrow{\pi} M_G = M/G$ is a principal fibration. Furthermore, the vertical G-direction covariant derivatives are bounded operators when restricted to G-invariant subspaces. This eventually pushes everything down to M_G.

In summary, we get a self-adjoint Spin^c-Dirac type operator D_Q on M_G acting on $\Omega^{0,*}(M_G, L_G)$, having the properties given in Theorem 4.1. To our surprise, it turns out to be non identical to the Spin^c-Dirac operator D^{L_G}.

THEOREM 4.1. – *There exist $c > 0$, $T_0 > 0$ such that there are no nonzero eigenvalues of D_Q^2 in $[0, c]$, and such that for any $T \geq T_0$, the number of eigenvalues of $D_T^2|_{\Omega_G^{0,*}(M,L)}$ in $[0, c]$ is equal to* $\dim(\ker D_Q)$.

Now, all arguments used to prove Theorem 4.1 preserve the $\mathbf{Z}_2$-grading of the Spin^c-bundles. Theorem 1.1 then follows from Theorem 4.1 easily.

Remark 4.2. – For a precise form of D_Q in the holomorphic category, *see* (6.1).

Remark 4.3. – If G does not act freely on $\mu^{-1}(0)$, then M_G is an orbifold. In this case, the above arguments can be modified easily to prove the orbifold version of Theorem 1.1.

Remark 4.4. – Alternatively, one can first take the principal fibration $G \to U \to U/G$ and then apply [1] to U/G to prove Theorem 4.1.

5. Two immediate extensions

Arguments in Sections 2 to 4 also lead immediately to further extensions of Theorem 1.1. Here we only state two of them. The first is a dual version of Theorem 1.1.

THEOREM 5.1. – *The following identity holds:*

$$\dim RR(M,\, L^{-1} \otimes \det(T^{(0,1)}M))^G = (-1)^{\dim G} \dim RR(M_G,\, L_G^{-1} \otimes \det(T^{(0,1)}M_G)).$$

The second result can be viewed as an invariance property of symplectic quotients. It has also been obtained independently by Meinrenken and Sjamaar.

THEOREM 5.2. – *If $\mu^{-1}(0)$ is not empty, then we have the equality of Todd genus,* $\langle \mathrm{Td}(TM),\, [M] \rangle = \langle \mathrm{Td}(TM_G),\, [M_G] \rangle$.

6. Holomorphic Morse inequalities

We now assume that $(M,\, \omega)$ is Kähler and work in the holomorphic category. Then $(M_G,\, \omega_G)$ is also Kähler. The line bundle L (resp. L_G) is now holomorphic over M (resp. M_G).

Let $h : M_G \to \mathbf{R}_+$ be defined by $h(x) = \mathrm{vol}(G_x) = \mathrm{vol}(\pi^{-1}(x))$. Then the Dirac type operator D_Q in Section 4 can be written precisely here as

$$D_Q = \sqrt{2}\,(h^{1/2}\,\bar\partial^{L_G}\,h^{-1/2} + h^{-1/2}\,(\bar\partial^{L_G})^*\,h^{1/2}). \tag{6.1}$$

From (2.2), (6.1), and proceeding as in Sections 2 through 4, one actually gets a $\mathbf{Z}$-graded refined version of Theorem 4.1. This culminates in the following refinement of Theorem 1.1, which is stated for Dolbeault cohomologies, where we use the upperscript G to denote the G-invariant part.

THEOREM 6.1. – *The following Morse type inequalities hold:*

(i) *For any* $0 \le p \le \frac{\dim M}{2}$,

$$\dim H^{0,p}(M,\, L)^G \le \dim H^{0,p}(M_G,\, L_G);$$

(ii) *For any* $0 \le p \le \frac{\dim M}{2}$,

$$\sum_{i=0}^{p} (-1)^i \dim H^{0,p-i}(M,\, L)^G \le \sum_{i=0}^{p} (-1)^i \dim H^{0,p-i}(M_G,\, L_G).$$

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Géométrie différentielle/*Differential Geometry*

Rigidity and vanishing theorems in K-theory

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Abstract. In this Note we announce some new rigidity and vanishing results in the equivariant K-theory. These results generalize the famous Witten rigidity theorems. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Théorèmes de rigidité et d'annulation dans la K-théorie

Résumé. *Dans cette Note, nous annonçons des résultats de rigidité et d'annulation dans la K-théorie équivariante. Ces résultats étendent les théorèmes de rigidité de Witten dans le contexte de la K-théorie équivariante.* © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Soit X une variété compacte, orientée et de dimension paire. On suppose que X admet une action de S^1 et que X est munie d'une structure spinorielle S^1-invariante.

Soit $g^{\mathrm{T}X}$ une métrique S^1-invariante sur $\mathrm{T}X$. Soit $\mathrm{S}(\mathrm{T}X)=\mathrm{S}^+(\mathrm{T}X)\oplus\mathrm{S}^-(\mathrm{T}X)$ le fibré des spineurs $\mathbf{Z}_2$-gradués sur $(\mathrm{T}X, g^{\mathrm{T}X})$. Suivant Witten [9], on pose

$$\Theta'_q(\mathrm{T}X)=\bigotimes_{n=1}^{\infty}\Lambda_{q^n}(\mathrm{T}X\otimes_{\mathbf{R}}\mathbf{C})\bigotimes_{n=1}^{\infty}\mathrm{Sym}_{q^n}(\mathrm{T}X\otimes_{\mathbf{R}}\mathbf{C})=\sum_{n=0}^{+\infty}R_nq^n,$$

avec $R_n\in K(X)$.

Witten a conjecturé dans [9] que pour tout $n\in\mathbf{N}$, le nombre de Lefschetz $\mathrm{L}(g)_n$ de l'opérateur de Dirac twisté, qui envoit $\Gamma(\mathrm{S}^+(\mathrm{T}X)\otimes\mathrm{S}(\mathrm{T}X)\otimes R_n)$ dans $\Gamma(\mathrm{S}^-(\mathrm{T}X)\otimes\mathrm{S}(\mathrm{T}X)\otimes R_n)$, ne dépend pas $g\in\mathrm{S}^1$.

La conjecture de Witten a été démontrée par Taubes [8], Bott–Taubes [2] et Liu [5] etc.

Note présentée par Jean-Michel BISMUT.

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Dans [6], Liu et Ma ont étendu la conjecture de Witten à une situation en famille. Ils ont démontré des résultats de rigidité et d'annulation au niveau du caractère de Chern équivariant pour la famille d'opérateurs de Dirac twistés décrit ci-dessus.

Dans cette Note, nous annonçons des résultats de rigidité et d'annulation au niveau de la K-théorie équivariante, qui raffinent les résultats de Liu–Ma [6]. Les détails de la preuve et les extensions sont développés dans [7].

In this Note, we announce the proofs of the K-theory versions of the famous rigidity and vanishing theorems for elliptic genera. Details and further extensions will be developed in [7].

1. A family rigidity theorem for the Witten elements

For simplicity, we will focus on the discussion of the rigidity for one of the elliptic genera. For more general rigidity and vanishing results, we refer the reader to [7].

Let $\pi : M \to B$ be a smooth fibration of compact manifolds with fibre X and $\dim X = 2\ell$. Let $\mathrm{T}X$ be the vertical tangent bundle of the fibration $\pi : M \to B$. We make the assumption that S^1 acts fiberwise on M, and that $\mathrm{T}X$ admits an S^1-equivariant spin structure. Let $g^{\mathrm{T}X}$ be an S^1-invariant metric on $\mathrm{T}X$. Let $\mathrm{S}(\mathrm{T}X) = \mathrm{S}^+(\mathrm{T}X) \oplus \mathrm{S}^-(\mathrm{T}X)$ be the $\mathbf{Z}_2$-graded bundle of spinors of $(\mathrm{T}X, g^{\mathrm{T}X})$.

For a complex (resp. real) vector bundle E over M, let

$$\mathrm{Sym}_t(E) = 1 + tE + t^2 \mathrm{Sym}^2 E + \cdots,$$
$$\Lambda_t(E) = 1 + tE + t^2 \Lambda^2 E + \cdots$$

be the symmetric and exterior power operations of E (resp. $E \otimes_{\mathbf{R}} \mathbf{C}$) in $K(M)[[t]]$ respectively. Following Witten [9], set

$$\Theta'_q(\mathrm{T}X) = \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\mathrm{T}X) \bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(\mathrm{T}X) = \sum_{n=0}^{+\infty} R_n q^n, \tag{1.1}$$

with each $R_n \in K(M)$.

For any $n \in \mathbf{N}$, $b \in B$, let $D^X_b \otimes R_n$ denote the twisted *signature* operator on $X_b = \pi^{-1}(b)$ mapping from $\Gamma\big((\mathrm{S}^+(\mathrm{T}X) \otimes \mathrm{S}(\mathrm{T}X) \otimes R_n)|_{X_b}\big)$ to $\Gamma\big((\mathrm{S}^-(\mathrm{T}X) \otimes \mathrm{S}(\mathrm{T}X) \otimes R_n)|_{X_b}\big)$. Then $\{D^X_b \otimes R_n\}_{b\in B}$ is a smooth family of twisted signature operators which we denote by $D^X \otimes R_n$. The family operator $D^X \otimes R_n$ is clearly S^1-equivariant. Thus, its index bundle $\mathrm{Ind}(D^X \otimes R_n)$, in the sense of Atiyah and Singer [1], lies in $K_{\mathrm{S}^1}(B)$. Let $\big(\mathrm{Ind}(D^X \otimes R_n)\big)^{\mathrm{S}^1} \in K(B)$ denote the S^1-invariant part of $\mathrm{Ind}(D^X \otimes R_n)$. We say that $D^X \otimes R_n$ is *rigid on the equivariant K-theory level* if $\mathrm{Ind}(D^X \otimes R_n) = \big(\mathrm{Ind}(D^X \otimes R_n)\big)^{\mathrm{S}^1}$.

We can now state the main result of this Note as follows:

THEOREM 1.1. – *For any $n \in \mathbf{N}$, the family operator $D^X \otimes R_n$ is rigid on the equivariant K-theory level.*

Remark 1.2. – When B is a point, Theorem 1.1 was conjectured by Witten [9] and was proved by Taubes [8], Bott–Taubes [2] and Liu [5], etc. When B is not a point, Theorem 1.1 refines a result of Liu–Ma [6] to the equivariant K-theory level.

In order to outline a proof of Theorem 1.1, we will first state in the next section a K-theory version of the equivariant family index theorem for the considered operators.

2. An equivariant family index theorem for circle actions

Let F be the fixed point set of the S^1-action on M. Then $\pi : F \to B$ is a fibration with compact fibre denoted by Y. One has the following splitting of $\mathrm{T}X$ over F,

$$\mathrm{T}X|_F = \mathrm{T}Y \bigoplus_{v\neq 0} N_{v,\mathbf{R}}, \tag{2.1}$$

where $N_{v,\mathbf{R}}$ denotes the underlying real bundle of the complex vector bundle N_v on which S^1 acts by sending g to g^v. Since we can choose either N_v or $\overline{N}_v$ as the complex vector bundle for $N_{v,\mathbf{R}}$, in what follows we may and we will assume that

$$\mathrm{T}X|_F = \mathrm{T}Y \bigoplus_{0<v} N_v, \tag{2.2}$$

where N_v is the complex vector bundle on which S^1 acts by sending g to g^v (here N_v can be zero).

Let $\mathrm{T}Y$ carry the orientation induced from those of $\mathrm{T}X$ and the N_v's via (2.2). Let D^Y be the family signature operator along the fibers Y. If E is an S^1-equivariant Hermitian vector bundle over F carrying with an S^1-invariant Hermitian connection, we denote by $D^Y \otimes E$ the associated family twisted signature operator. Then the index bundle of $D^Y \otimes E$ lies in $K_{\mathrm{S}^1}(B)$. For any $h \in \mathbf{Z}$, let $\mathrm{Ind}(D^Y \otimes E, h)$ denote the component of $\mathrm{Ind}(D^Y \otimes E)$ of weight h with respect to the induced S^1-representation. In what follows, if $R(q) = \sum_{m\in\mathbf{Z}} R_m q^m \in K_{\mathrm{S}^1}(M)[[q]]$, we will also denote $\mathrm{Ind}(D^X \otimes R_m, h)$ by $\mathrm{Ind}(D^X \otimes R(q), m, h)$.

The main result of this section can be stated as follows:

THEOREM 2.1. – *For $m,\ h \in \mathbf{Z}$, we have the following identity in $K(B)$,*

$$\begin{aligned}&\mathrm{Ind}\big(D^X \otimes \Theta'_q(\mathrm{T}X), m, h\big)\\ &\quad = \sum_\alpha (-1)^{\Sigma_{0<v}\dim N_v} \mathrm{Ind}\Big(D^{Y_\alpha} \otimes \Theta'_q(\mathrm{T}X) \otimes \mathrm{Sym}\Big(\bigoplus_{0<v} N_v\Big) \otimes \Lambda\Big(\bigoplus_{0<v} N_v\Big), m, h\Big),\end{aligned} \tag{2.3}$$

where α runs over the connected components of F.

Proof. – Theorem 2.1 is proved in [7] by using the analytic arguments in [10] and [11]. □

3. Proof of Theorem 1.1

For $p \in \mathbf{N}$, we define the following elements in $K_{\mathrm{S}^1}(F)[[q]]$:

$$\begin{aligned}\mathcal{F}_p(X) &= \bigotimes_{0<v}\Big(\bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(N_v) \bigotimes_{n>pv} \mathrm{Sym}_{q^n}(\overline{N}_v)\Big) \bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(\mathrm{T}Y),\\ \mathcal{F}'_p(X) &= \bigotimes_{\substack{0<v\\ 0\leqslant n\leqslant pv}} \big(\mathrm{Sym}_{q^{-n}}(N_v) \otimes \det N_v\big),\end{aligned}$$

$$\mathcal{F}^{-p}(X) = \mathcal{F}_p(X) \otimes \mathcal{F}'_p(X) \otimes \Lambda\Big(\bigoplus_{0<v} N_v\Big) \otimes \Big(\det\Big(\bigoplus_{0<v} N_v\Big)\Big)^{-1} \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(\mathrm{T}X). \tag{3.1}$$

Then

$$\mathcal{F}^0(X) = \Theta'_q(\mathrm{T}X) \otimes \mathrm{Sym}\Big(\bigoplus_{0<v} N_v\Big) \otimes \Lambda\Big(\bigoplus_{0<v} N_v\Big).$$

Set

$$e(N)=\sum_{0<v} v^2 \dim N_v, \qquad d'(N)=\sum_{0<v} v \dim N_v. \tag{3.2}$$

We now state two intermediate results on the relations between the family indices on the fixed point set.

PROPOSITION 3.1. – *For $h,\ p,\ m\in\mathbf{Z}$, $p>0$, we have the following identity in $K(B)$,*

$$\begin{aligned}&\sum_\alpha(-1)^{\Sigma_{0<v}\dim N_v}\operatorname{Ind}\left(D^{Y_\alpha}\otimes\Theta_q'(\mathrm{T}X)\otimes\operatorname{Sym}\left(\bigoplus_{0<v}N_v\right)\otimes\Lambda\left(\bigoplus_{0<v}N_v\right),m,h\right)\\&\quad=\sum_\alpha(-1)^{\Sigma_{0<v}\dim N_v}\operatorname{Ind}\left(D^{Y_\alpha}\otimes\mathcal{F}^{-p}(X),m+\frac{1}{2}p^2e(N)+\frac{1}{2}pd'(N),h\right).\end{aligned}\tag{3.3}$$

PROPOSITION 3.2. – *For $h,\ p\in\mathbf{Z}$, $p>0$, $m\in\mathbf{Z}$, on each connected component F_α of F, we have the following identity in $K(B)$,*

$$\operatorname{Ind}\left(D^{Y_\alpha}\otimes\mathcal{F}^{-p}(X),m+\frac{1}{2}p^2e(N)+\frac{1}{2}pd'(N),h\right)=\operatorname{Ind}(D^{Y_\alpha}\otimes\mathcal{F}^{0}(X),m+ph,h).$$

Propositions 3.1 and 3.2 are proved in [7], where, inspired by Taubes [8], we introduce certain shifting operations for vector bundles over F and study the behaviour of the involved family indices under the shifting operations. Moreover, in the proof of Proposition 3.1, we make use a key idea in [8] to reduce the problem to the fixed point set of the induced $\mathbf{Z}_n$-actions. For more details, *see* [7].

Proof of Theorem 1.1. – By (3.1), Theorem 2.1 and Propositions 3.1, 3.2, for $p\in\mathbf{Z}$, $p>0$, we get the following identity in $K(B)$,

$$\operatorname{Ind}\big(D^X\otimes\Theta_q'(\mathrm{T}X),m,h\big)=\operatorname{Ind}(D^X\otimes\Theta_q'(\mathrm{T}X),m',h),\tag{3.4}$$

with

$$m'=m+ph.\tag{3.5}$$

Note that by (1.1), if $m<0$, for $h\in\mathbf{Z}$, we have

$$\operatorname{Ind}\big(D^X\otimes\Theta_q'(\mathrm{T}X),m,h\big)=0\quad\text{in }K(B).\tag{3.6}$$

Let $m_0,\ h\in\mathbf{Z}$ with $h\neq 0$ be fixed:
(i) if $h>0$, we take $m'=m_0$, then when p is big enough, we get $m<0$;
(ii) if $h<0$, we take $m=m_0$, then as p is big enough we get $m'<0$.
From (3.4), (3.6) and the above discussion, we get Theorem 1.1. □

4. Vanishing results and further remarks

In some sense, our proof given in [7] may be considered as a K-theory version of the proof given by Bott–Taubes [2] of the Witten rigidity theorem, which was also inspired by the ideas in Taubes' proof [8]. While on the other hand, the proof in [7] is self-contained and the arguments in [7], even in the case where the base B is a point, are different from the ones in the papers of Bott–Taubes [2], Liu [5] and Taubes [8]. Moreover, our method in [7] is quite general and allows us to deal with systematically more general situations than what was described in this note. We refer to [7] for more results and discussions. Here, for the conclusion of this Note, we only state one of the vanishing results, which follows from our techniques together with an observation of Dessai [3].

THEOREM 4.1. – *Assume that M is connected and that $\frac{1}{2}p_1(\mathrm{T}X) = 0$, where $p_1(\mathrm{T}X)$ is the first Pontryagin class of $\mathrm{T}X$. If the S^1-action on M is non-trivial, and is induced from a fiberwise S^3-action on M which also preserves the spin structure on $\mathrm{T}X$, then the index bundle of the family twisted* Dirac *operator $\mathcal{D}^X \bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(\mathrm{T}X)$ is identically zero in $K_{\mathrm{S}^1}(B)$.*

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Differential Geometry

Spin^c-manifolds and elliptic genera

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Presented by Jean-Michel Bismut

Abstract

We present an extension of the "miraculous cancellation" formulas of Alvarez-Gaumé, Witten and Kefeng Liu to a twisted version where an extra complex line bundle is involved. Relations to the Ochanine congruence formula on $8k+4$ dimensional Spin^c manifolds are discussed. ***To cite this article: F. Han, W. Zhang, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Variétés Spin^c et genre elliptique. Nous présentons une extension de formules d'annulation d'Alvarez-Gaumé, Witten et Liu lorsqu'on tensorise les fibrés considérés par un fibré en droites complexe. On discute le lien entre nos formules et les formules de congruence d'Ochanine pour les variétés Spin^c de dimension $8k+4$. ***Pour citer cet article : F. Han, W. Zhang, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

1. Introduction

Let M be a Riemannian manifold. Let ∇^{TM} be the associated Levi-Civita connection, and let $R^{TM}=\nabla^{TM,2}$ be the curvature of ∇^{TM}. Then ∇^{TM} extends canonically to a Hermitian connection $\nabla^{T_{\mathbf{C}}M}$ on $T_{\mathbf{C}}M=TM\otimes\mathbf{C}$.

Let $\widehat{A}(TM,\nabla^{TM})$, $\widehat{L}(TM,\nabla^{TM})$ be the Hirzebruch characteristic forms defined by

$$\widehat{A}\big(TM,\nabla^{TM}\big)=\det{}^{1/2}\left(\frac{(\sqrt{-1}/4\pi)R^{TM}}{\sinh((\sqrt{-1}/4\pi)R^{TM})}\right),\qquad \widehat{L}(TM,\nabla^{TM})=\det{}^{1/2}\left(\frac{(\sqrt{-1}/2\pi)R^{TM}}{\tanh((\sqrt{-1}/4\pi)R^{TM})}\right),\quad (1)$$

and let $\mathrm{ch}(T_{\mathbf{C}}M,\nabla^{T_{\mathbf{C}}M})$ denote the Chern character form associated to $(T_{\mathbf{C}}M,\nabla^{T_{\mathbf{C}}M})$ (cf. [9, Section 1.6]).

When $\dim M=12$, the following equation for 12-forms was proved by Alvarez-Gaumé and Witten in [1], which they called "miraculous cancellation",

$$\big\{\widehat{L}\big(TM,\nabla^{TM}\big)\big\}^{(12)}=\big\{8\widehat{A}\big(TM,\nabla^{TM}\big)\,\mathrm{ch}\big(T_{\mathbf{C}}M,\nabla^{T_{\mathbf{C}}M}\big)-32\widehat{A}\big(TM,\nabla^{TM}\big)\big\}^{(12)}. \quad (2)$$

These authors also discussed the applications of such formulas to physics.

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In [5], Kefeng Liu generalized (2) to arbitrarily $8k+4$ dimensional manifolds by developing modular invariance properties of characteristic numbers.

In this Note, we present an extension of Liu's formula in the presence of an extra complex line bundle (or equivalently, a rank two real oriented vector bundle). In dimension 12, this extension can be described as follows: let ξ be a rank two real oriented Euclidean vector bundle, equipped with a Euclidean connection ∇^{ξ}, let $c = e(\xi, \nabla^{\xi})$ be the associated Euler form (cf. [9, Section 3.4]). Then the following equation for 12-forms holds,

$$\left\{\frac{\hat{L}(TM, \nabla^{TM})}{\cosh^2(c/2)}\right\}^{(12)} = \Big\{\big[8\hat{A}\big(TM, \nabla^{TM}\big)\,\mathrm{ch}\big(T_{\mathbf{C}}M, \nabla^{T_{\mathbf{C}}M}\big) - 32\hat{A}\big(TM, \nabla^{TM}\big) - 24\hat{A}\big(TM, \nabla^{TM}\big)\big(\mathrm{e}^{c} + \mathrm{e}^{-c} - 2\big)\big]\cosh\left(\frac{c}{2}\right)\Big\}^{(12)}. \tag{3}$$

Clearly, when ξ is trivial and $c = 0$, (3) reduces to the formula (2) of Alvarez-Gaumé and Witten. Our work was motivated by the Ochanine congruence formula [7].

2. Main results

Let M be a $8k+4$ dimensional Riemannian manifold. Let ∇^{TM} be the associated Levi-Civita connection. Let V be a rank $2l$ real Euclidean vector bundle over M equipped with a Euclidean connection ∇^{V}. Let ξ be a rank two real oriented Euclidean vector bundle over M carrying with a Euclidean connection ∇^{ξ}. Let $c = e(\xi, \nabla^{\xi})$ be the Euler form of ξ canonically associated to ∇^{ξ}.

Set $V_{\mathbf{C}} = V \otimes \mathbf{C}$ and $\xi_{\mathbf{C}} = \xi \otimes \mathbf{C}$. Then $V_{\mathbf{C}}$ and $\xi_{\mathbf{C}}$ are complex vector bundles over M, each of which is equipped with a Hermitian metric, and a unitary connection.

If W is a Hermitian vector bundle over M equipped with a Hermitian connection ∇^{W}, we denote by $\mathrm{ch}(W, \nabla^{W})$ the associated Chern character form (cf. [9, Section 1.6]). Also, for any complex number t, set $\Lambda_t(W) = \mathbf{C}|_M + tW + t^2\Lambda^2(W) + \cdots$ and $S_t(W) = \mathbf{C}|_M + tW + t^2S^2(W) + \cdots$, where for any integer $j \geqslant 1$, $\Lambda^j(W)$ (resp. $S^j(W)$) is the j-th exterior (resp. symmetric) power of W. Set $\widetilde{W} = W - \mathbf{C}^{\mathrm{rk}(W)}$.

Let $q = \mathrm{e}^{2\pi\sqrt{-1}\tau}$ with $\tau \in \mathbf{H}$, the upper half plane. Set

$$\Theta_1(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}) = \bigotimes_{n=1}^{\infty} S_{q^n}\big(\widetilde{T_{\mathbf{C}}M}\big) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}\big(\widetilde{V}_{\mathbf{C}} - 2\tilde{\xi}_{\mathbf{C}}\big) \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-1/2}}(\tilde{\xi}_{\mathbf{C}}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{-q^{s-1/2}}\big(\tilde{\xi}_{\mathbf{C}}\big), \tag{4}$$

$$\Theta_2(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}) = \bigotimes_{n=1}^{\infty} S_{q^n}\big(\widetilde{T_{\mathbf{C}}M}\big) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{-q^{m-1/2}}\big(\widetilde{V}_{\mathbf{C}} - 2\tilde{\xi}_{\mathbf{C}}\big) \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-1/2}}\big(\tilde{\xi}_{\mathbf{C}}\big) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{q^s}\big(\tilde{\xi}_{\mathbf{C}}\big). \tag{5}$$

Clearly, $\Theta_1(M, V_{\mathbf{C}}, \xi_{\mathbf{C}})$ and $\Theta_2(M, V_{\mathbf{C}}, \xi_{\mathbf{C}})$ admit formal Fourier expansion in $q^{1/2}$ as

$$\Theta_1(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}) = A_0(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}) + A_1(M, V_{\mathbf{C}}, \xi_{\mathbf{C}})q^{1/2} + \cdots, \tag{6}$$

$$\Theta_2(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}) = B_0(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}) + B_1(M, V_{\mathbf{C}}, \xi_{\mathbf{C}})q^{1/2} + \cdots, \tag{7}$$

where the A_j's and B_j's are elements in the semi-group generated by Hermitian vector bundles over M. These vector bundles A_j, B_j are naturally equipped with Hermitian metrics and unitary connections ∇^{A_j}, ∇^{B_j}.

Let $R^V = \nabla^{V,2}$ denote the curvature of ∇^V. We can now state our main result as follows.

Theorem 2.1. *If the equality for the first Pontryagin forms $p_1(TM, \nabla^{TM}) = p_1(V, \nabla^V)$ holds, then one has the equation for $(8k+4)$-forms,*

$$\left\{\frac{\hat{A}(TM, \nabla^{TM})\,\det^{1/2}(2\cosh((\sqrt{-1}/4\pi)R^V))}{\cosh^2(c/2)}\right\}^{(8k+4)} = 2^{l+2k+1}\sum_{r=0}^{k} 2^{-6r}\left\{b_r\cosh\left(\frac{c}{2}\right)\right\}^{(8k+4)}, \tag{8}$$

where each b_r, $0 \leqslant r \leqslant k$, *is a finite canonical integral linear combination of the characteristic forms* $\hat{A}(TM,\nabla^{TM})\,\mathrm{ch}(B_j,\nabla^{B_j})$, $j \geqslant 0$.

When $\xi = \mathbf{R}^2$ and $c = 0$, Theorem 2.1 is exactly Liu's result in [5, Theorem 1].

If we take $V = TM$ and $\nabla^V = \nabla^{TM}$ in (8), we get

$$\left\{\frac{\hat{L}(TM,\nabla^{TM})}{\cosh^2(c/2)}\right\}^{(8k+4)} = 8\sum_{r=0}^{k} 2^{6k-6r}\left\{b_r\cosh\left(\frac{c}{2}\right)\right\}^{(8k+4)}. \tag{9}$$

In the case where $k = 1$, one obtains (3) from (9).

Now assume that M is closed, oriented and carries a Spin^c structure with $[c] \equiv w_2(TM) \bmod 2$, where $w_2(TM)$ is the second Stiefel–Whitney class of TM. Let B be a connected closed oriented $8k+2$ submanifold in M such that $[B] \in H_{8k+2}(M,\mathbf{Z})$ is Poincaré dual to $[c]$. Let $B\cdot B$ be the self-intersection of B in M which can be thought of as a closed oriented $8k$ manifold. Then by [7], we know that

$$\int_M \frac{\hat{L}(TM,\nabla^{TM})}{\cosh^2(c/2)} = \mathrm{Sign}(M) - \mathrm{Sign}(B\cdot B). \tag{10}$$

On the other hand, by [2], we know that each $\int_M b_r\cosh(\frac{c}{2})$, $0 \leqslant r \leqslant k$, is an integer. Combining this argument with (9) and (10), we deduce that

$$\frac{\mathrm{Sign}(M) - \mathrm{Sign}(B\cdot B)}{8} \equiv \int_M b_k\cosh\left(\frac{c}{2}\right) \mod 64. \tag{11}$$

By combining (11) with the Rokhlin type congruences proved in [8], one can give a direct proof of the analytic version of the Ochanine congruence formula [7] stated in [6]. Moreover, if M is spin, then using (11), the Ochanine divisibility result (cf. [7]) and [2], we get

$$\mathrm{Sign}(B\cdot B) \equiv 0 \bmod 8, \tag{12}$$

a result which seems to be of interest by itself.

On the other hand, there are twisted cancellation formulas similar to (8), (9) on $8k$ manifolds, generalizing the (untwisted) cancellation formulas stated in [5, p. 32].

More details and further applications will be given in [4].

3. Proof of Theorem 2.1

The methods of [5, Section 3] can be adapted here, with obvious modifications which take into account the presence of ξ and c. Here, we only indicate the main steps of the proof.

First, since (8) is a local assertion, we may and we will assume that both TM and V are oriented. As in [5], we use the notation of formal Chern roots $\{\pm 2\pi\sqrt{-1}\,y_v\}$ and $\{\pm 2\pi\sqrt{-1}\,x_j\}$ for $(V_{\mathbf{C}},\nabla^{\mathbf{C}})$ and $(T_{\mathbf{C}}M,\nabla^{T_{\mathbf{C}}M})$ respectively. We also set $c = 2\pi\sqrt{-1}\,u$.

For $\tau \in \mathbf{H}$ and $q = \mathrm{e}^{2\pi\sqrt{-1}\tau}$, set

$$P_1(\tau) = \left\{\frac{\hat{A}(TM,\nabla^{TM})\det^{1/2}(2\cosh((\sqrt{-1}/4\pi)R^V))}{\cosh^2(c/2)}\,\mathrm{ch}\big(\Theta_1(M,V_{\mathbf{C}},\xi_{\mathbf{C}}),\nabla^{\Theta_1(M,V_{\mathbf{C}},\xi_{\mathbf{C}})}\big)\right\}^{(8k+4)}, \tag{13}$$

$$P_2(\tau) = \left\{\hat{A}\big(TM,\nabla^{TM}\big)\,\mathrm{ch}\big(\Theta_2(M,V_{\mathbf{C}},\xi_{\mathbf{C}}),\nabla^{\Theta_2(M,V_{\mathbf{C}},\xi_{\mathbf{C}})}\big)\cosh\left(\frac{c}{2}\right)\right\}^{(8k+4)}, \tag{14}$$

where $\nabla^{\Theta_i(M,V_{\mathbf{C}},\xi_{\mathbf{C}})}$, $i = 1,2$, are the induced Hermitian connections with $q^{1/2}$-coefficients on $\Theta_i(M,V_{\mathbf{C}},\xi_{\mathbf{C}})$ one gets from the ∇^{A_j}, ∇^{B_j}'s (compare with (6) and (7)). Then a direct computation shows that

$$P_1(\tau)=2^l\left\{\prod_{j=1}^{4k+2}\left(x_j\frac{\theta'(0,\tau)}{\theta(x_j,\tau)}\right)\left(\prod_{v=1}^{l}\frac{\theta_1(y_v,\tau)}{\theta_1(0,\tau)}\right)\frac{\theta_1^2(0,\tau)}{\theta_1^2(u,\tau)}\frac{\theta_3(u,\tau)}{\theta_3(0,\tau)}\frac{\theta_2(u,\tau)}{\theta_2(0,\tau)}\right\}^{(8k+4)}, \tag{15}$$

$$P_2(\tau)=\left\{\prod_{j=1}^{4k+2}\left(x_j\frac{\theta'(0,\tau)}{\theta(x_j,\tau)}\right)\left(\prod_{v=1}^{l}\frac{\theta_2(y_v,\tau)}{\theta_2(0,\tau)}\right)\frac{\theta_2^2(0,\tau)}{\theta_2^2(u,\tau)}\frac{\theta_3(u,\tau)}{\theta_3(0,\tau)}\frac{\theta_1(u,\tau)}{\theta_1(0,\tau)}\right\}^{(8k+4)}, \tag{16}$$

where $\theta(z,\tau)$ and $\theta_i(z,\tau),\ i=1,2,3$, are the classical Jacobi theta functions (cf. [3]).

Since $p_1(TM,\nabla^{TM})=p_1(V,\nabla^V)$, i.e., $\sum_{j=1}^{4k+2}x_j^2=\sum_{v=1}^{l}y_v^2$, by (15), (16) and by the transformation laws of theta functions (cf. [3]), one verifies directly that $P_2(\tau)$ is a modular form of weight $4k+2$ over $\Gamma^0(2)$. Moreover,

$$P_1\left(-\frac{1}{\tau}\right)=2^l\tau^{4k+2}P_2(\tau). \tag{17}$$

On the other hand, following [5], write $\theta_j=\theta_j(0,\tau),\ 1\leqslant j\leqslant 3$, and set $\delta_1(\tau)=\frac{1}{8}(\theta_2^4+\theta_3^4)$, $\varepsilon_1(\tau)=\frac{1}{16}\theta_2^4\theta_3^4$, $\delta_2(\tau)=-\frac{1}{8}(\theta_1^4+\theta_3^4)$ and $\varepsilon_2(\tau)=\frac{1}{16}\theta_1^4\theta_3^4$. They admit Fourier expansion

$$\delta_1(\tau)=\frac{1}{4}+6q+\cdots,\qquad \varepsilon_1(\tau)=\frac{1}{16}-q+\cdots, \tag{18}$$

$$\delta_2(\tau)=-\frac{1}{8}-3q^{1/2}+\cdots,\qquad \varepsilon_2(\tau)=q^{1/2}+\cdots, \tag{19}$$

where the "$\cdots$" terms are higher degree terms all having integral coefficients.

By [5, Lemma 2], we know that δ_2 (resp. ε_2) is a modular form of weight 2 (resp. 4) over $\Gamma^0(2)$, and that $8\delta_2,\ \varepsilon_2$ generate the ring of modular forms with integral coefficients over $\Gamma^0(2)$. Combining this argument with (7), (14) and (19), we obtain

$$P_2(\tau)=h_0(8\delta_2)^{2k+1}+h_1(8\delta_2)^{2k-1}\varepsilon_2+\cdots+h_k(8\delta_2)\varepsilon_2^k, \tag{20}$$

where each $h_r,\ 0\leqslant r\leqslant k$, is a canonically defined finite integral linear combination of the forms $\{\hat{A}(TM,\nabla^{TM})\operatorname{ch}(B_j,\nabla^{B_j})\cosh(\frac{c}{2})\}^{(8k+4)},\ j\geqslant 0$. For example, h_0 and h_1 can be written explicitly as $h_0=-\{\hat{A}(TM,\nabla^{TM})\cosh(\frac{c}{2})\}^{(8k+4)}$ and $h_1=\{\hat{A}(TM,\nabla^{TM})[24(2k+1)-\operatorname{ch}(B_1,\nabla^{B_1})]\cosh(\frac{c}{2})\}^{(8k+4)}$.

Now recall that by [5, p. 36], $\delta_i,\ \varepsilon_i, i=1,2$, verify the transformation laws $\delta_2(-\frac{1}{\tau})=\tau^2\delta_1(\tau),\ \varepsilon_2(-\frac{1}{\tau})=\tau^4\varepsilon_1(\tau)$. Using also (17) and (20), we find that

$$P_1(\tau)=2^l\left[h_0(8\delta_1)^{2k+1}+h_1(8\delta_1)^{2k-1}\varepsilon_1+\cdots+h_k(8\delta_1)\varepsilon_1^k\right]. \tag{21}$$

By (6), (13), (18), (21), and taking $q=0$, we get (8). □

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COMPTES RENDUS MATHEMATIQUE

Differential Geometry

Circle actions and $\mathbf{Z}/k$-manifolds

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Abstract

We establish an S^1-equivariant index theorem for Dirac operators on $\mathbf{Z}/k$-manifolds. As an application, we generalize the Atiyah–Hirzebruch vanishing theorem for S^1-actions on closed spin manifolds to the case of $\mathbf{Z}/k$-manifolds. ***To cite this article: W. Zhang, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Actions du cercle et $\mathbf{Z}/k$ variétés. On établit un théorème d'indice S^1-équivariant pour les opérateurs de Dirac sur des $\mathbf{Z}/k$ variétés. On donne une application de ce résultat, qui généralise le théorème d'Atiyah–Hirzebruch sur les actions de S^1 aux $\mathbf{Z}/k$ variétés. ***Pour citer cet article : W. Zhang, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

1. S^1-actions and the vanishing theorem

Let X be a closed connected smooth spin manifold admitting a non-trivial circle action. A classical theorem of Atiyah and Hirzebruch [1] states that $\hat{A}(X) = 0$, where $\hat{A}(X)$ is the Hirzebruch $\hat{A}$-genus of X. In this Note we present an extension of the above result to the case of $\mathbf{Z}/k$-manifolds, which were introduced by Sullivan in his studies of geometric topology. We recall the basic definition for completeness (cf. [6]).

Definition 1.1. A compact connected $\mathbf{Z}/k$-manifold is a compact manifold X with boundary ∂X, which admits a decomposition $\partial X = \bigcup_{i=1}^{k} (\partial X)_i$ into k disjoint manifolds and k diffeomorphisms $\pi_i : (\partial X)_i \to Y$ to a closed manifold Y.

Let $\pi : \partial X \to Y$ be the induced map. In what follows, we will call an object α (e.g., metrics, connections, etc.) of X a $\mathbf{Z}/k$-object if there will be a corresponding object β on Y such that $\alpha|_{\partial X} = \pi^*\beta$. We make the assumption that X is $\mathbf{Z}/k$ oriented, $\mathbf{Z}/k$ spin and is of even dimension.

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Let g^{TX} be a $\mathbf{Z}/k$ Riemannian metric of X which is of product structure near ∂X. Let R^{TX} be the curvature of the Levi-Civita connection associated to g^{TX}. Let E be a $\mathbf{Z}/k$ complex vector bundle over X. Let g^E be a $\mathbf{Z}/k$ Hermitian metric on E which is a product metric near ∂X. Let ∇^E be a $\mathbf{Z}/k$ connection on E preserving g^E such that ∇^E is of product structure near ∂X. Let R^E be the curvature of ∇^E. Let $D_+^E : \Gamma(S_+(TX)\otimes E) \to \Gamma(S_-(TX)\otimes E)$ be the associated Dirac operator on X and $D^E_{+,\partial X}$ (and then D^E_Y) be its induced Dirac operator on ∂X (and then on Y). Let $\bar\eta(D^E_Y)$ be the reduced η-invariant of D^E_Y in the sense of [2]. Then

$$\hat{A}_{(k)}(X,E) = \int_X \det{}^{1/2}\left(\frac{\sqrt{-1}R^{TX}/4\pi}{\sinh(\sqrt{-1}R^{TX}/4\pi)}\right)\operatorname{tr}\left[\mathrm{e}^{(\sqrt{-1}/2\pi)R^E}\right] - k\bar\eta\left(D^E_Y\right) \mod k\mathbf{Z} \tag{1}$$

does not depend on (g^{TX}, g^E, ∇^E) and determines a topological invariant in $\mathbf{Z}/k\mathbf{Z}$ (cf. [2] and [6]). Moreover, Freed and Melrose [7] have proved a mod k index theorem, giving $\hat{A}_{(k)}(X,E)\in\mathbf{Z}/k\mathbf{Z}$ a purely topological interpretation. When $E=\mathbf{C}$ is the trivial vector bundle over X, we usually omit the superscript E.

Theorem 1.2. *If X admits a nontrivial $\mathbf{Z}/k$ circle action preserving the orientation and the Spin structure on TX, then $\hat{A}_{(k)}(X)=0$. Moreover, the equivariant mod k index in the sense of Freed and Melrose vanishes.*

It turns out that the original method in [1] is difficult to extend to the case of manifolds with boundary to prove Theorem 1.2. Thus we will instead make use of an extension of the method of Witten [10]. Analytic localization techniques developed by Bismut and Lebeau [3, Section 9] and their extensions to manifolds with boundary developed in [5] play important roles in our proof.

2. A mod k localization formula for circle actions

We make the assumption that the $\mathbf{Z}/k$ circle action on X lifts to a $\mathbf{Z}/k$ circle action on E. Without loss of generality, we may and we will assume that this $\mathbf{Z}/k$ circle action preserves g^{TX}, g^E and ∇^E. Let $D^E_{+,APS} : \Gamma(S_+(TX)\otimes E)\to\Gamma(S_-(TX)\otimes E)$ be the elliptic operator obtained by imposing the standard Atiyah–Patodi–Singer boundary condition [2] on D^E_+.

Let H be the Killing vector field on X generated by the S^1 action on X. Then $H|_{\partial X}\subset\partial X$ induces a Killing vector field H_Y on Y. Let $\mathcal{L}_H$ denote the corresponding Lie derivative acting on $\Gamma(S_\pm(TX)\otimes E)$. Then $\mathcal{L}_H$ commutes with $D^E_{+,APS}$.

For any $n\in\mathbf{Z}$, let $F^n_\pm$ be the eigenspaces of $\Gamma(S_\pm(TX)\otimes E)$ with respect to the eigenvalue $2\pi n$ of $\frac{1}{\sqrt{-1}}\mathcal{L}_H$. Let $D^E_{+,APS}(n) : F^n_+\to F^n_-$ be the restriction of $D^E_{+,APS}$ on F^n_+. Then $D^E_{+,APS}(n)$ is Fredholm. We denote its index by $\operatorname{ind}(D^E_{+,APS}(n))\in\mathbf{Z}$.

Let X_H (resp. Y_H) be the zero set of H (resp. H_Y) on X (resp. Y). Then X_H is a $\mathbf{Z}/k$-manifold and there is a canonical map $\pi_{X_H} : \partial X_H\to Y_H$ induced from π. We fix a connected component $X_{H,\alpha}$ of X_H, and we omit the subscript α if there is no confusion.

We identify the normal bundle to X_H in X to the orthogonal complement of TX_H in $TX|_{X_H}$. Then $TX|_{X_H}$ admits an S^1-invariant orthogonal decomposition $TX|_{X_H} = N_{m_1}\oplus\cdots\oplus N_{m_l}\oplus TX_H$, where each N_γ, $\gamma\in\mathbf{Z}$, is a complex vector bundle on which $g\in S^1\subset\mathbf{C}$ acts by multiplication by g^γ. By using the same notation as in [8, (1.8)], we simply write that $TX|_{X_H} = \bigoplus_{v\neq 0} N_v\oplus TX_H$. Similarly, let $E|_{X_H}$ admits the S^1-invariant decomposition $E|_{X_H}=\bigoplus_v E_v$.

Let $S(TX_H, (\det N)^{-1})$ be the complex spinor bundle over X_H associated to the canonically induced Spin^c structure on TX_H. It is a $\mathbf{Z}/k$ Hermitian vector bundle and carries a canonically induced $\mathbf{Z}/k$ Hermitian connection.

Recall that by [1, 2.4], one has $\sum_v v\dim N_v\equiv 0 \bmod 2\mathbf{Z}$. Following [8, (1.15)], set

$$R(q) = q^{1/2\sum_v |v|\dim N_v}\bigotimes_{v>0}\left(\operatorname{Sym}_{q^v}(N_v)\otimes\det N_v\right)\bigotimes_{v<0}\operatorname{Sym}_{q^{-v}}(\overline{N}_v)\otimes\sum_v q^v E_v = \bigoplus_n R_n q^n,$$

$$R'(q)=q^{-1/2\sum_v|v|\dim N_v}\bigotimes_{v>0}\mathrm{Sym}_{q^{-v}}(\overline{N}_v)\bigotimes_{v<0}\big(\mathrm{Sym}_{q^v}(N_v)\otimes\det N_v\big)\otimes\sum_v q^vE_v=\bigoplus_n R'_nq^n.$$

Then each R_n (resp. R'_n) is a $\mathbf{Z}/k$ Hermitian vector bundle over X_H carrying a canonically induced $\mathbf{Z}/k$ Hermitian connection. For any $n\in\mathbf{Z}$, let $D^{R_n}_{X_H,+}:\Gamma(S_+(TX_H,(\det N)^{-1})\otimes R_n)\to\Gamma(S_-(TX_H,(\det N)^{-1})\otimes R_n)$ be the canonical twisted Spin^c Dirac operator on X_H. Let $D^{R_n}_{X_H,+,APS}$ be the corresponding elliptic operator associated to the Atiyah–Patodi–Singer boundary condition [2]. We will use similar notation for R'_n.

Theorem 2.1. *For any integer $n\in\mathbf{Z}$, the following identities hold,*

$$\operatorname{ind}D^E_{+,APS}(n)\equiv\sum_\alpha(-1)^{\sum_{0<v}\dim N_v}\operatorname{ind}D^{R_n}_{X_{H,\alpha},+,APS}\mod k\mathbf{Z},\tag{2}$$

$$\operatorname{ind}D^E_{+,APS}(n)\equiv\sum_\alpha(-1)^{\sum_{v<0}\dim N_v}\operatorname{ind}D^{R'_n}_{X_{H,\alpha}+,APS}\mod k\mathbf{Z}.\tag{3}$$

Proof. For any $T\in\mathbf{R}$, following Witten [10], let $D^E_{T,+}:\Gamma(S_+(TX)\otimes E)\to\Gamma(S_-(TX)\otimes E)$ be the Dirac type operator defined by $D^E_{T,+}=D^E_++\sqrt{-1}Tc(H)$. Let $D^E_{T,+,APS}$ be the corresponding elliptic operator associated to the Atiyah–Patodi–Singer boundary condition [2]. Clearly, $D^E_{T,+,APS}$ also commutes with the S^1-action. For any integer n, let $D^E_{T,+,APS}(n)$ be the restriction of $D^E_{T,+,APS}$ on F^n_+. Then $D^E_{T,+,APS}(n)$ is still Fredholm. By an easy extension of [5, Theorem 1.2] to the current equivariant and $\mathbf{Z}/k$ situation, one sees that $\operatorname{ind}(D^E_{T,+,APS}(n))\bmod k\mathbf{Z}$ does not depend on $T\in\mathbf{R}$ (compare with [9, Theorem 4.2]).

Let $D^E_{T,+,\partial X}:\Gamma((S_+(TX)\otimes E)|_{\partial X})\to\Gamma((S_+(TX)\otimes E)|_{\partial X})$ be the induced Dirac type operator of $D^E_{T,+}$ on ∂X. For any integer n, let $D^E_{T,+,\partial X}(n):F^n_+|_{\partial X}\to F^n_+|_{\partial X}$ be the restriction of $D^E_{T,+,\partial X}$ on $F^n_+|_{\partial X}$. Also, the induced Dirac operators $D^{R_n}_{+,\partial X_H}$ and $D^{R_n}_{Y_H}$ can be defined in the same way as in Section 1.

Let $a_n>0$ be such that $\mathrm{Spec}(D^{R_n}_{Y_H})\cap[-2a_n,2a_n]\subseteq\{0\}$. By combining the techniques in [3, Section 9], [4, Section 4b]) and [8, Section 1.2], one can prove the following analogue of [4, Theorem 3.9], stating that there exists $T_1>0$ such that for any $T\geqslant T_1$,

$$\#\big\{\lambda\in\mathrm{Spect}\big(D^E_{T,+,\partial X}(n)\big):\ -a_n\leqslant\lambda\leqslant a_n\big\}=\dim\big(\ker D^{R_n}_{+,\partial X_H}\big)=k\dim\big(\ker D^{R_n}_{Y_H}\big).\tag{4}$$

If $\dim(\ker D^{R_n}_{Y_H})=0$, then by (4), one sees that when $T\geqslant T_1$, $D^E_{T,+,\partial X}(n)$ is invertible. Then $\operatorname{ind}(D^E_{T,+,APS}(n))$ itself does not depend on $T\geqslant T_1$. Moreover, by combining the techniques in [8, Section 1.2] and [5, Section 3], one can further prove that there exists $T_2>0$ such that when $T\geqslant T_2$,

$$\operatorname{ind}\big(D^E_{T,+,APS}(n)\big)=\sum_\alpha(-1)^{\sum_{0<v}\dim N_v}\operatorname{ind}D^{R_n}_{X_{H,\alpha},+,APS}\tag{5}$$

(compare with [5, (2.13)]). From (5) and the mod k invariance of $\operatorname{ind}(D^E_{T,+,APS}(n))$ with respect to $T\in\mathbf{R}$, one gets (2).

In general, $\dim(\ker D^{R_n}_{Y_H})$ need not be zero, and the eigenvalues of $D^E_{T,+,\partial X}(n)$ lying in $[-a_n,a_n]$ are not easy to control. Thus the above arguments no longer apply directly. Instead, we observe that $\dim(\ker(D^{R_n}_{Y_H}-a_n))=0$, and we use the method in [5] to perturb the Dirac type operators under consideration.

To do this, let $\varepsilon>0$ be sufficiently small so that g^{TX}, g^E and ∇^E are of product structure on $[0,\varepsilon]\times\partial X\subset X$. Let $f:X\to\mathbf{R}$ be an S^1-invariant smooth function such that $f\equiv1$ on $[0,\varepsilon/3]\times\partial X$ and $f\equiv0$ outside of $[0,2\varepsilon/3]\times\partial X$. Let r denote the parameter in $[0,\varepsilon]$. Let $D^{R_n}_{X_H,-a_n,+}$ be the Dirac type operator acting on $\Gamma(S_+(TX_H,(\det N)^{-1})\otimes R_n)$ defined by $D^{R_n}_{X_H,-a_n,+}=D^{R_n}_{X_H,+}-a_nfc(\frac{\partial}{\partial r})$. Let $D^{R_n}_{X_H,-a_n,+,APS}$ be the corresponding elliptic operator associated to the Atiyah–Patodi–Singer boundary condition [2]. By an easy extension of [5, Theorem 1.2] (compare with [9, Theorem 4.2]), we see that,

$$\sum_\alpha (-1)^{\sum_{0<v}\dim N_v}\,\mathrm{ind}\, D^{R_n}_{X_{H,\alpha},-a_n,+,APS} \equiv \sum_\alpha (-1)^{\sum_{0<v}\dim N_v}\,\mathrm{ind}\, D^{R_n}_{X_{H,\alpha},+,APS} \mod k\mathbf{Z}. \tag{6}$$

For any $T\in\mathbf{R}$, let $D^E_{T,-a_n,+}:\Gamma(S_+(TX)\otimes E)\to\Gamma(S_-(TX)\otimes E)$ be the Dirac type operator defined by $D^E_{T,-a_n,+}=D^E_{T,+}-a_n fc(\frac{\partial}{\partial r})$. Let $D^E_{T,-a_n,+,APS}$ be the corresponding elliptic operator associated to the Atiyah–Patodi–Singer boundary condition. Let $D^E_{T,-a_n,+,APS}(n)$ be its restriction on F^n_+. Then $D^E_{T,-a_n,+,APS}(n)$ is still Fredholm. By another extension of [5, Theorem 1.2], one has

$$\mathrm{ind}\, D^E_{T,-a_n,+,APS}(n)\equiv \mathrm{ind}\, D^E_{T,+,APS}(n) \mod k\mathbf{Z}. \tag{7}$$

Moreover, since $D^{R_n}_{Y_H}-a_n$, which is the induced Dirac type operator from $D^{R_n}_{X_H,-a_n,+}$ through π_{X_H}, is invertible, by combining the arguments in [8, Section 1.2] with those in [5, Section 3], one deduces that there exists $T_3>0$ such that for any $T\geqslant T_3$, the following analogue of (5) holds,

$$\mathrm{ind}\, D^E_{T,-a_n,+,APS}(n)=\sum_\alpha (-1)^{\sum_{0<v}\dim N_v}\,\mathrm{ind}\, D^{R_n}_{X_{H,\alpha},-a_n,+,APS}. \tag{8}$$

From (6)–(8) and the mod k invariance of $\mathrm{ind}(D^E_{T,+,APS}(n))$ with respect to $T\in\mathbf{R}$, one gets (2).

Similarly, by taking $T\to-\infty$, one gets (3). □

3. Proof of Theorem 1.2

We apply Theorem 2.1 to the case $E=\mathbf{C}$.

First, if $X_H=\emptyset$, by Theorem 2.1, it is obvious that for each $n\in\mathbf{Z}$,

$$\mathrm{ind}\big(D_{+,APS}(n)\big)\equiv 0 \mod k\mathbf{Z}. \tag{9}$$

When $X_H\neq\emptyset$, we see that $\sum_v |v|\dim N_v>0$ (i.e., at least one of the N_v's is nonzero) on each connected component of X_H. Then by (2) and by the definition of the R_n's, we deduce that for any integer $n\leqslant 0$, (9) holds. Similarly, by (3) and by the definition of the R'_n's, one deduces that (9) holds for any integer $n\geqslant 0$.

In summary, for any $n\in\mathbf{Z}$, (9) holds.

From (1) and (9), by the Atiyah–Patodi–Singer index theorem [2], and using the obvious fact that $\mathrm{ind}(D_{+,APS})=\sum_n \mathrm{ind}(D_{+,APS}(n))$, one gets $\hat{A}_{(k)}(X)=0$. □

Remark 1. By combining Theorem 2.1 with the arguments in [8, Sections 2–4], one should be able to prove an extension of the Witten rigidity theorem, of which a K-theoretic version has been worked out in [8], to $\mathbf{Z}/k$-manifolds. This, together with some other consequences of Theorem 1.2, will be carried out elsewhere.

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Differential Geometry

Bergman kernels and symplectic reduction

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Abstract

We present several results concerning the asymptotic expansion of the invariant Bergman kernel of the spin^c Dirac operator associated with high tensor powers of a positive line bundle on a compact symplectic manifold. ***To cite this article: X. Ma, W. Zhang, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Noyaux de Bergman et réduction symplectique. Nous annonçons des résultats sur le développement asymptotique du noyau de Bergman G-invariant de l'opérateur de Dirac spin^c associé à une puissance tendant vers l'infini d'un fibré en droites positif sur une variété symplectique compacte. ***Pour citer cet article : X. Ma, W. Zhang, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

Soit (X, ω) une variété symplectique compacte, et soit (L, h^L) un fibré en droites hermitien muni d'une connexion hermitienne ∇^L telle que $\frac{\sqrt{-1}}{2\pi}(\nabla^L)^2 = \omega$. Soit (E, h^E) un fibré vectoriel hermitien sur X muni d'une connexion hermitienne ∇^E. Soit g^{TX} une métrique riemannienne sur X, et soit J une structure presque complexe compatible séparément à g^{TX} et ω. Alors les données géométriques ci-dessus définissent canoniquement un opérateur de Dirac spin^c D_p agissant sur $\Omega^{0,\bullet}(X, L^p \otimes E)$, l'espace de $(0, \bullet)$-formes à valeurs dans $L^p \otimes E$.

Soit G un groupe de Lie compact connexe et soit $\mathfrak{g}$ son algèbre de Lie. On suppose que G agit sur X, et que son action se relève à L et E en préservant J, les métriques et les connexions ci-dessus. Alors $\operatorname{Ker} D_p$ est une G-représentation de dimension finie. Soit $(\operatorname{Ker} D_p)^G$ la partie G-invariante de $\operatorname{Ker} D_p$. Le noyau de Bergman

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G-invariant $P_p^G(x,x')$ $(x,x'\in X)$ est le noyau $\mathscr{C}^\infty$ de la projection orthogonale P_p^G de $\Omega^{0,\bullet}(X,L^p\otimes E)$ sur $(\operatorname{Ker} D_p)^G$ associé à la forme de volume riemannienne $dv_X(x')$.

Dans cette Note, nous annonçons des résultats sur le développement asymptotique de $P_p^G(x,x')$ quand p tend vers l'infini. Le détail des démonstrations et des applications de nos résultats est donné dans [6].

1. Introduction

Let (X,ω) be a compact symplectic manifold of real dimension $2n$. Assume that there exists a Hermitian line bundle L over X endowed with a Hermitian connection ∇^L with the property that $\frac{\sqrt{-1}}{2\pi}R^L=\omega$, where $R^L=(\nabla^L)^2$ is the curvature of (L,∇^L). Let (E,h^E) be a Hermitian vector bundle on X with Hermitian connection ∇^E and its curvature R^E.

Let g^{TX} be a Riemannian metric on X. Let $\mathbf{J}:TX\to TX$ be the skew-adjoint linear map which satisfies the relation $\omega(u,v)=g^{TX}(\mathbf{J}u,v)$ for $u,v\in TX$. Let J be an almost complex structure such that $g^{TX}(Ju,Jv)=g^{TX}(u,v)$, $\omega(Ju,Jv)=\omega(u,v)$, and that $\omega(\cdot,J\cdot)$ defines a metric on TX. Then J commutes with $\mathbf{J}$, thus $J=\mathbf{J}(-\mathbf{J}^2)^{-1/2}$. Let ∇^{TX} be the Levi-Civita connection on (TX,g^{TX}) with curvature R^{TX} and scalar curvature r^X. Then ∇^{TX} induces a natural connection $\nabla^{\det}$ on $\det(T^{(1,0)}X)$ with curvature $R^{\det}$, and the Clifford connection ∇^{Cliff} on the Clifford module $\Lambda(T^{*(0,1)}X)$ with curvature R^{Cliff}. The spin^c Dirac operator D_p acts on $\Omega^{0,\bullet}(X,L^p\otimes E)=\bigoplus_{q=0}^n\Omega^{0,q}(X,L^p\otimes E)$, the direct sum of spaces of $(0,q)$-forms with values in $L^p\otimes E$. We denote by D_p^+ the restriction of D_p on $\Omega^{0,\mathrm{even}}(X,L^p\otimes E)$. By [4, Theorem 2.5], when p is large enough, $\operatorname{Coker} D_p^+$ vanishes.

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and $\dim G=n_0$. Suppose that G acts on X and its action on X lifts on L,E. Moreover, we assume the G-action preserves the above connections and metrics on TX,L,E and J. Then $\operatorname{Ker} D_p$ is a finite dimensional representation space of G.

The action of G on L induces naturally a moment map $\mu:X\to\mathfrak{g}^*$. Now we assume that $0\in\mathfrak{g}^*$ is a regular value of μ. Then the Marsden–Weinstein symplectic reduction $(X_G=\mu^{-1}(0)/G,\omega_{X_G})$ is a symplectic manifold when G acts freely on $\mu^{-1}(0)$. Moreover, (L,∇^L), (E,∇^E) descend to (L_G,∇^{L_G}), (E_G,∇^{E_G}) over X_G so that the corresponding curvature condition $\frac{\sqrt{-1}}{2\pi}R^{L_G}=\omega_G$ holds (cf. [3]). The G-invariant almost complex structure J also descends to an almost complex structure J_G on TX_G, and h^L,h^E,g^{TX} descend to h^{L_G},h^{E_G},g^{TX_G} respectively. Thus we can construct the corresponding spin^c Dirac operator $D_{G,p}$ on X_G.

Let $(\operatorname{Ker} D_p)^G$ denote the G-invariant part of $\operatorname{Ker} D_p$. Let P_p^G be the orthogonal projection from $\Omega^{0,\bullet}(X,L^p\otimes E)$ on $(\operatorname{Ker} D_p)^G$. The G-invariant Bergman kernel is $P_p^G(x,x')$ $(x,x'\in X)$, the smooth kernel of P_p^G with respect to the Riemannian volume form $dv_X(x')$. Let pr_1 and pr_2 be the projections from $X\times X$ onto the first and second factor X respectively. Then $P_p^G(x,x')$ is a smooth section of $\mathrm{pr}_1^*E_p\otimes\mathrm{pr}_2^*E_p^*$ on $X\times X$ with $E_p=\Lambda(T^{*(0,1)}X)\otimes L^p\otimes E$. In particular, $P_p^G(x,x)\in\operatorname{End}(E_p)_x=\operatorname{End}(\Lambda(T^{*(0,1)}X)\otimes E)_x$.

The G-invariant Bergman kernel $P_p^G(x,x')$ is a local analytic version of $(\operatorname{Ker} D_p)^G$.

In this Note, we present several results concerning the asymptotic expansions of $P_p^G(x,x')$ as $p\to+\infty$. More details will appear in [6].

2. Main results

The first result shows that one can reduce our problem to a problem near $\mu^{-1}(0)$.

Theorem 2.1. *For any open G-neighborhood U of $\mu^{-1}(0)$ in X, $\varepsilon_0>0$, $l,m\in\mathbb{N}$, there exists $C_{l,m}>0$ (depend on U,ε_0) such that for $p\geqslant 1$, $x,x'\in X$, $d(Gx,x')\geqslant\varepsilon_0$ or $x,x'\in X\setminus U$,*

$$\left|P_p^G(x,x')\right|_{\mathscr{C}^m} \leqslant C_{l,m}\,p^{-l}, \tag{1}$$

where $\mathscr{C}^m$ is the $\mathscr{C}^m$-norm induced by ∇^L, ∇^E, ∇^{TX}, h^L, h^E, g^{TX}.

Assume for simplicity that G acts freely on $\mu^{-1}(0)$. Let U be an open neighborhood of $\mu^{-1}(0)$ such that G acts freely on U. For any G-equivariant bundle (F,∇^F) on U, we denote by F_B the bundle on $U/G=B$ induced naturally by G-invariant sections of F on U. The connection ∇^F induces canonically a connection ∇^{F_B} on F_B. Let R^{F_B} be its curvature. We denote also by $\mu^F(K)=\nabla^F_K-L_K\in\mathrm{End}(F)$ for $K\in\mathfrak{g}$. Note that $P_p^G\in(\mathscr{C}^\infty(U\times U,\mathrm{pr}_1^*E_p\otimes\mathrm{pr}_2^*E_p^*))^{G\times G}$, thus we can view $P_p^G(x,x')$ as a smooth section of $(\mathrm{pr}_1^*E_p)_B\otimes(\mathrm{pr}_2^*E_p^*)_B$ on $B\times B$.

Let g^{TB} be the Riemannian metric on $U/G=B$ induced by g^{TX}. Let ∇^{TB} be the Levi-Civita connection on (TB,g^{TB}) with curvature R^{TB}. Let N_G be the normal bundle to X_G in B. We identify N_G with the orthogonal complement of TX_G in $(TB|_{X_G},g^{TB})$. Let P^{TX_G}, P^{N_G} be the orthogonal projections from $TB|_{X_G}$ on TX_G, N_G respectively. Set

$$\nabla^{N_G}=P^{N_G}\big(\nabla^{TB}|_{X_G}\big)P^{N_G},\quad {}^0\nabla^{TB}=P^{TX_G}\big(\nabla^{TB}|_{X_G}\big)P^{TX_G}\oplus\nabla^{N_G},\quad A=\nabla^{TB}-{}^0\nabla^{TB}. \tag{2}$$

Then ∇^{N_G}, ${}^0\nabla^{TB}$ are Euclidean connections on N_G, $TB|_{X_G}$ on X_G, and A is the associated second fundamental form. We denote by $\mathrm{vol}(Gx)$ $(x\in U)$ the volume of the orbit Gx equipped with the metric induced by g^{TX}. Following [9, (3.10)], let $h(x)$ be the function on U defined by

$$h(x)=\big(\mathrm{vol}(Gx)\big)^{1/2}. \tag{3}$$

Then h reduces to a function on B. We denote by $I_{\mathbb{C}\otimes E}$ the projection from $\Lambda(T^{*(0,1)}X)\otimes E$ onto $\mathbb{C}\otimes E$ under the decomposition $\Lambda(T^{*(0,1)}X)=\mathbb{C}\oplus\Lambda^{>0}(T^{*(0,1)}X)$, and $I_{\mathbb{C}\otimes E_B}$ the corresponding projection on B.

For any $x_0\in X_G$, $Z\in T_{x_0}B$, we write $Z=Z^0+Z^\perp$, with $Z^0\in T_{x_0}X_G$, $Z^\perp\in N_{G,x_0}$. Let $\tau_{Z^0}Z^\perp\in N_{G,\exp^{X_G}_{x_0}(Z^0)}$ be the parallel transport of $Z^\perp$ with respect to the connection ∇^{N_G} along the geodesic in X_G, $[0,1]\ni t\to\exp^{X_G}_{x_0}(tZ^0)$. For $\varepsilon_0>0$ small enough, we identify $Z\in T_{x_0}B$, $|Z|<\varepsilon_0$ with $\exp^B_{\exp^{X_G}_{x_0}(Z^0)}(\tau_{Z^0}Z^\perp)\in B$, then for $x_0\in X_G$, $Z,Z'\in T_{x_0}B$, $|Z|,|Z'|<\varepsilon_0$, the map

$$\Psi:TB|_{X_G}\times TB|_{X_G}\to B\times B,\quad \Psi(Z,Z')=\big(\exp^B_{\exp^{X_G}_{x_0}(Z^0)}(\tau_{Z^0}Z^\perp),\exp^B_{\exp^{X_G}_{x_0}(Z'^0)}(\tau_{Z'^0}Z'^\perp)\big)$$

is well defined. We identify $(E_p)_{B,Z}$ to $(E_p)_{B,x_0}$ by using parallel transport with respect to $\nabla^{(E_p)_B}$ along $[0,1]\ni u\to uZ$. Let $\pi_B:TB|_{X_G}\times TB|_{X_G}\to X_G$ be the natural projection from the fiberwise product of $TB|_{X_G}$ on X_G onto X_G. From Theorem 2.1, we only need to understand $P_p^G\circ\Psi$, and under our identification, $P_p^G\circ\Psi(Z,Z')$ is a smooth section of $\pi_B^*(\mathrm{End}(E_p)_B)=\pi_B^*(\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_B)$ on $TB|_{X_G}\times TB|_{X_G}$. Let $|\ \ |_{\mathscr{C}^{m'}(X_G)}$ be the $\mathscr{C}^{m'}$-norm on $\mathscr{C}^\infty(X_G,\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_B)$ induced by $\nabla^{\mathrm{Cliff}_B}$, ∇^{E_B}, h^E and g^{TX}. The norm $|\ \ |_{\mathscr{C}^{m'}(X_G)}$ induces naturally a $\mathscr{C}^{m'}$-norm along X_G on $\mathscr{C}^\infty(TB|_{X_G}\times TB|_{X_G},\pi_B^*(\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_B))$, we still denote it by $|\ \ |_{\mathscr{C}^{m'}(X_G)}$.

Let g^{TX_G}, g^{N_G} be the metric on TX_G, N_G induced by g^{TX}. Let dv_{X_G}, dv_{N_G} be the Riemannian volume forms on (X_G,g^{TX_G}), (N_G,g^{N_G}). Let $\kappa\in\mathscr{C}^\infty(TB|_{X_G},\mathbb{R})$, with $\kappa=1$ on X_G, be defined by that for $Z\in T_{x_0}B$, $x_0\in X_G$,

$$dv_B(x_0,Z)=\kappa(x_0,Z)\,dv_{T_{x_0}B}(Z)=\kappa(x_0,Z)\,dv_{X_G}(x_0)\,dv_{N_{G,x_0}}. \tag{4}$$

Theorem 2.2. *Assume that G acts freely on $\mu^{-1}(0)$ and $\mathbf{J}=J$ on $\mu^{-1}(0)$. Then there exist $\mathcal{Q}_r(Z,Z')\in\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0}$ $(x_0\in X_G, r\in\mathbb{N})$, polynomials in Z,Z' with the same parity as r, whose coefficients are polynomials in A, R^{TB}, R^{Cliff_B}, R^{E_B}, μ^E, μ^{Cliff} (resp. r^X, $R^{\det}$, R^E; resp. h, R^L, R^{L_B}; resp. μ) and their derivatives at x_0 to order $r-1$ (resp. $r-2$; resp. r, resp. $r+1$), such that if we denote by*

$$P^{(r)}_{x_0}(Z,Z')=\mathcal{Q}_r(Z,Z')P(Z,Z'),\quad \mathcal{Q}_0(Z,Z')=I_{\mathbb{C}\otimes E_B}, \tag{5}$$

with

$$P(Z,Z')=\exp\Big(-\frac{\pi}{2}\big|Z^0-Z'^0\big|^2-\pi\sqrt{-1}\big\langle J_{x_0}Z^0,Z'^0\big\rangle\Big)2^{n_0/2}\exp\big(-\pi\big(|Z^\perp|^2+|Z'^\perp|^2\big)\big), \tag{6}$$

then there exists $C''>0$ *such that for any* $k,m,m',m''\in\mathbb{N}$, *there exists* $C>0$ *such that for* $x_0\in X_G$, $Z,Z'\in T_{x_0}B$, $|Z|,|Z'|\leqslant\varepsilon_0$,

$$\begin{aligned}&\big(1+\sqrt{p}\,|Z^\perp|+\sqrt{p}\,|Z'^\perp|\big)^{m''}\sup_{|\alpha|+|\alpha'|\leqslant m}\Bigg|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\\ &\qquad\Bigg(p^{-n+n_0/2}\big(h\kappa^{1/2}\big)(Z)\big(h\kappa^{1/2}\big)(Z')P_p^G\circ\Psi(Z,Z')-\sum_{r=0}^k P_{x_0}^{(r)}(\sqrt{p}\,Z,\sqrt{p}\,Z')p^{-r/2}\Bigg)\Bigg|_{\mathscr{C}^{m'}(X_G)}\\ &\quad\leqslant Cp^{-(k+1-m)/2}\big(1+\sqrt{p}\,\big|Z^0\big|+\sqrt{p}\,\big|Z'^0\big|\big)^{2(n+k+2)+m}\exp\big(-\sqrt{C''}\sqrt{p}\,|Z-Z'|\big)+\mathscr{O}\big(p^{-\infty}\big).\end{aligned} \tag{7}$$

Furthermore, the expansion is uniform in the following sense: *for any fixed* $k,m,m',m''\in\mathbb{N}$, *assume that the derivatives of* g^{TX}, h^L, ∇^L, h^E, ∇^E, *and* J *with order* $\leqslant 2n+2k+m+m'+4$ *run over a set bounded in the* $\mathscr{C}^{m'}$*-norm taken with respect to the parameters and, moreover,* g^{TX} *runs over a set bounded below. Then the constant* C *is independent of* g^{TX}; *and the* $\mathscr{C}^{m'}$*-norm in* (7) *includes also the derivatives on the parameters.*

In (7), the term $\mathscr{O}(p^{-\infty})$ means that for any $l,l_1\in\mathbb{N}$, there exists $C_{l,l_1}>0$ such that its $\mathscr{C}^{l_1}$-norm is dominated by $C_{l,l_1}p^{-l}$. The kernel $P(Z,Z')$ is the product of two kernels: along $T_{x_0}X_G$, it is the classical Bergman kernel on $T_{x_0}X_G$ with complex structure J_{x_0}, while along N_G, it is the kernel of a harmonic oscillator on N_{G,x_0}.

Remark 1. (i) When $G=\{1\}$, Theorem 2.2 has been proved in [2, Theorem 3.18$'$].

(ii) If we take $Z=Z'=0$ in (7), then we get for $x_0\in X_G$,

$$P_{x_0}^{(0)}(0,0)=2^{n_0/2}I_{\mathbb{C}\otimes E_B}, \tag{8}$$

and

$$\Bigg|p^{-n+n_0/2}h^2(x_0)P_p^G(x_0,x_0)-\sum_{r=0}^k P_{x_0}^{(2r)}(0,0)p^{-r}\Bigg|_{\mathscr{C}^{m'}(X_G)}\leqslant Cp^{-k-1}. \tag{9}$$

In fact, (8) and (9) can be obtained as direct consequences of the full off-diagonal asymptotic expansion of the Bergman kernel proved in [2, Theorem 3.18$'$].

Remark 2. Assume that (X,ω) is a Kähler manifold and $\mathbf{J}=J$ on X. Assume also that (L,∇^L), (E,∇^E) are holomorphic vector bundles with holomorphic Hermitian connections. Then D_p^2 preserves the $\mathbb{Z}$-graduation of $\Omega^{0,\bullet}(X,L^p\otimes E)$ and $\operatorname{Ker}D_p=H^0(X,L^p\otimes E)$ when p is large enough, and so $P_p^G(x_0,x_0)\in\operatorname{End}(E)$. In particular $P_{x_0}^{(0)}(0,0)=2^{n_0/2}\mathrm{Id}_E$ in (8). In the special case of $E=\mathbb{C}$, $P_p^G(x_0,x_0)$ is a function on X_G, (9) has been proved in [7, Theorem 1] without knowing the informations on $P_{x_0}^{(2r)}(0,0)$, while in [8, Theorem 1], it was claimed that $P_{x_0}^{(0)}(0,0)=1$.

Let $\mathscr{I}_p$ be a section of $\operatorname{End}(\Lambda(T^{*(0,1)}X)\otimes E)_B$ on X_G defined by

$$\mathscr{I}_p(x_0)=\int_{Z\in N_G,\ |Z|\leqslant\varepsilon_0}h^2(x_0,Z)P_p^G\circ\Psi\big((x_0,Z),(x_0,Z)\big)\kappa(x_0,Z)\,\mathrm{d}v_{N_G}(Z). \tag{10}$$

By Theorem 2.1, modulo $\mathscr{O}(p^{-\infty})$, $\mathscr{I}_p(x_0)$ does not depend on ε_0, and

$$\dim(\operatorname{Ker} D_p)^G = \int_X \operatorname{Tr}\big[P_p^G(y,y)\big]\,dv_X(y) = \int_{X_G} \operatorname{Tr}\big[\mathscr{I}_p(x_0)\big]\,dv_{X_G}(x_0) + \mathscr{O}\big(p^{-\infty}\big). \tag{11}$$

A direct consequence of Theorem 2.2 is the following corollary.

Corollary 2.3. *Taken $Z = Z' \in N_{G,x_0}$, $m = 0$ in* (7), *we get*

$$\begin{aligned}&\left| p^{-n+n_0/2} h^2(Z) P_p^G(Z,Z) - \sum_{r=0}^{k} P_{x_0}^{(r)}(\sqrt{p}\,Z, \sqrt{p}\,Z)\,p^{-r/2} \right|_{\mathscr{C}^{m'}(X_G)} \\ &\quad \leqslant C p^{-(k+1)/2}\big(1+\sqrt{p}\,|Z|\big)^{-m''} + \mathscr{O}\big(p^{-\infty}\big).\end{aligned} \tag{12}$$

In particular, there exist $\Phi_r \in \operatorname{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{B,x_0}$ ($r \in \mathbb{N}$) which are polynomials in A, R^{TB}, R^{Cliff_B}, R^{E_B}, μ^E, μ^{Cliff}, (resp. r^X, $R^{\det}$, R^E; resp. h, R^{L_B}, R^L; resp. μ), and their derivatives at x_0 to order $2r-1$ (resp. $2r-2$; resp. $2r$; resp. $2r+1$), and $\Phi_0 = I_{\mathbb{C}\otimes E_B}$, such that for any $k, m' \in \mathbb{N}$, there exists $C_{k,m'} > 0$ such that for any $x_0 \in X_G$, $p \in \mathbb{N}$,

$$\left| p^{-n+n_0} \mathscr{I}_p(x_0) - \sum_{r=0}^{k} \Phi_r(x_0)\,p^{-r} \right|_{\mathscr{C}^{m'}} \leqslant C_{k,m'}\,p^{-k-1}. \tag{13}$$

Theorem 2.4. *If (X,ω) is a Kähler manifold and L, E are holomorphic vector bundles with holomorphic Hermitian connections ∇^L, ∇^E, $\mathbf{J} = J$ on U, and G acts freely on $\mu^{-1}(0)$, then in* (13), *$\Phi_r(x_0) \in \operatorname{End}(E_G)_{x_0}$ are polynomials in A, R^{TB}, R^{E_B}, μ^E, R^E (resp. h, R^{L_B}; resp. μ) and their derivatives at x_0 to order $2r-1$ (resp. $2r$; resp. $2r+1$), and $\Phi_0 = \mathrm{Id}_{E_G}$. Moreover*

$$\Phi_1(x_0) = \frac{1}{8\pi} r_{x_0}^{X_G} + \frac{3}{4\pi}\Delta_{X_G}\log(h|_{X_G}) + \frac{\sqrt{-1}}{4\pi} R_{x_0}^{E_G}\big(e_j^0, J_G e_j^0\big). \tag{14}$$

Here r^{X_G} is the Riemannian scalar curvature of (TX_G, g^{TX_G}), Δ_{X_G} is the Bochner–Laplacian on X_G, and $\{e_j^0\}$ is an orthonormal basis of TX_G.

Still making the same assumptions as in Theorem 2.4, let $\tilde{h}$ denote the restriction to X_G of the function h defined in (3). In view of [9, (3.11), (3.54)], set $\tilde{h}^{E_G} = \tilde{h}^2 h^{E_G}$.

Let $\widetilde{P}_p^{X_G}$ denote the orthogonal projection from $\mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$ onto $H^0(X, L_G^p \otimes E_G)$ associated to the metrics h^{L_G}, $\tilde{h}^{E_G}$, g^{TX_G}. Let $\widetilde{P}_p^{X_G}(x_0, x_0')$ $(x_0, x_0' \in X_G)$ denote the corresponding Bergman kernel with respect to $dv_{X_G}(x_0')$.

Then by [2, Theorem 1.3], we have the following theorem.

Theorem 2.5. *Under the assumption of Theorem 2.4, there exist smooth coefficients $\widetilde{\Phi}_r(x_0) \in \operatorname{End}(E_G)_{x_0}$ which are polynomials in R^{TX_G}, R^{E_G} (resp. h), and their derivatives at x_0 to order $2r-1$ (resp. $2r$), and $\widetilde{\Phi}_0 = \mathrm{Id}_{E_G}$, such that for any $k, l \in \mathbb{N}$, there exists $C_{k,l} > 0$ such that for any $x_0 \in X_G$, $p \in \mathbb{N}$,*

$$\left| p^{-n+n_0} \widetilde{P}_p^{X_G}(x_0,x_0) - \sum_{r=0}^{k} \widetilde{\Phi}_r(x_0)\,p^{-r} \right|_{\mathscr{C}^l} \leqslant C_{k,l}\,p^{-k-1}. \tag{15}$$

Moreover, the following identity holds,

$$\widetilde{\Phi}_1(x_0) = \frac{1}{8\pi} r_{x_0}^{X_G} + \frac{1}{2\pi}\Delta_{X_G}\log\tilde{h} + \frac{\sqrt{-1}}{4\pi} R_{x_0}^{E_G}\big(e_j^0, J_G e_j^0\big). \tag{16}$$

Remark 3. From (14) and (16), one sees that in general $\Phi_1 \neq \widetilde{\Phi}_1$, if $\widetilde{h}$ is not constant on X_G. This reflects a subtle difference between the Bergman kernel and the geometric quantization.

The proof of the above theorems uses techniques adapted from [1, §11], [2,5], along with a deformation of D_p^2 by the Casimir operator (i.e., to consider $D_p^2 - p\mathrm{Cas}$, which plays a key role in proving Theorems 2.1, 2.2). We refer to [6] for more details.

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Differential Geometry

Superconnection and family Bergman kernels

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Abstract

We establish an asymptotic expansion for families of Bergman kernels. The key idea is to use the superconnection formalism as in the local family index theorem. ***To cite this article: X. Ma, W. Zhang, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Superconnexion et noyaux de Bergman en famille. Nous annonçons des résultats sur le développement asymptotique du noyau de Bergman en famille. L'idée principale est d'utiliser le formalisme des superconnexions comme dans la preuve du théorème de l'indice local en famille. ***Pour citer cet article : X. Ma, W. Zhang, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Soit $\pi : W \to S$ une submersion holomorphe de variétés compactes de fibre X, et soit L un fibré en droites holomorphes sur W qui est positif le long de la fibre X. Soit E un fibré vectoriel holomorphe sur W. Pour p assez grand, la première classe de Chern du fibré vectoriel holomorphe $H^0(X, L^p \otimes E)$ est calculé par le théorème de Grothendieck–Riemann–Roch (G.R.R.).

En considérant la courbure de $H^0(X, L^p \otimes E)$ comme un opérateur agissant le long de la fibre, nous étudions dans cette Note le développement asymptotique de son noyau quand la puissance p tend vers $+\infty$. Nos résultats raffinent le développement asymptotique de la première classe de Chern donnée par le théorème de G.R.R. au niveau des formes différentielles comme dans la version locale du théorème de l'indice en famille. L'idée principale est d'utiliser « un morceau » de la superconnexion introduite par Bismut dans la preuve du théorème de l'indice local en famille.

Les résultats annoncés dans cette Note sont démontrés dans [12].

1. Introduction

Let W, S be smooth compact complex manifolds. Let $\pi : W \to S$ be a holomorphic submersion with compact fiber X and $\dim_{\mathbb{C}} X = n$. Let E be a holomorphic vector bundle on W. Let L be a holomorphic line bundle on W.

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We suppose that L is positive along the fiber X.

We will add a subscript $\mathbb{R}$ for the corresponding real objects. Thus TX is the holomorphic relative tangent bundle of π, and $T_{\mathbb{R}}X$ is the corresponding real vector bundle. Let $J^{T_{\mathbb{R}}X}$ be the complex structure on $T_{\mathbb{R}}X$.

By the Kodaira vanishing theorem, there exists $p_0 \in \mathbb{N}$ such that the higher fiberwise cohomologies vanish and that $H^0(X, (L^p \otimes E)|_X)$ forms a vector bundle, denoted by $H^0(X, L^p \otimes E)$, on S for $p > p_0$. From now on, we always assume $p > p_0$.

By the Grothendieck–Riemann–Roch Theorem, in $H^2(S, \mathbb{R})$, as $p \to +\infty$, we have

$$c_1\big(H^0\big(X, L^p \otimes E\big)\big) = \mathrm{rk}(E) \int_X \frac{c_1(L)^{n+1}}{(n+1)!} p^{n+1} + \int_X \bigg(c_1(E) + \frac{\mathrm{rk}(E)}{2} c_1(TX)\bigg) \frac{c_1(L)^n}{n!} p^n + \mathscr{O}\big(p^{n-1}\big). \tag{1}$$

Now, in view of the Bismut local family index theorem [2], it is natural to ask whether a local version of (1) still holds which involves the curvature of the vector bundle $H^0(X, L^p \otimes E)$.

Let us introduce our geometric data now. Let h^L be a Hermitian metric on L such that the restriction of $\sqrt{-1}\,R^L$ along the fiber X is a positive $(1,1)$-form, here R^L is the curvature of the holomorphic Hermitian connection ∇^L on (L, h^L). Let $\omega := c_1(L, h^L)$ be the Chern–Weil representative of the first Chern class $c_1(L)$ of (L, h^L), then

$$\omega = c_1\big(L, h^L\big) = \frac{\sqrt{-1}}{2\pi} R^L. \tag{2}$$

Thus ω is a smooth real 2-form of complex type $(1,1)$ on W. Moreover, ω defines a Kähler form along the fiber X, i.e.

$$g^{T_{\mathbb{R}}X}(u, v) = \omega\big(u, J^{T_{\mathbb{R}}X} v\big) \tag{3}$$

defines a Riemannian metric on $T_{\mathbb{R}}X$. We denote by h^{TX} the corresponding Hermitian metric on TX.

Let dv_X be the Riemannian volume form on $(X, g^{T_{\mathbb{R}}X})$.

Let h^E be a Hermitian metric on E. Let ∇^E be the holomorphic Hermitian connection on (E, h^E) with its curvature R^E.

Let $h^{H^0(X,L^p\otimes E)}$ be the L^2-metric on $H^0(X, L^p \otimes E)$ induced by h^L, h^E and $g^{T_{\mathbb{R}}X}$. Let $\nabla^{H^0(X,L^p\otimes E)}$ be the holomorphic Hermitian connection on $(H^0(X, L^p \otimes E), h^{H^0(X,L^p\otimes E)})$. Let $R^{H^0(X,L^p\otimes E)} = (\nabla^{H^0(X,L^p\otimes E)})^2$ be the curvature of $\nabla^{H^0(X,L^p\otimes E)}$. Then

$$R^{H^0(X,L^p\otimes E)} \in \Lambda^2\big(T_{\mathbb{R}}^*S\big) \otimes \mathrm{End}\big(H^0\big(X, L^p \otimes E\big)\big).$$

For $s \in S$, let $P_{p,s}$ be the orthogonal projection from $\mathscr{C}^\infty(X_s, (L^p \otimes E)|_{X_s})$ onto $H^0(X_s, (L^p \otimes E)|_{X_s})$. In the sequel, we write instead P_p.

We will identify $R^{H^0(X,L^p\otimes E)}$ to

$$P_p R^{H^0(X,L^p\otimes E)} P_p \in \Lambda^2\big(T_{\mathbb{R}}^*S\big) \otimes \mathrm{End}\big(\mathscr{C}^\infty\big(X, \big(L^p \otimes E\big)\big|_X\big)\big).$$

Let $R^{H^0(X,L^p\otimes E)}(x, x')$ $(x, x' \in X_s, s \in S)$ be the smooth kernel of the operator $R^{H^0(X,L^p\otimes E)}$ with respect to $dv_{X_s}(x')$. Then

$$R^{H^0(X,L^p\otimes E)}(x, x) \in \pi^*\big(\Lambda^2\big(T_{\mathbb{R}}^*S\big)\big) \otimes \mathrm{End}(E_x). \tag{4}$$

The purpose of this Note is to evaluate the asymptotics as $p \to \infty$ of the kernel of $R^{H^0(X,L^p\otimes E)}$.

2. Main result

Let T^HW be the orthogonal bundle to TX with respect to ω. Then T^HW is a sub-bundle of TW such that

$$TW = T^HW \oplus TX. \tag{5}$$

Let P^{TX} be the projection from $TW = T^HW \oplus TX$ onto TX. For $U \in T_{\mathbb{R}}S$, let $U^H \in T_{\mathbb{R}}^HW$ be the horizontal lift of U.

Let $T \in \Lambda^2(T^*_{\mathbb{R}}W) \otimes T_{\mathbb{R}}X$ be the tensor defined in the following way: for $U, V \in T_{\mathbb{R}}S$, $X, Y \in T_{\mathbb{R}}X$,

$$\begin{aligned} &T(U^H, V^H) := -P^{TX}[U^H, V^H], \quad T(X,Y) := 0, \\ &T(U^H, X) := \frac{1}{2}\big(g^{T_{\mathbb{R}}X}\big)^{-1}\big(\mathcal{L}_{U^H} g^{T_{\mathbb{R}}X}\big)X. \end{aligned} \tag{6}$$

Let R^{TX} be the curvature of the holomorphic Hermitian connection ∇^{TX} on (TX, h^{TX}). Then the Chern–Weil representative of the first Chern class of (TX, h^{TX}) is $c_1(TX, h^{TX}) = \frac{\sqrt{-1}}{2\pi}\operatorname{Tr}[R^{TX}]$.

Let $\{g_\alpha\}$ be a frame of TS and $\{g^\alpha\}$ its dual frame.

Clearly, (5) induces canonically a decomposition $\Lambda(T^*_{\mathbb{R}}W) = \pi^*(\Lambda(T^*_{\mathbb{R}}S))\widehat{\otimes}\Lambda(T^*_{\mathbb{R}}X)$. We will denote by $A^{(i)}$ the component in $\pi^*(\Lambda^i(T^*_{\mathbb{R}}S))\widehat{\otimes}\Lambda(T^*_{\mathbb{R}}X)$, of a differential form A on W. Then $dv_X = (\omega^n)^{(0)}/n!$.

Theorem 2.1. *There exist smooth sections $b_{2,r}(x) \in \mathscr{C}^\infty(W, \Lambda^2(T^*_{\mathbb{R}}S) \otimes \operatorname{End}(E_x))$ which are polynomials in R^{TX}, T, R^E (and R^L), and their derivatives of order $\leqslant 2r-1$ (resp. $2r$) along the fiber X such that for any $k, l \in \mathbb{N}$, there exists $C_{k,l} > 0$ such that for any $p \in \mathbb{N}$, $p > p_0$,*

$$\left| R^{H^0(X, L^p\otimes E)}(x,x) - \sum_{r=0}^{k} b_{2,r}(x)\, p^{n-r+1} \right|_{\mathscr{C}^l(W)} \leqslant C_{k,l}\, p^{n-k}, \tag{7}$$

with

$$\begin{aligned} &\frac{\sqrt{-1}}{2\pi} b_{2,0} = \frac{(\omega^{n+1})^{(2)}}{(n+1)(\omega^n)^{(0)}}\operatorname{Id}_E = g^\alpha \wedge \bar{g}^\beta \omega\big(g^H_\alpha, \bar{g}^H_\beta\big)\operatorname{Id}_E, \\ &\frac{\sqrt{-1}}{2\pi} b_{2,1} = \left(\left(\frac{1}{2}c_1\big(TX, h^{TX}\big) + \frac{\sqrt{-1}}{2\pi}R^E - \frac{1}{8\pi} g^\alpha \wedge \bar{g}^\beta \Delta_X\big(\omega\big(g^H_\alpha, \bar{g}^H_\beta\big)\big)\right)\omega^n\right)^{(2)} \Big/ \big(\omega^n\big)^{(0)}, \end{aligned} \tag{8}$$

where Δ_X is the (positive) Laplace operator of the fiber X_s.

If we take the trace of this asymptotic (7) on E and integrate along X, we get a refinement of (1) on the level of differential forms, in the spirit of the local family index theorem.

3. Idea of the proof

Proof. By using the full off-diagonal asymptotic expansion of the Bergman kernel [6] with the parameter $s \in S$, it is not hard to prove the existence of an expansion with leading term p^{n+2}, but further work is needed to get the vanishing of the first coefficient, and it is difficult to compute the other coefficients this way.

Our main idea here is to use the superconnection formalism to prove Theorem 2.1. This gives us a conceptually clear way to get our result: an important feature of our superconnection is that its curvature is a second order differential operator along the fiber X, while the superconnection itself involves derivatives along the horizontal direction. Just as in the Bismut local family index theorem [2], this property of our superconnection plays an important role in our proof.

We now explain briefly the superconnection formalism.

Let $\bar\partial^{L^p\otimes E,*}$ be the formal adjoint of the fiberwise Dolbeault operator $\bar\partial^{L^p\otimes E}$ on the Dolbeault complex $\Omega^{0,\bullet}(X, L^p \otimes E)$. Set

$$D_p = \sqrt{2}\big(\bar\partial^{L^p\otimes E} + \bar\partial^{L^p\otimes E,*}\big). \tag{9}$$

Let ∇^{E_p} be the connection on $E_p := \Lambda(T^{*(0,1)}X) \otimes L^p \otimes E$ induced by the holomorphic Hermitian connections ∇^{TX}, ∇^L, ∇^E on TX, L, E respectively.

For $U \in T_{\mathbb{R}}S$, if σ is a smooth section of $\Omega^{0,\bullet}(X, L^p\otimes E)$ over S, i.e. $\sigma \in \mathscr{C}^\infty(W, E_p)$, set

$$\nabla^{\Omega}_U \sigma = \nabla^{E_p}_{U^H}\sigma. \tag{10}$$

Then ∇^Ω is a Hermitian connection on $\Omega^{0,\bullet}(X, L^p \otimes E)$ over S. Let B_p be the superconnection on $\Lambda(T^*_{\mathbb{R}}S)\,\widehat{\otimes}\,\Omega^{0,\bullet}(X, L^p\otimes E)$ defined by

$$B_p = D_p + \nabla^{\Omega}. \tag{11}$$

We now describe the explicit geometric construction of $\nabla^{H^0(X,L^p\otimes E)}$ given in [4, Theorem 3.4] (cf. [3, Theorem 3.11]). Let $\nabla^{L^p\otimes E}$ be the connection on $L^p\otimes E$ induced by ∇^L, ∇^E. For $U\in T_{\mathbb{R}}S$, $\sigma\in\mathscr{C}^\infty(S, H^0(X, L^p\otimes E))$, then

$$\nabla_U^{H^0(X,L^p\otimes E)}\sigma = P_p\nabla_{U^H}^{L^p\otimes E}\sigma, \tag{12}$$

where σ is considered as a section of $L^p\otimes E$ on W.

From (11), (12) and the spectral gap property of D_p^2 (cf. [5,9]), for p large enough, we have

$$R^{H^0(X,L^p\otimes E)} = \frac{1}{2\pi\sqrt{-1}}\bigg[\int\limits_{|\lambda|=2\pi p}\big(\lambda - B_p^2\big)^{-1}\lambda\,\mathrm{d}\lambda\bigg]^{(2)}. \tag{13}$$

Now, by using the formal power series trick developed in [10], we get a general and algorithmic way to compute the coefficients in the expansion. More details will appear in [12]. □

Remark 3.1. In this Note, we have only formulated our results in the case of holomorphic line bundles which are fiberwise positive. Actually, the results hold also for symplectic line bundles. In [12], we also prove the existence of an off-diagonal asymptotic expansion which implies, for example, that $R^{H^0(X,L^p\otimes E)}$ is a Toeplitz operator with values in $\Lambda^2(T_{\mathbb{R}}^*S)$ in the sense of [11, Chapter 7].

By (7), (8), the curvatures $R^{H^0(X,L^p\otimes E)}(x,x)$ provide a natural approximation of the Monge–Ampère operator on the space of Kähler metrics. It should have relations with the existence problem of geodesics on the space of Kähler metrics (cf. [7,8,14,13]).

From Eq. (8), for large p, we can obtain more precise positivity estimates for $H^0(X, L^p\otimes E)$ than in [1, §6].

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HIGHER SPECTRAL FLOW

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Abstract. For a continuous curve of families of Dirac type operators we define a higher spectral flow as a K-group element. We show that this higher spectral flow can be computed analytically by $\hat{\eta}$-forms, and is related to the family index in the same way as the spectral flow is related to the index. We also introduce a notion of Toeplitz family and relate its index to the higher spectral flow.

We introduce the notion of *higher spectral flow*, generalizing the usual spectral flow (cf [APS1]). The higher spectral flow is defined for a continuous one parameter family (i.e. a curve) of *families of Dirac type operators* parametrized by a compact space and is an element of the K-group of the parameter space. The (virtual) dimension of this K-group element is precisely the (usual) spectral flow. The definition makes use of the concept of *spectral section* introduced recently by Melrose-Piazza [MP].

We show that this higher version of spectral flow satisfies the basic properties of spectral flow. For example, its Chern character can be expressed analytically in terms of a generalization of the $\hat{\eta}$ form of Bismut-Cheeger [BC1]. The higher spectral flow is also related to the family index, in the same way as the spectral flow is related to the index.

We also introduce a notion of Toeplitz family and relate its index to the higher spectral flow. This generalizes a result of Booss-Wojciechowski [BW, Theorem 17.17]. Finally we use higher spectral flow to prove a generalization of the family index theorem for manifolds with boundary [BC2], [BC3], [MP].

The details and the proofs will appear in [DZ2].

1. Spectral flow and spectral section

We take the definition of spectral flow as in [APS1]. Thus, if D_s, $0 \leq s \leq 1$, is a curve of self-adjoint Fredholm operators, the spectral flow

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$sf\{D_s\}$ counts the net number of eigenvalues of D_s which change sign when s varies from 0 to 1. (Throughout the paper a family always means a *continuous* family, and a curve always means a one parameter family.)

The notion of spectral section can be defined for a family of self adjoint first order elliptic pseudodifferential operators [MP].

Definition. A spectral section for a family of self adjoint first order elliptic pseudodifferential operators D_z ($z \in B$) is a family of self adjoint pseudodifferential projections P_z such that for some smooth function $R: B \longrightarrow \mathbb{R}$ and every $z \in B$

$$D_z u = \lambda u \Rightarrow \begin{cases} P_z u = u \text{ if } \lambda > R(z) \\ P_z u = 0 \text{ if } \lambda < -R(z). \end{cases}$$

As is proved in [MP], when the parameter space is compact, the existence of a spectral section is equivalent to the vanishing of the (analytic) index of the family. Thus the existence for a one parameter family is always assured.

Now let D_s be a curve of self adjoint elliptic pseudodifferential operators. Let Q_s be the spectral projection onto the direct sum of eigenspaces of D_s with nonnegative eigenvalues (the APS projection). The following theorem provides a link between the above two notions.

Theorem 1.1. *Let P_s be a spectral section of D_s. Then $[Q_1 - P_1]$ defines an element of $K^0(pt) \cong \mathbb{Z}$ and so does $[Q_0 - P_0]$. Moreover the difference $[Q_1 - P_1] - [Q_0 - P_0]$ is independent of the choice of the spectral section P_s, and it computes the spectral flow of D_s:*

$$sf\{D_s\} = [Q_1 - P_1] - [Q_0 - P_0].$$

Proof. The independence of the choice of spectral section follows from a construction in [MP, Proposition 2]. For (1.1), we use spectral sections to reduce it to the finite dimensional case, where it can be easily verified.

This leads us to the notion of

2. Higher spectral flow

Let $\pi : X \longrightarrow B$ be a smooth fibration with the typical fiber Z an odd dimensional closed manifold and B compact. A family of self adjoint elliptic pseudodifferential operators on Z, parametrized by B, will be called a B-family. Consider a curve of B-families, $D_u = \{D_{b,u}\}$, $u \in [0, 1]$.

Assuming that the index bundle of D_0 vanishes, the homotopy invariance of the index then implies that the index bundle of each D_u vanishes. Let Q_0, Q_1 be spectral sections of D_0, D_1 respectively. If we consider the total family $\tilde{D} = \{D_{b,u}\}$ parametrized by $B \times I$, then there is a total spectral section $\tilde{P} = \{P_{b,u}\}$. Let P_u be the restriction of $\tilde{P}$ over $B \times \{u\}$.

According to [MP] difference of spectral sections defines K-group element of the parameter space.
Definition. The (higher) spectral flow $sf\{(D_0, Q_0),\ (D_1, Q_1)\}$ between the pairs (D_0, Q_0), (D_1, Q_1) is an element in $K(B)$ defined by

$$sf\{(D_0, Q_0),\ (D_1, Q_1)\} = [Q_1 - P_1] - [Q_0 - P_0] \in K(B).$$

The definition is independent of the choice of the (total) spectral section P, as it follows again from [MP]. When D_u, $u \in S^1$ is a periodic family, we choose $Q_1 = Q_0$. In this case the spectral flow turns out to be independent of $Q_0 = Q_1$ and therefore defines an invariant of the family, denoted $sf\{D_u\}$.

For the most part of this paper we are going to restrict our attention to B-families of Dirac type operators, defined as follows. For simplicity we assume that the vertical tangent bundle $TZ \longrightarrow X$ is spin and carries a fixed spin structure[1]. Let g^{TZ} be a metric on TZ. Let E be a complex vector bundle over X with an hermitian metric g^E and compatible connection ∇^E. Corresponding to these geometric data we have a family of Dirac operators D_b^E, $b \in B$. This is a family of self adjoint elliptic operators on Z parametrized by B.
Definition. By a B-family of self adjoint *Dirac type operators* we mean a family of self adjoint elliptic operators $\tilde{D}_b$ parametrized by B whose principal symbol is the same as that of D_b^E.
Basic assumption. We assume that the family D_b^E has vanishing index bundle.

A typical example satisfying our basic assumption is the family of signature operators. More generally a B-family whose kernels have constant dimension will always satisfy the basic assumption. Another class of examples comes from the boundary family of a family of Dirac operators on manifolds with boundary.

Now we can speak of the higher spectral flow of a curve of B-families of Dirac type operators, given the basic assumption.

3. $\hat{\eta}$-form and higher spectral flow

By [MP], given a spectral section $P = \{P_b\}$ of a B-family $D = \{D_b\}$, there is a family of zeroth order finite rank pseudodifferential operators $A = \{A_b\}$ such that each $\tilde{D}_b = D_b + A_b$ is invertible and that P_b is precisely the APS projection of $\tilde{D}_b$. (We will call A a Melrose-Piazza operator associated to the spectral section.)

[1] Our discussion extends without difficulty to the more general case when there are smoothly varying $\mathbb{Z}_2$-graded Hermitian Clifford modules over the fibers, with graded unitary connections.

For our purpose we define a $\hat{\eta}$-form which generalizes both [BC1] and [MP]. The main point is that we need to use general superconnections, not just Bismut superconnection [B]. We first introduce some notations.

Let X be the total space of the fibration whose typical fiber is Z and the base B. Choosing a connection amounts to a splitting

$$TX = TZ \oplus T^H X.$$

We also have the identification $T^H X = \pi^* TB$.

Endow B with a metric g^{TB} and let g^{TX} be the metric defined by

$$g^{TX} = g^{TZ} \oplus \pi^* g^{TB}.$$

Let P, $P^\perp$ be the orthogonal projections of TX onto TZ, $T^H X$ respectively and denote by ∇^{TX}, ∇^{TB} the Levi-Civita connections. Following Bismut [B], let ∇^{TZ} be the connection on the vertical bundle defined by $\nabla^{TZ} = P\nabla^{TX} P$. This is a connection compatible with the metric g^{TZ} and independent of the choice of the metric g^{TB}.

We use $S(TZ)$ to denote the spinor bundle of TZ. Then the connection lifts to a connection on the spinor bundle. Also following Bismut we view $\Gamma(S(TZ) \otimes E)$ as the space of sections of an infinite dimensional vector bundle H_∞ over B, with fiber

$$H_{\infty,b} = \Gamma(S(TZ_b) \otimes E_b).$$

Then $\nabla^{S(TZ)\otimes E}$ determines a connection on H_∞ by the prescription:

$$\tilde{\nabla}_X h = \nabla_{X^H} h.$$

Now, let $\{D_b\}$ be a B-family of self adjoint Dirac type operators as defined previously, and A a Melrose-Piazza operator associated to a spectral section P.

Definition. For any $B_i \in \Omega^i(T^*B) \otimes \Gamma(\mathrm{End}(S(TZ) \otimes E)) = \Omega^i(T^*B) \otimes cl(TZ) \otimes \mathrm{End}(E)$ odd, and any $t > 0$, we define the superconnection B_t to be

$$B_t = \tilde{\nabla} + \sqrt{t}(D + \rho(t)A) + \sum_{i=0}^{\dim B} t^{\frac{1-i}{2}} B_i,$$

where ρ is a cut-off function in t: $\rho(t) = 1$ when $t > 8$ and $\rho(t) = 0$ when $t < 2$.

Definition. We define, for $\Re(s) \gg 0$,

$$\hat{\eta}(P, A, s) = \frac{1}{2\sqrt{\pi}} \int_0^\infty t^{\frac{s}{2}} \mathrm{Tr}^{even}[\frac{dB_t}{dt} e^{-B_t^2}] dt.$$

This defines a differential form on B which depends holomorphically on s for $\Re(s) \gg 0$. By the standard argument it extends to a meromorphic function of s in the whole complex plane with only simple poles. Note that our definition has an extra factor of $\frac{1}{2}$.

Theorem 3.1. *The residue of $\hat{\eta}(P, A, s)$ at $s = 0$ is exact.*

Definition. The $\hat{\eta}$-form is defined as

$$\hat{\eta}(D, P) = \{\hat{\eta}(P, A, s) - \frac{\mathrm{Res}_{s=0}\{\hat{\eta}(P, A, s)\}}{s}\}_{s=0}.$$

Remark. The $\hat{\eta}$-form for a general superconnection is defined in [BC1] in the finite dimensional case.

A variational argument shows

Proposition 3.2. *The value of $\hat{\eta}(D, P)$ in $\Omega^*(B)/d\Omega^*(B)$ is independent of the choice of the cut-off function ρ and the Melrose-Piazza operator A.*

The dependence of $\hat{\eta}$-form on the spectral section is also well understood. The following is a slight generalization of a result of [MP].

Theorem 3.3. *If P_0 and P_1 are spectral sections of the family D, then their difference defines an element $[P_1 - P_0]$ in $K(B)$. Moreover*

$$ch(P_1 - P_0) = \hat{\eta}(D, P_1) - \hat{\eta}(D, P_0) \text{ in } H^*(B).$$

The following theorem generalizes the well known relationship between the spectral flow and the eta invariant.

Theorem 3.4. *Let D_u be a curve of B-families of Dirac type operators and Q_0, Q_1 spectral sections of D_0, D_1 respectively. Let $\tilde{B}_t(u) = \tilde{\nabla} + \sqrt{t}(D_u + \rho(t)\tilde{A}) + \sum_{i\geq 1} t^{\frac{1-i}{2}} B_i(u)$ be a curve of superconnections. Then we have the following identity in $H^*(B)$:*

$$ch(sf\{(D_0, Q_0),\ (D_1, Q_1)\}) = \hat{\eta}(D_1, Q_1) - \hat{\eta}(D_0, Q_0) - \int_0^1 \frac{d\hat{\eta}(D_u, P_u)}{du} du,$$

where the last term is a local invariant computable from the asymptotic expansion

$$\frac{1}{\sqrt{\pi}} \frac{\partial}{\partial s}\{ Tr^{even}[\exp(-\tilde{B}_t^2(u) - s\frac{\partial \tilde{B}_t(u)}{\partial u})]\}|_{s=0} = \sum_{i\geq -k} a_i t^i,$$

i.e., we have

$$\frac{d\hat{\eta}(D_u, P_u)}{du} = a_0(u).$$

This theorem provides a way to compute, analytically, the Chern character of higher spectral flow. We will use Formula (3.4) to compute the higher spectral flow of both a periodic family and a Toeplitz family.

4. Higher spectral flow and family index

We consider a periodic family $D_{b,u}, b \in B,\ u \in S^1$ of B-families of Dirac type operators, where B is a closed manifold. The total family $\tilde{D}_{b,u}$ is a family of Dirac operators on an even dimensional space (the total space of a fibration over S^1 with the typical fiber Z). Thus it defines an index bundle $\mathrm{ind}(\tilde{D}) \in K(B)$. The following theorem generalizes the well known relationship between the spectral flow and the index.

Theorem 4.1. *We have*

$$\mathrm{ch}(\mathrm{ind}(\tilde{D})) = \mathrm{ch}(sf\{D_u\}) \ \textit{in}\ H^*(B).$$

Proof. We apply Theorem 3.4 to compute the higher spectral flow and show that the result agrees with the Atiyah-Singer formula [AS] for the family index.

Remark. It is very likely that the higher spectral flow and the family index actually equal as K-group elements, but at the moment we do not see how to prove this.

As a consequence, we deduce the following relation between the spectral flow and higher spectral flow.

Corollary 4.2. *Let D_u^Z be a periodic family of B-families of Dirac operators and let D_u^X be the one parameter family of the total Dirac operators. Then we have*

$$sf\{D_u^X\} = \int_B \hat{A}(B)ch(sf\{D_u^Z\}).$$

5. Higher spectral flow and Toeplitz family

Now let$\mathbb{C}^N$ be a trivial vector bundle over X with its canonical (trivial) metric and connection. Let

$$g: \ X \longrightarrow \mathrm{GL}(N, \mathbb{C}).$$

Then g acts on the trivial bundle. Moreover it extends in the obvious way to an operator

$$g_b: \ \mathrm{L}^2(S(TZ_b) \otimes E_b) \otimes \mathbb{C}^N \longrightarrow \mathrm{L}^2(S(TZ_b) \otimes E_b) \otimes \mathbb{C}^N.$$

Let P be a spectral section of the B-family D. Then P also extends to an operator

$$P_b: \ \mathrm{L}^2(S(TZ_b) \otimes E_b) \otimes \mathbb{C}^N \longrightarrow \mathrm{L}^2(S(TZ_b) \otimes E_b) \otimes \mathbb{C}^N$$

by acting as identity on the factor $\mathbb{C}^N$. Let $L_{P,b}$ be the image space of $\mathrm{L}^2(S(TZ_b) \otimes E_b) \otimes \mathbb{C}^N$ under P_b.

Definition. For any $b \in B$, define $T(g)_b$ to be the bounded linear operator

$$T(g)_b = P_b g_b : \; L_{P,b} \longrightarrow L_{P,b}.$$

This is a generalization of the notion of Toeplitz operator. $T(g)$ is a continuous family of Fredholm operators parametrized by B, and by Atiyah-Singer [AS], it defines an element

$$\mathrm{ind}_a(T(g)) \in K(B),$$

its index bundle. The index bundle of the Toeplitz family does not depend on the geometric data and the spectral section.

Without loss of generality, we now assume that g is unitary. Extend D to $\mathrm{L}^2(S(TZ_b) \otimes E_b) \otimes \mathbb{C}^N$ in the obvious way. Then P is still a spectral section for D. Moreover, gPg^{-1} is a spectral section for gDg^{-1}. Connecting D and gDg^{-1} by the linear path, we have the following generalization of [BW, Theorem 17.17].

Theorem 5.1. *We have an equality*

$$\mathrm{ch}(\mathrm{ind}_a(T(g))) = \mathrm{ch}(sf\{(D,P),(gDg^{-1},gPg^{-1})\}).$$

Remark. In this theorem one can verify directly that the right hand side does not depend on P.

Proof. Once again we apply Theorem 3.4 to compute the higher spectral flow. On the other hand one can still use the argument of [BD] to compute the family index of the Toeplitz family. The computations show that they are equal. We point out that we shall use a special modification of the Bismut superconnection to make sure that the local index type calculation can be carried out.

As an interesting example we can take D a B-family of signature operators. In this case the higher spectral flow in Theorem 5.1 is usually nontrivial, although the higher spectral flow of a one parameter family of B-families of signature operators is always zero.

Theorem 5.1 is in fact true in K-theory, see [DZ2] for detail[2].

6. An extension of family index theorem

An important property of the spectral flow is that it measures the change of the index for manifolds with boundary under continuous deformation [DZ1]. We show that higher spectral flow measures the change of the family index for manifolds with boundary under continuous deformation.

[2]We thank Krysztof Wojciechowski for helpful discussions.

Let $\pi : M \longrightarrow B$ be a smooth fibration with the typical fiber Y an even dimensional manifold with boundary Z. For simplicity[3] we assume that the vertical tangent bundle $TY \longrightarrow M$ is spin and carries a fixed spin structure. Let g^{TY} be a metric on TY which is of product type near the boundary. Let E be a complex vector bundle over M with an hermitian metric g^E and compatible connection ∇^E. Corresponding to these geometric data we have a family of Dirac operators D_b^E, $b \in B$. This is a family of elliptic operators on the manifold with boundary Y, parametrized by B.

Definition. By a B-family of Dirac type operators on the manifold with boundary Y we mean a family of elliptic operators $\tilde{D}_b$ parametrized by B which is of product type near the boundary and whose principal symbol is the same as that of D_b^E. Moreover, we assume that the boundary family is also a B-family of Dirac type operators.

It follows that spectral sections always exist for the boundary family. Choosing a spectral section, we can then define the family index for this B-family of Dirac type operators on the manifold with boundary as in [MP].

Theorem 6.1. *Let $\tilde{D}_{b,u}$ be a one parameter family of B-families of Dirac type operators on the manifold with boundary Y. Let $\tilde{D}^{\partial}_{b,u}$ denote the boundary family. Let $Q_{b,0}$, $Q_{b,1}$ be spectral sections for $\tilde{D}^{\partial}_{b,0}$, $\tilde{D}^{\partial}_{b,1}$ respectively. Then*

$$\mathrm{ind}(\tilde{D}_1, Q_1) - \mathrm{ind}(\tilde{D}_0, Q_0) = -sf\{(\tilde{D}^{\partial}_0, Q_0),\ (\tilde{D}^{\partial}_1, Q_1)\} \text{ in } K(B).$$

This is a generalization of [DZ1, Theorem 1.1]. Using this result we can prove the following generalization of the family index theorem for manifolds with boundary [BC2], [BC3], [MP].

Theorem 6.2. *Let $\tilde{D}$ be a B-family of Dirac type operators on the manifold with boundary Y and $\tilde{D}^{\partial}$ its boundary family. Let Q be a spectral section of $\tilde{D}^{\partial}$. Then*

$$\mathrm{ch}(\mathrm{ind}(\tilde{D}, Q)) = \int_Y a_0 - \hat{\eta}(\tilde{D}^{\partial}, Q),$$

where a_0 is the constant term in the asymptotic expansion of

$$Tr_s[\exp(-\tilde{B}_t^2)] = \sum_{i \geq -\frac{\dim Y}{2} - [\frac{\dim B}{2}]} a_i t^i,$$

[3] Once again our discussion extends to the more general case.

with an arbitrary total superconnection $\tilde{B}_t$ now extended to the double of the fibration, while $\hat{\eta}(\tilde{D}^\partial, Q)$ is the $\hat{\eta}$-form defined using the boundary superconnection induced from $\tilde{B}_t$.

Proof. We use the family index theorem of [MP] for the Dirac family and flow to the more general Dirac type family.
Remark. This is the analogue in the family case for the general index formula of Atiyah-Patodi-Singer for manifold with boundary [APS2].

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Heat Kernels and the Index Theorems on Even and Odd Dimensional Manifolds*

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Abstract

In this talk, we review the heat kernel approach to the Atiyah-Singer index theorem for Dirac operators on closed manifolds, as well as the Atiyah-Patodi-Singer index theorem for Dirac operators on manifolds with boundary. We also discuss the odd dimensional counterparts of the above results. In particular, we describe a joint result with Xianzhe Dai on an index theorem for Toeplitz operators on odd dimensional manifolds with boundary.

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1. Introduction

As is well-known, the index theorem proved by Atiyah and Singer [AS1] in 1963, which expresses the analytically defined index of elliptic differential operators through purely topological terms, has had a wide range of implications in mathematics as well as in mathematical physics. Moreover, there have been up to now many different proofs of this celebrated result.

The existing proofs of the Atiyah-Singer index theorem can roughly be divided into three categories:

(i) The cobordism proof: this is the proof originally given in [AS1]. It uses the cobordism theory developed by Thom and modifies Hirzebruch's proof of his Signature theorem as well as his Riemann-Roch theorem;

(ii) The K-theoretic proof: this is the proof given by Atiyah and Singer in [AS2]. It modifies Grothendieck's proof of the Hirzebruch-Riemann-Roch theorem and relies on the topological K-theory developed by Atiyah and Hirzebruch. The Bott periodicity theorem plays an important role in this proof;

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(iii) The heat kernel proof: this proof originates from a simple and beautiful formula due to Mckean and Singer [MS], and has closer relations with differential geometry as well as mathematical physics. It also lead directly to the important Atiyah-Patodi-Singer index theorem for Dirac operators on manifolds with boundary.

In this article, we will survey some of the developments concerning the heat kernel proofs of various index theorems, including a recent result with Dai [DZ2] on an index theorem for Toeplitz operators on odd dimensional manifolds with boundary.

2. Heat kernels and the index theorems on even dimensional manifolds

We start with a smooth closed oriented $2n$-dimensional manifold M and two smooth complex vector bundles E, F over M, on which there is an elliptic differential operator between the spaces of smooth sections, $D_+ : \Gamma(E) \to \Gamma(F)$.

If we equip TM with a Riemannian metric and E, F with Hermitian metrics repectively, then $\Gamma(E)$ and $\Gamma(F)$ will carry canonically induced inner products.

Let $D_- : \Gamma(F) \to \Gamma(E)$ be the formal adjoint of D_+ with respect to these inner products. Then the index of D_+ is given by

$$\operatorname{ind} D_+ = \dim (\ker D_+) - \dim (\ker D_-) . \tag{2.1}$$

It is a topological invariant not depending on the metrics on TM, E and F.

The famous Mckean-Singer formula [MS] says that $\operatorname{ind} D_+$ can also be computed by using the heat operators associated to the Laplacians D_-D_+ and D_+D_-. That is, for any $t > 0$, one has

$$\operatorname{ind} D_+ = \operatorname{Tr}\left[\exp\left(-tD_-D_+\right)\right] - \operatorname{Tr}\left[\exp\left(-tD_+D_-\right)\right] . \tag{2.2}$$

By introducing the $\mathbf{Z}_2$-graded vector bundle $E \oplus F$ and setting $D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$, we can rewrite the difference of the two traces in the right hand side of (2.2) as a single "supertrace" as follows,

$$\operatorname{ind} D_+ = \operatorname{Tr}_s\left[\exp\left(-tD^2\right)\right] , \quad \text{for any } t > 0. \tag{2.2$'$}$$

Let $P_t(x, y)$ be the smooth kernel of $\exp(-tD^2)$ with respect to the volume form on M. For any $f \in \Gamma(E \oplus F)$, one has

$$\exp\left(-tD^2\right) f(x) = \int_M P_t(x, y) f(y) dy. \tag{2.3}$$

In particular,

$$\operatorname{Tr}_s\left[\exp\left(-tD^2\right)\right] = \int_M \operatorname{Tr}_s\left[P_t(x, x)\right] dx. \tag{2.4}$$

Now, for simplicity, we assume that the elliptic operator D is of order one. Then by a standard result, which goes back to Minakshisundaram and Pleijiel [MP], one has that when $t>0$ tends to 0,

$$P_t(x,x) = \frac{1}{(4\pi t)^n}\left(a_{-n} + a_{-n+1}t + \cdots + a_0 t^n + o_x\left(t^n\right)\right), \tag{2.5}$$

where $a_i \in \mathrm{End}((E\oplus F)_x)$, $i=-n,\ldots,0$.

By (2.2)', (2.4) and (2.5), and by taking $t>0$ small enough, one deduces that

$$\int_M \mathrm{Tr}_s[a_i]dx = 0, \quad -n\le i<0,$$

$$\mathrm{ind}\, D_+ = \left(\frac{1}{4\pi}\right)^n \int_M \mathrm{Tr}_s[a_0]dx. \tag{2.6}$$

Mckean and Singer conjectured in [MS] that for certain geometric operators, there should be some "fantastic cancellation" so that the following far reaching refinement of (2.6) holds,

$$\mathrm{Tr}_s[a_i] = 0, \quad -n\le i<0,$$

and moreover, $\mathrm{Tr}_s[a_0]$ can be calculated simply in the Chern-Weil geometric theory of characteristic classes.

In fact, as a typical example, let M be an even dimensional compact smooth oriented *spin* manifold carrying a Riemannian metric g^{TM}. Let R^{TM} be the curvature of the Levi-Civita connection associated to g^{TM}. Let $S(TM)=S_+(TM)$ $\oplus S_-(TM)$ be the Hermitian vector bundle of (TM,g^{TM})-spinors, and $D_+ : \Gamma(S_+(TM)) \to \Gamma(S_-(TM))$ the associated *Dirac* operator.

One then has the formula (cf. [BGV, Chap. 4, 5]),

$$\lim_{t\to 0} \mathrm{Tr}_s\left[P_t(x,x)\right]dx = \left\{\widehat{A}\left(\frac{R^{TM}}{2\pi}\right)\right\}^{\max} := \left\{\det{}^{1/2}\left(\frac{\frac{\sqrt{-1}}{4\pi}R^{TM}}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^{TM}\right)}\right)\right\}^{\max}, \tag{2.7}$$

which implies the Atiyah-Singer index theorem [AS1] for D_+:

$$\mathrm{ind}\, D_+ = \widehat{A}(M) := \int_M \widehat{A}\left(\frac{R^{TM}}{2\pi}\right). \tag{2.8}$$

A result of type (2.7) is called a *local index theorem.* The first proof of such a local result was given by V. K. Patodi [P] for the de Rham-Hodge operator $d+d^*$. Other direct heat kernel proofs of (2.7) have been given by Berline-Vergne, Bismut, Getzler and Yu respectively. We refer to [BGV] and [Yu] for more details.

The heat kernel proof of the local index theorem leads to a generalization of the index theorem for Dirac operators to the case of manifolds with boundary. This was achieved by Atiyah, Patodi and Singer in [APS], and will be reviewed in the next section.

3. The index theorem for Dirac operators on even dimensional manifolds with boundary

Let M be a smooth compact oriented even dimensional *spin* manifold with (nonempty) smooth boundary ∂M. Then ∂M is again oriented and *spin.*

Let g^{TM} be a metric on TM. Let $g^{T\partial M}$ be its restriction on $T\partial M$. We assume for simplicity that g^{TM} is of *product structure* near the boundary ∂M. Let $S(TX) = S_+(TX) \oplus S_-(TX)$ be the $\mathbf{Z}_2$-graded Hermitian vector bundle of (TX, g^{TX})-spinors.

Since now M has a nonempty boundary ∂M, the associated Dirac operator $D_+ : \Gamma(S_+(TM)) \to \Gamma(S_-(TM))$ is *not* elliptic. To get an elliptic problem, one needs to introduce an elliptic boundary condition for D_+, and this was achieved by Atiyah, Patodi and Singer in [APS]. It is remarkable that this boundary condition, to be described right now, is *global* in nature.

First of all, the Dirac operator D_+ induces canonically a formally self-adjoint first order elliptic differential operator

$$D_{\partial M} : \Gamma\left(S_+(TM)|_{\partial M}\right) \to \Gamma\left(S_+(TM)|_{\partial M}\right),$$

which is called the induced Dirac operator on the boundary ∂M.

Clearly, the L^2-completion of $S_+(TM)|_{\partial M}$ admits an orthogonal decomposition

$$L^2\left(S_+(TM)|_{\partial X}\right) = \bigoplus_{\lambda \in \mathrm{Spec}(D_{\partial M})} E_\lambda, \tag{3.1}$$

where E_λ is the eigenspace of λ.

Let $L^2_{\geq 0}(S_+(TM)|_{\partial M})$ denote the direct sum of the eigenspaces E_λ associated to the eigenvalues $\lambda \geq 0$. Let $P_{\geq 0}$ denote the orthogonal projection from $L^2(S_+(TM)|_{\partial M})$ to $L^2_{\geq 0}(S_+(TM)|_{\partial M})$. We call $P_{\geq 0}$ the *Atiyah-Patodi-Singer projection* associated to $D_{\partial M}$, to emphasize its role in [APS].

Then by [APS], the boundary problem

$$(D_+, P_{\geq 0}) : \{u : u \in \Gamma(S_+(TM)),\ P_{\geq 0}\left(u|_{\partial M}\right) = 0\} \to \Gamma(S_-(TM)), \tag{3.2}$$

is Fredholm. We call this elliptic boundary problem the Atiyah-Patodi-Singer boundary problem associated to D_+. We denote by $\mathrm{ind}\,(D_+, P_{\geq 0})$ the index of the Fredholm operator (3.2).

The Atiyah-Patodi-Singer index theorem *The following identity holds,*

$$\mathrm{ind}\ (D_+, P_{\geq 0}) = \int_M \widehat{A}\left(\frac{R^{TM}}{2\pi}\right) - \overline{\eta}\left(D_{\partial M}\right). \tag{3.3}$$

The boundary correction term $\overline{\eta}(D_{\partial M})$ appearing in the right hand side of (3.3) is a spectral invariant associated to the induced Dirac operator $D_{\partial M}$ on ∂M. It is defined as follows: for any complex number $s \in \mathbf{C}$ with $\mathrm{Re}(s) > \dim M$, define

$$\eta\left(D_{\partial M}, s\right) = \sum_{\lambda \in \mathrm{Spec}(D_{\partial M})} \frac{\mathrm{sgn}(\lambda)}{|\lambda|^s}. \tag{3.4}$$

By using the heat kernel method, one can show easily that $\eta(D_{\partial M}, s)$ can be extended to a meromorphic function on $\mathbf{C}$, which is holomorphic at $s = 0$. Following [APS], we then define

$$\overline{\eta}\left(D_{\partial M}\right) = \frac{\dim\left(\ker D_{\partial M}\right) + \eta\left(D_{\partial M}, 0\right)}{2} \tag{3.5}$$

and call it the (reduced) eta invariant of $D_{\partial M}$.

The eta invariants of Dirac operators have played important roles in many aspects of topology, geometry and mathematical physics.

In the next sections, we will discuss the role of eta invariants in the heat kernel approaches to the index theorems on odd dimensional manifolds.

4. Heat kernels and the index theorem on odd dimensional manifolds

Let M be now an *odd* dimensional smooth closed oriented *spin* manifold. Let g^{TM} be a Riemannian metric on TM and $S(TM)$ the associated Hermitian vector bundle of (TM, g^{TM})-spinors.[1] In this case, the associated *Dirac* operator $D : \Gamma(TM) \to \Gamma(TM)$ is (formally) *self-adjoint*.[2] Thus, one can proceed as in Section 3 to construct the Atiyah-Patodi-Singer projection

$$P_{\geq 0} : L^2(S(TM)) \to L^2_{\geq 0}(S(TM)).$$

Now consider the trivial vector bundle $\mathbf{C}^N$ over M. We equip $\mathbf{C}^N$ with the canonical trivial metric and connection. Then $P_{\geq 0}$ extends naturally to an orthogonal projection from $L^2(S(TM) \otimes \mathbf{C}^N)$ to $L^2_{\geq 0}(S(TM) \otimes \mathbf{C}^N)$ by acting as identity on $\mathbf{C}^N$. We still denote this extension by $P_{\geq 0}$.

On the other hand, let

$$g : M \to U(N)$$

be a smooth map from M to the unitary group $U(N)$. Then g can be interpreted as automorphism of the trivial complex vector bundle $\mathbf{C}^N$. Moreover g extends naturally to an action on $L^2(S(TM) \otimes \mathbf{C}^N)$ by acting as identity on $L^2(S(TM))$. We still denote this extended action by g.

With the above data given, one can define a *Toeplitz* operator T_g as follows,

$$T_g = P_{\geq 0} g P_{\geq 0} : L^2_{\geq 0}\left(S(TM) \otimes \mathbf{C}^N\right) \longrightarrow L^2_{\geq 0}\left(S(TM) \otimes \mathbf{C}^N\right). \tag{4.1}$$

The first important fact is that T_g is a Fredholm operator. Moreover, it is equivalent to an elliptic pseudodifferential operator of order zero. Thus one can compute its index by using the Atiyah-Singer index theorem [AS2], as was indicated in the paper of Baum and Douglas [BD], and the result is

$$\operatorname{ind} T_g = -\left\langle \widehat{A}(TM)\operatorname{ch}(g), [M] \right\rangle, \tag{4.2}$$

[1] Since now M is of odd dimension, the bundle of spinors does not admit a $\mathbf{Z}_2$-graded structure.

[2] In fact, if M bounds an even dimensional spin manifold, then D can be thought of as the induced Dirac operator on boundary appearing in the previous section.

where $\mathrm{ch}(g)$ is the odd Chern character associated to g.

There is also an analytic proof of (4.2) by using heat kernels. For this one first applies a result of Booss and Wojciechowski (cf. [BW]) to show that the computation of $\mathrm{ind}\, T_g$ is equivalent to the computation of the spectral flow of the linear family of self-adjoint elliptic operators, acting of $\Gamma(S(TM) \otimes \mathbf{C}^N)$, which connects D and gDg^{-1}. The resulting spectral flow can then be computed by variations of η-invariants, where the heat kernels are naturally involved.

The above ideas have been extended in [DZ1] to give a heat kernel proof of a family extension of (4.2).

5. An index theorem for Toeplitz operators on odd dimensional manifolds with boundary

In this section, we describe an extension of (4.2) to the case of manifolds with boundary, which was proved recently in my paper with Xianzhe Dai [DZ2]. This result can be thought of as an odd dimensional analogue of the Atiyah-Patodi-Singer index theorem described in Section 3.

This section is divided into three subsections. In Subsection 4.1, we extend the definition of Toeplitz operators to the case of manifolds with boundary. In Subsection 4.2, we define an η-invariant for cylinders which will appear in the statement of the main result to be described in Subsection 4.3.

5.1. Toeplitz operators on manifolds with boundary

Let M be an odd dimensional oriented *spin* manifold with (nonempty) boundary ∂M. Then ∂M is also oriented and spin. Let g^{TM} be a Riemannian metric on TM such that it is of product structure near the boundary ∂M. Let $S(TM)$ be the Hermitian bundle of spinors associated to (M, g^{TM}). Since $\partial M \neq \emptyset$, the *Dirac* operator $D : \Gamma(S(TM)) \to \Gamma(S(TM))$ is no longer elliptic. To get an elliptic operator, one needs to impose suitable boundary conditions, and it turns out that again we will adopt the boundary conditions introduced by Atiyah, Patodi and Singer [APS].

Let $D_{\partial M} : \Gamma(S(TM)|_{\partial M}) \to \Gamma(S(TM)|_{\partial M})$ be the canonically induced Dirac operator on the boundary ∂M. Then $D_{\partial M}$ is elliptic and (formally) self-adjoint. For simplicity, we assume here that $D_{\partial M}$ is *invertible*, that is, $\ker D_{\partial M} = 0$.

Let $P_{\partial M,\geq 0}$ denote the Atiyah-Patodi-Singer projection from $L^2(S(TM)|_{\partial M})$ to $L^2_{\geq 0}(S(TM)|_{\partial M})$. Then $(D, P_{\partial M,\geq 0})$ forms a *self-adjoint* elliptic boundary problem. We will also denote the corresponding elliptic self-adjoint operator by $D_{P_{\partial M,\geq 0}}$.

Let $L^2_{P_{\partial M,\geq 0},\geq 0}(S(TM))$ be the space of the direct sum of eigenspaces of nonnegative eigenvalues of $D_{P_{\partial M,\geq 0}}$. Let $P_{P_{\partial M,\geq 0},\geq 0}$ denote the orthogonal projection from $L^2(S(TM))$ to $L^2_{P_{\partial M,\geq 0},\geq 0}(S(TM))$.

Now let $\mathbf{C}^N$ be the trivial complex vector bundle over M of rank N, which carries the trivial Hermitian metric and the trivial Hermitian connection. We extend $P_{P_{\partial M,\geq 0},\geq 0}$ to act as identity on $\mathbf{C}^N$.

Let $g : M \to U(N)$ be a smooth unitary automorphism of $\mathbf{C}^N$. Then g extends to an action on $S(TM) \otimes \mathbf{C}^N$ by acting as identity on $S(TM)$.

Since g is unitary, one verifies easily that the operator $gP_{\partial M,\geq 0}g^{-1}$ is an orthogonal projection on $L^2((S(TM)\otimes\mathbf{C}^N)|_{\partial M})$, and that $gP_{\partial M,\geq 0}g^{-1}-P_{\partial M,\geq 0}$ is a pseudodifferential operator of order less than zero. Moreover, the pair $(D, gP_{\partial M,\geq 0}g^{-1})$ forms a *self-adjoint* elliptic boundary problem. We denote its associated elliptic self-adjoint operator by $D_{gP_{\partial M,\geq 0}g^{-1}}$.

Let $L^2_{gP_{\partial M,\geq 0}g^{-1},\geq 0}(S(TM)\otimes\mathbf{C}^N)$ be the space of the direct sum of eigenspaces of nonnegative eigenvalues of $D_{gP_{\partial M,\geq 0}g^{-1}}$. Let $P_{gP_{\partial M,\geq 0}g^{-1},\geq 0}$ denote the orthogonal projection from $L^2(S(TM)\otimes\mathbf{C}^N)$ to $L^2_{gP_{\partial M,\geq 0}g^{-1},\geq 0}(S(TM)\otimes\mathbf{C}^N)$.

Clearly, if $s\in L^2(S(TM)\otimes\mathbf{C}^N)$ verifies $P_{\partial M,\geq 0}(s|_{\partial M})=0$, then gs verifies

$$gP_{\partial M,\geq 0}g^{-1}\left((gs)|_{\partial M}\right)=0.$$

Definition 5.1 The *Toeplitz operator* T_g is defined by

$$T_g=P_{gP_{\partial M,\geq 0}g^{-1},\geq 0}\,g\,P_{P_{\partial M,\geq 0},\geq 0}:$$
$$L^2_{P_{\partial M,\geq 0},\geq 0}\left(S(TM)\otimes\mathbf{C}^N\right)\to L^2_{gP_{\partial M,\geq 0}g^{-1},\geq 0}\left(S(TM)\otimes\mathbf{C}^N\right).$$

One verifies that T_g is a Fredholm operator. The main result of this section evaluates the index of T_g by more geometric quantities.

5.2. An η-invariant associated to g

We consider the cylinder $[0,1]\times\partial M$. Clearly, the restriction of g on ∂M extends canonically to this cylinder.

Let $D|_{[0,1]\times\partial M}$ be the restriction of D on $[0,1]\times\partial M$. We equip the boundary condition $P_{\partial M,\geq 0}$ at $\{0\}\times\partial M$ and the boundary condition $\mathrm{Id}-gP_{\partial M,\geq 0}g^{-1}$ at $\{1\}\times\partial M$. Then $(D|_{[0,1]\times\partial M}, P_{\partial M,\geq 0}, \mathrm{Id}-gP_{\partial M,\geq 0}g^{-1})$ forms a self-adjoint elliptic boundary problem. We denote the corresponding elliptic self-adjoint operator by $D_{P_{\partial M,\geq 0},gP_{\partial M,\geq 0}g^{-1}}$.

Let $\eta(D_{P_{\partial M,\geq 0},gP_{\partial M,\geq 0}g^{-1}},s)$ be the η-function of $s\in\mathbf{C}$ which, when $\mathrm{Re}(s)>>0$, is defined by

$$\eta\left(D_{P_{\partial M,\geq 0},gP_{\partial M,\geq 0}g^{-1}},s\right)=\sum_{\lambda\neq 0}\frac{\mathrm{sgn}(\lambda)}{|\lambda|^s},$$

where λ runs through the nonzero eigenvalues of $D_{P_{\partial M,\geq 0},gP_{\partial M,\geq 0}g^{-1}}$.

It is proved in [DZ2] that under our situation, $\eta(D_{P_{\partial M,\geq 0},gP_{\partial M,\geq 0}g^{-1}},s)$ can be extended to a meromorphic function on $\mathbf{C}$ which is holomorphic at $s=0$.

Let $\overline{\eta}(D_{P_{\partial M,\geq 0},gP_{\partial M,\geq 0}g^{-1}})$ be the reduced η-invariant defined by

$$\overline{\eta}\left(D_{P_{\partial M,\geq 0},gP_{\partial M,\geq 0}g^{-1}}\right)$$
$$=\frac{\dim\ker\left(D_{P_{\partial M,\geq 0},gP_{\partial M,\geq 0}g^{-1}}\right)+\eta\left(D_{P_{\partial M,\geq 0},gP_{\partial M,\geq 0}g^{-1}}\right)}{2}.$$

5.3. An index theorem for T_g

Let ∇^{TM} be the Levi-Civita connection associated to the Riemannian metric g^{TM}. Let $R^{TM} = (\nabla^{TM})^2$ be the curvature of ∇^{TM}. Also, we use d to denote the trivial connection on the trivial vector bundle $\mathbf{C}^N$ over M. Then $g^{-1}dg$ is a $\Gamma(\mathrm{End}(\mathbf{C}^N))$ valued 1-form over M.

Let $\mathrm{ch}(g,d)$ denote the odd Chern character form (cf. [Z]) of (g,d) defined by

$$\mathrm{ch}(g,d) = \sum_{n=0}^{(\dim M-1)/2} \frac{n!}{(2n+1)!}\left(\frac{1}{2\pi\sqrt{-1}}\right)^{n+1} \mathrm{Tr}\left[\left(g^{-1}dg\right)^{2n+1}\right].$$

Let $\mathcal{P}_M$ denote the Calderón projection associated to D on M (cf. [BW]). Then $\mathcal{P}_M$ is an orthogonal projection on $L^2((S(TM)\otimes\mathbf{C}^N)|_{\partial M})$, and that $\mathcal{P}_M - P_{\partial M,\geq 0}$ is a pseudodifferential operator of order less than zero.

Let $\tau_\mu(P_{\partial M,\geq 0}, gP_{\partial M,\geq 0}g^{-1}, \mathcal{P}_M) \in \mathbf{Z}$ be the Maslov triple index in the sense of Kirk and Lesch [KL, Definition 6.8].

We can now state the main result of [DZ2], which generalizes an old result of Douglas and Wojciechowski [DoW], as follows.

Theorem 5.2 *The following identity holds,*

$$\mathrm{ind}\, T_g = -\int_M \widehat{A}\left(\frac{R^{TM}}{2\pi}\right)\mathrm{ch}(g,d) + \overline{\eta}\left(D_{P_{\partial M,\geq 0}, gP_{\partial M,\geq 0}g^{-1}}\right)$$
$$-\tau_\mu\left(P_{\partial M,\geq 0}, gP_{\partial M,\geq 0}g^{-1}, \mathcal{P}_M\right).$$

The following immediate consequence is of independent interests.

Corollary 5.3 *The number*

$$\int_M \widehat{A}\left(\frac{R^{TM}}{2\pi}\right)\mathrm{ch}(g,d) - \overline{\eta}\left(D_{P_{\partial M,\geq 0}, gP_{\partial M,\geq 0}g^{-1}}\right)$$

is an integer.

The strategy of the proof of Theorem 5.2 given in [DZ2] is the same as that of the heat kernel proof of (4.2). However, due to the appearance of the boundary ∂M, one encounters new difficulties. To overcome these difficulties, one makes use of the recent result on the splittings of η invariants (cf. [KL]) as well as some ideas involved in the Connes-Moscovici local index theorem in noncommutative geometry [CM] (see also [CH]). Moreover, the local index calculations appearing near ∂M is highly nontrivial. We refer to [DZ2] for more details.

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