

ON THE MOD k INDEX THEOREM OF FREED AND MELROSE

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The purpose of this short note is to present an alternative approach to a formula of Freed-Melrose [6, Corollary 5.4], which expresses the topological index of vector bundles over Z/k -manifolds through geometric data.

Recall that Freed and Melrose proved their formula by first establishing a general index theorem for Z/k -manifolds and then making an application of the Atiyah-Patodi-Singer index theorem for manifolds with boundary [2].

Our approach is based on a result established jointly by Bismut and the author in [4] concerning the behaviour of the η -invariants under real embeddings. By such approach the use of the Atiyah-Patodi-Singer index theorem mentioned above is avoided. From our argument, it turns out immediately that for certain special dimensions, one can refine the Z/k index formula to a $2Z/2k$ formula. Furthermore, our method also suggests a promised new approach to the Atiyah-Patodi-Singer index theorem itself.

This paper is organized as follows. In Section 1, we recall the basic notation and facts about Z/k -manifolds. In Section 2, we give our approach to the Freed-Melrose formula in which we are interested. Section 3 contains a $2Z/2k$ refinement for dimension $8k + 4$. In the final Section 4, we discuss the Atiyah-Patodi-Singer index theorem for manifolds with boundary [2] from the point of view of our approach.

1. The topological index for Z/k -manifolds

Z/k -manifolds were introduced by Sullivan in his studies of geometric

Received April 8, 1994.

topology. We recall the basic definitions for completeness (cf. Freed [5] and Freed-Melrose [6]).

Definition 1.1. A compact Z/k -manifold is a compact manifold X with boundary ∂X , which has a decomposition $\partial X = \bigcup_{i=1}^k (\partial X)_i$ into k disjoint manifolds and k diffeomorphism $\pi_i : (\partial X)_i \rightarrow Y$ to a closed manifold Y .

Let $\pi : \partial X \rightarrow Y$ be the induced map. We use (X, Y, π) to denote this Z/k -manifold.

Convention 1.2. In what follows, we will call a (covariant) object α (e.g. vector bundles, metrics, connections, etc.) of X a Z/k object if there will be a corresponding object β of Y such that $\alpha|_{\partial X} = \pi^* \beta$.

For simplicity, we make the assumption that X is spin and of even dimension. Then Y is an odd dimensional manifold carrying an induced spin structure.

Let (X', Y', π') be another spin Z/k -manifold. We call an embedding $i_X : X \rightarrow X'$ a Z/k -embedding if there is an embedding $i_Y : Y \rightarrow Y'$ such that

$$(1.1) \quad \pi' i_X|_{\partial X} = i_Y \pi.$$

Let $K(X, Y, \pi)$ be the K -group of (X, Y, π) generated by Z/k complex vector bundles over (X, Y, π) . Denote by $\widetilde{K}(X, Y, \pi)$ the corresponding reduced K -group. The classical construction of direct images in K -theory also works for $K(X, Y, \pi)$ (cf. [5], [6]). In particular, if $E \in K(X, Y, \pi)$ and $i : (X, Y, \pi) \rightarrow (X', Y', \pi')$ is an embedding between even dimensional compact spin Z/k -manifolds, then the direct image $i_* E$ lies in $\widetilde{K}(X', Y', \pi')$.

Example 1.2. Let n be a positive integer. Let $S^{n,k}$ be the Z/k -manifold obtained by removing k open balls D^n from the n -sphere. The identification map, which is obviously defined, will be denoted by $\pi_{n,k}$.

Lemma 1.3 (cf. Freed-Melrose [6]). *One has*

$$(1.2) \quad \widetilde{K}(S^{n,k}, S^{n-1}, \pi_{n,k}) = Z/k.$$

Now let E be a Z/k complex vector bundle over (X, Y, π) , and let $i : (X, Y, \pi) \hookrightarrow (S^{n,k}, S^{n-1}, \pi_{n,k})$ be a Z/k embedding with n even. The existence of such an embedding is clear.

Definition 1.4. (cf. Freed [5] and Freed-Melrose [6]). The Z/k topological index of E is an element in Z/k given by

$$(1.3) \quad \text{ind}_{(k)}(E) = [i_!E] \in Z/k = \widetilde{K}(S^{m,k}, S^{n-1}, \pi_{n,k}).$$

Standard techniques in K -theory can be adapted here to show that $\text{ind}_{(k)}(E)$ does not depend on the embedding i . Furthermore, the following Riemann-Roch property still holds.

Proposition 1.5. *Let E be a Z/k complex vector bundle over (X, Y, π) . Let $i_X : (X, Y, \pi) \hookrightarrow (X', Y', \pi')$ be a Z/k embedding between even dimensional compact spin Z/k manifolds. Then, one has*

$$(1.4) \quad \text{ind}_{(k)}(E) = \text{ind}_{(k)}(i_{X!}E).$$

2. The Freed-Melrose formula for the Z/k index

Let (X, Y, π) be as in Section 1 an even dimensional compact spin Z/k -manifold.

Let g^{TY} be a metric on TY . Let g^{TX} be a Z/k metric on TX such that g^{TX} is a product metric near ∂X and that

$$(2.1) \quad g^{TX} |_{T(\partial X)} = \pi^* g^{TY}.$$

Let ∇^{TX} (resp. ∇^{TY}) be the Levi-Civita connection of g^{TX} (resp. g^{TY}).

Let F be a Z/k complex vector bundle over Y . Let g^F be a metric on F and let ∇^F be a connection on F preserving g^F .

Let E be a Z/k vector bundle over (X, Y, π) such that $E|_{\partial X} = \pi^* F$. Let g^E be a metric on E such that it is a product metric near ∂X and that $g^E|_{\partial X} = \pi^* g^F$. Let ∇^E be a Z/k connection on E preserving g^E such that $\nabla^E|_{\partial X} = \pi^* \nabla^F$ and that ∇^E is a product connection near ∂X .

Let $D_{Y,F}$ be the Dirac operator coupled with F on Y .

If D is a self-adjoint Dirac operator, denote by $\bar{\eta}(D)$ the reduced η -invariant introduced by Atiyah-Patodi-Singer [2].

We will use the same notation as in Bismut-Zhang [4] to express the characteristic forms.

Theorem 2.1. (Freed-Melrose [6, Corollary 5.4]). *The following identity holds,*

$$(2.2) \quad \text{ind}_{(k)}(E) = \int_X \hat{A}(TX, \nabla^{TX}) \text{ch}(E, \nabla^E) - k\bar{\eta}(D_{Y,F}), \pmod k.$$

Remark 2.2. The integrality of the right-hand side of (2.2) is non-trivial. It can be seen as a consequence of the index theorem of Atiyah-Patodi-Singer [2] for manifolds with boundary.

In what follows, we will give a proof of Theorem 2.1 directly, without referring to the Atiyah-Patodi-Singer index theorem.

Proof of Theorem 2.1. Let $i_X : (X, Y, \pi) \hookrightarrow (S^{n,k}, S^{n-1}, \pi_{n,k})$ be a Z/k embedding with n even. Note $i_Y : Y \hookrightarrow S^{n-1}$ the corresponding embedding of Y in S^{n-1} .

Let $\pi_X : N_X \rightarrow X$ (resp. $\pi_Y : N_Y \rightarrow Y$) be the normal bundle to X (resp. Y) in $S^{n,k}$ (resp. S^{n-1}). Then one has

$$(2.3) \quad \pi^*N_Y = N_X |_{\partial X}.$$

Let ξ_+, ξ_- be two Z/k complex vector bundles of a same dimension over $(S^{n,k}, S^{n-1}, \pi_{n,k})$ such that $\xi_+ - \xi_- \in \widetilde{K}(S^{n,k}, S^{n-1}, \pi_{n,k})$ is a representative of $i_!E$. Let μ_+, μ_- be two complex vector bundles over Y satisfying that $\xi_{\pm} |_{\partial S^{n,k}} = \pi_{n,k}^* \mu_{\pm}$. Then by (2.3), $\mu_+ - \mu_- \in \widetilde{K}(S^{n-1})$ is a representative of $i_!F$.

In view of Bismut-Zhang [4, Remark 1.1], we can and will assume that the following analogue of [4, 1.10], to which we also refer for relevant notation, holds, after constructing suitable Z/k metrics and connections on $(TS^{n,k}, TS^{n-1})$ and $\xi = \xi_+ \oplus \xi_-, \mu = \mu_+ \oplus \mu_-$ and a Z/k self-adjoint element $V_X \in \text{End}(\xi)$ with corresponding element $V_Y \in \text{End}(\mu)$,

$$(2.4) \quad \begin{aligned} & (\pi_X^* \text{Ker } V_X, \pi_X^* h^{\text{Ker } V_X}, j_Z V_X(x)) \\ & \simeq (\pi_X^*(F^{N_X^*} \otimes E), \pi_X^* g^{F^{N_X^*} \otimes E}, \tau^{N_X^*} \tilde{c}(Z)), \text{ on } X \setminus \partial X, \end{aligned}$$

$$(2.4') \quad \begin{aligned} & (\pi_Y^* \text{Ker } V_Y, \pi_Y^* h^{\text{Ker } V_Y}, j_Z V_Y(y)) \\ & \simeq (\pi_Y^*(F^{N_Y^*} \otimes F), \pi_Y^* g^{F^{N_Y^*} \otimes F}, \tau^{N_Y^*} \tilde{c}(Z)), \text{ on } Y, \end{aligned}$$

Furthermore, V_X, V_Y are invertible on $S^{n,k} \setminus X, S^{n-1} \setminus Y$ respectively. Also, we can and will impose the product condition near $\partial S^{n,k}$ and ∂X for all objects, and the condition that the embedding $(X, Y) \hookrightarrow (S^{n,k}, S^{n-1})$ is totally geodesic.

Now let $\gamma^{S^{n,k}}, \gamma^{S^{n-1}}$ be the Chern-Simons currents on $S^{n,k}, S^{n-1}$ constructed in [3] and [4], corresponding to (2.4), (2.4') respectively. Recall that they satisfy the following transgression formulas,

$$(2.5) \quad d\gamma^{S^{n,k}} = \text{ch}(\xi, \nabla^\xi) - \hat{A}^{-1}(N_X, \nabla^{N_X}) \text{ch}(E, \nabla^E) \delta_X,$$

$$(2.5') \quad d\gamma^{S^{n-1}} = \text{ch}(\mu, \nabla^\mu) - \hat{A}^{-1}(N_Y, \nabla^{N_Y}) \text{ch}(F, \nabla^F) \delta_Y,$$

where ∇^{N_X} (resp. ∇^{N_Y}) is the orthogonal projection of $\nabla^{TS^{n,k}}|_X$ (resp. $\nabla^{TS^{n-1}}|_Y$) on N_X (resp. N_Y).

Furthermore one has

$$(2.6) \quad \gamma^{S^{n,k}}|_{\partial S^{n,k}} = \pi_{n,k}^* \gamma^{S^{n-1}}.$$

From (2.5), (2.6), one deduces that

$$(2.7) \quad \begin{aligned} & \int_{S^{n,k}} \hat{A}(TS^{n,k}, \nabla^{TS^{n,k}}) \text{ch}(\xi, \nabla^\xi) - \int_X \hat{A}(TX, \nabla^{TX}) \text{ch}(E, \nabla^E) \\ &= k \int_{S^{n-1}} \hat{A}(TS^{n-1}, \nabla^{TS^{n-1}}) \gamma^{S^{n-1}}. \end{aligned}$$

On the other hand, use of [4, Theorem 2.2] yields

$$(2.8) \quad \bar{\eta}(D_{S^{n-1}, \mu}) \equiv \bar{\eta}(D_{Y, F}) + \int_{S^{n-1}} \hat{A}(TS^{n-1}, \nabla^{TS^{n-1}}) \gamma^{S^{n-1}}, \pmod{Z}.$$

By (2.7), (2.8), we get

$$(2.9) \quad \begin{aligned} & \int_{S^{n,k}} \hat{A}(TS^{n,k}, \nabla^{TS^{n,k}}) \text{ch}(\xi, \nabla^\xi) - k \bar{\eta}(D_{S^{n-1}, \mu}) \\ & \equiv \int_X \hat{A}(TX, \nabla^{TX}) \text{ch}(E, \nabla^E) - k \bar{\eta}(D_{Y, F}) \pmod{k}. \end{aligned}$$

Now combining (2.9) with Proposition 1.5, we can reduce the proof of Theorem 2.1 to the case of the Z/k manifold $(S^{n,k}, S^{n-1}, \pi_{n,k})$. This,

in turn, by using (2.9) again for the pair $(X, Y, \pi) = (S^{2,k}, S^1, \pi_{2,k})$, and by the Z/k Thom isomorphism theorem, which can be proved by adapting directly the usual proof of Thom isomorphism in ordinary K -theory, can be reduced to the case of $(S^{2,k}, S^1, \pi_{2,k})$. The proof of Theorem 2.1 can then be completed by the easy calculations already worked out in [5, 1.14]. q.e.d.

Remark 2.3. Although our proof is written out for spin manifolds, the same strategy applies to spin^c-manifolds as well. We leave this to the interested reader.

Remark 2.4. There is also a proof of (2.2) by N. Higson [7], using the K -theory of C^* -algebras and also the Atiyah-Patodi-Singer index theorem for manifolds with boundary [2].

3. A mod 2 refinement for real vector bundles

The purpose of this section is to establish a mod 2 refinement of Theorem 2.1. In fact, by taking $k = 0$ in (2.2), one gets the well-known integrality of the characteristic number $\langle \hat{A}(TX) \text{ch}(E), [X] \rangle$.

Now if $\dim X \equiv 4 \pmod{8}$, $k = 0$ and E is the complexification of a real vector bundle, one has the mod 2 refinement due to Atiyah and Hirzerbruch [1] that $\langle \hat{A}(TX) \text{ch}(E), [X] \rangle$ is an even integer. Our improvement of Theorem 2.1 parallels this refinement of Atiyah-Hirzerbruch in the $k \neq 0$ case.

Thus from now on, we assume that (X, Y, π) is a compact spin Z/k -manifold of dimension $8m + 4$.

Let $\widetilde{KO}(X, Y, \pi)$ be the corresponding reduced KO group of Z/k real vector bundles over (X, Y, π) .

Proposition 3.1. *Let n be a positive integer such that $n \equiv 4 \pmod{8}$. Then one has $\widetilde{KO}(S^{n,k}, S^{n-1}, \pi_{n,k}) = 2Z/2k (= Z/k)$.*

Proof. This can be proved in the same way as Lemma 1.3 from the classical fact that $\widetilde{KO}(S^n) = 2Z$. q.e.d.

According to Proposition 3.1, and using the fact that an $8l$ dimensional spinor space is the complexification of real vector space, we can define the Z/k topological index of a Z/k real vector bundle E over (X, Y, π) as an element in $2Z/2k$:

$$(3.1) \quad \text{ind}_{(k)}(E) \in 2Z/2k.$$

On the other hand, let E_C be the complexification of E . Then we

can define metrics, connections and Dirac operators as in Section 2 for E_C .

Now we can state our improvement of Theorem 2.1 as follows.

Theorem 3.2. *The following identity holds,*

$$(3.2) \quad \text{ind}_{(k)}(E) \equiv \int_X \hat{A}(TX, \nabla^{TX}) \text{ch}(E_C, \nabla^{E_C}) - k\bar{\eta}(D_{Y, F_C}),$$

$$(\text{mod } 2k),$$

where F_C is the complexification of F , which is the real vector bundle over Y corresponding to E .

Proof. The strategy of the proof of (3.2) is the same as our proof of Theorem 2.1. All that one needs to note is the following two points:

(i) Since an $8m + 3$ dimensional spinor space carries a quaternionic structure, $\bar{\eta}(D_{Y, F_C})$ is in fact mod 2 continuous. So as argued similarly in [8], the formula here corresponding to (2.8) holds mod $2Z$;

(ii) Instead of reducing the proof of (2.2) to $(S^{2,k}, S^1, \pi_{2,k})$, here we use the fact that an $8l$ dimensional spinor space is the complexification of a real space, to reduce (3.2) to $(S^{4,k}, S^3, \pi_{4,k})$ for which (3.2) can also be verified easily.

We leave the details to the interested reader. q.e.d.

Remark 3.3. It seems that a similar modification of Freed-Melrose's and/or Higson's argument can also lead to such a mod 2 refinement.

4. Comments in relations with the Atiyah-Patodi-Singer index theorem

We assume $k = 1$ in this Section.

Recall that in this case, Theorem 2.1 is an immediate consequence of the Atiyah-Patodi-Singer index theorem for manifolds with boundary [2].

Now that we have given the direct proof of Theorem 2.1, we naturally hope that the idea of our approach would also be helpful in understanding the Atiyah-Patodi-Singer index theorem itself.

More precisely, in using the notation as in Section 1 and 2, let $D_{S^{n,1}, \xi} + TV_X$ be the Dirac operator on $S^{n,1}$ coupled with the coefficient ξ satisfying the Atiyah-Patodi-Singer boundary condition [2]. Then one gets easily the following result.

Theorem 4.1. *Let T be a nonnegative real number. Then the following identity holds,*

$$(4.1) \quad \text{ind}(D_{S^{n,1},\xi} + TV_X) = \int_{S^{n,1}} \hat{A}(TS^{n,1}, \nabla^{TS^{n,1}}) \text{ch}(\xi, \nabla^\xi) - \bar{\eta}(D_{S^{n-1},\mu} + TV_Y).$$

Proof. Formula (4.1) follows from the Atiyah-Patodi-Singer index theorem [2] for $D_{S^{n,1},\xi} + TV_X$ and the easy local index calculation that

$$(4.2) \quad \lim_{t \rightarrow 0} \text{Tr}_S[\exp(-t(D_{S^{n,1},\xi} + TV_X)^2)(x, x)] d \text{vol}(x) = \{\hat{A}(TS^{n,1}, \nabla^{TS^{n,1}}) \text{ch}(\xi, \nabla^\xi)\}^{max}(x, x), x \in S^{n,1} - S^{n-1}.$$

q.e.d.

Now set $T = 0$ in (4.1). One has,

$$(4.3) \quad \text{ind}(D_{S^{n,1},\xi}) = \int_{S^{n,1}} \hat{A}(TS^{n,1}, \nabla^{TS^{n,1}}) \text{ch}(\xi, \nabla^\xi) - \bar{\eta}(D_{S^{n-1},\mu}).$$

By (4.2), (4.3), we get

$$(4.4) \quad \text{ind}(D_{S^{n,1},\xi} + TV_X) + \bar{\eta}(D_{S^{n-1},\mu} + TV_Y) = \text{ind}(D_{S^{n-1},\xi}) + \bar{\eta}(D_{S^{n-1},\mu}).$$

Clearly, the left-hand side of (4.4) does not depend on $T \in [0, +\infty)$.

Recall that the behaviour of $\bar{\eta}(D_{S^{n-1},\mu} + TV_Y)$ as T tends to ∞ has been studied in Bismut-Zhang [4]. This suggests that a new demonstration of the Atiyah-Patodi-Singer index theorem for Dirac operators could be achieved if we could

- i) prove (4.4) directly;
- ii) study the behaviour of $\text{ind}(D_{S^{n,1},\mu} + TV_X)$ as T is sufficiently large.

We believe that such a strategy, which would yield a K -theoretic proof of the Atiyah-Patodi-Singer index theorem [2], is promising and would inevitably lead to better understandings of the role of Atiyah-Patodi-Singer boundary conditions [2] appearing at so many places in differential geometry and mathematical physics. (Note added in proof: see X. Dai & W. Zhang, C. R. Acad. Sci. Paris, (1) 319 (1994) 1293-1297.)

Acknowledgements

The author is indebted to Professor Jean-Michel Bismut for his kindness and very helpful suggestions. This work was partially supported by NSF grant DMS 9022140 through MSRI, and also by the Chinese National Natural Science Foundation.

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