Journal of Geometry and Physics 62 (2012) 1038–1053

Contents lists available at SciVerse ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Modular forms and generalized anomaly cancellation formulas

Fei Han [a,*], Kefeng Liu [b,c], Weiping Zhang [d]

[a] Department of Mathematics, National University of Singapore, Block S17, 10 Lower Kent Ridge Road, Singapore 119076, Singapore
[b] Department of Mathematics, University of California at Los Angeles, Los Angeles, CA 90095, USA
[c] Center of Mathematical Sciences, Zhejiang University, 310027, PR China
[d] Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, PR China

ARTICLE INFO

Article history:
Received 26 September 2011
Received in revised form 19 January 2012
Accepted 30 January 2012
Available online 6 February 2012

Keywords:
Modular forms
Anomaly cancellation
Pontryagin forms

ABSTRACT

In this paper, we generalize the anomaly cancellation formulas given by Alvarez-Gaumé and Witten (1983), Liu (1995) and Han and Zhang (2004) [1,2,7] to the cases where an auxiliary bundle W and a complex line bundle ξ are involved with no conditions on the first Pontryagin forms being assumed.

0. Introduction

In [1], gravitational anomaly cancellation formulas are derived from direct computations. In particular, in dimension 12, the Alvarez-Gaumé and Witten "miraculous cancellation" formula can be written as

$$\{\widehat{L}(TX,\nabla^{TX})\}^{(12)}=\{8\widehat{A}(TX,\nabla^{TX})\mathrm{ch}(T_{\mathbf{C}}X,\nabla^{T_{\mathbf{C}}X})\}^{(12)}-32\{\widehat{A}(TX,\nabla^{TX})\}^{(12)},\qquad(0.1)$$

where X is a twelve-dimensional Riemannian manifold, ∇^{TX} is the associated Levi-Civita connection, $T_{\mathbf{C}}X$ is the complexification of TX (with the induced Hermitian connection $\nabla^{T_{\mathbf{C}}X}$) and $\widehat{L}(TX,\nabla^{TX})$, $\widehat{A}(TX,\nabla^{TX})$ are the Hirzebruch characteristic forms (see (1.1)).

In [2], Liu generalizes (0.1) to general $8m+4$ dimensions by developing modular invariance properties of characteristic forms. Actually, in [2], Liu obtains a more general cancellation formula by including an auxiliary bundle W. More precisely, assume X to be $8m+4$ dimensional and W be a rank $2l$ Euclidean vector bundle over X with a Euclidean connection ∇^W and curvature $R^W=\nabla^{W,2}$; if $p_1(TX,\nabla^{TX})=p_1(W,\nabla^W)$, then the following identity holds:

$$\left\{\widehat{A}(TX,\nabla^{TX})\det{}^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)\right\}^{(8m+4)}=\sum_{r=0}^{m}2^{l+2m+1-6r}\left\{\widehat{A}(TX,\nabla^{TX})\mathrm{ch}(b_r(T_{\mathbf{C}}X,W_{\mathbf{C}},\mathbf{C}^2))\right\}^{(8m+4)},\qquad(0.2)$$

where the $b_r(T_{\mathbf{C}}X,W_{\mathbf{C}},\mathbf{C}^2)$ are virtual complex vector bundles with connections over X canonically determined by (TX,∇^{TX}) and (W,∇^W). In dimension 12, by direct computation, (0.2) becomes

* Corresponding author.
E-mail addresses: mathanf@nus.edu.sg (F. Han), liu@math.ucla.edu (K. Liu), weiping@nankai.edu.cn (W. Zhang).

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2012.01.016

$$\left\{\widehat{A}(TX,\nabla^{TX})\det{}^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)\right\}^{(12)}$$
$$=2^{l-3}\left\{\widehat{A}(TX,\nabla^{TX})\mathrm{ch}(W_{\mathbf{C}},\nabla^{W_{\mathbf{C}}})\right\}^{(12)}-2^{l-2}(l-4)\left\{\widehat{A}(TX,\nabla^{TX})\right\}^{(12)}. \tag{0.3}$$

When $(TX,\nabla^{TX})=(W,\nabla^W)$, (0.2) gives

$$\frac{1}{8}\left\{\widehat{L}(TX,\nabla^{TX})\right\}^{(8m+4)}=\sum_{r=0}^{m}2^{6m-6r}\left\{\widehat{A}(TX,\nabla^{TX})\mathrm{ch}(b_r(T_{\mathbf{C}}X,T_{\mathbf{C}}X,\mathbf{C}^2))\right\}^{(8m+4)}. \tag{0.4}$$

We obtain, as an application [3], by the Atiyah–Hirzebruch divisibility [4], that (0.4) implies the Ochanine divisibility [5], which asserts that the signature of an $8k+4$-dimensional smooth closed spin manifold is divisible by 16.

To study higher dimensional Rokhlin congruence, Han and Zhang [6,7] extend the "miraculous cancellation" formulas of Alvarez-Gaumé, Witten and Liu to a twisted version where an extra complex line bundle (or equivalently a rank 2 real oriented vector bundle) is involved. More precisely, if ξ is a rank 2 real oriented Euclidean vector bundle equipped with a Euclidean connection ∇^ξ and $c=e(\xi,\nabla^\xi)$ is the associated Euler form, when $p_1(TX,\nabla^{TX})=p_1(W,\nabla^W)$, the following identity holds:

$$\left\{\frac{\widehat{A}(TX,\nabla^{TX})\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)}{\cosh^2\left(\frac{c}{2}\right)}\right\}^{(8m+4)}$$
$$=\sum_{r=0}^{m}2^{l+2m+1-6r}\left\{\widehat{A}(TX,\nabla^{TX})\mathrm{ch}(b_r(T_{\mathbf{C}}X,W_{\mathbf{C}},\xi_{\mathbf{C}}))\cosh\left(\frac{c}{2}\right)\right\}^{(8m+4)}, \tag{0.5}$$

where the $b_r(T_{\mathbf{C}}X,W_{\mathbf{C}},\xi_{\mathbf{C}})$ are virtual complex vector bundles with connections over X canonically determined by (TX,∇^{TX}), (W,∇^W) and (ξ,∇^ξ). Obviously, when ξ is trivial and $c=0$, (0.5) reduces to (0.2).

When $\dim X=12$ and $(TX,\nabla^{TX})=(W,\nabla^W)$, (0.5) gives

$$\left\{\frac{\widehat{L}(TX,\nabla^{TX})}{\cosh^2\left(\frac{c}{2}\right)}\right\}^{(12)}$$
$$=\left\{\left[8\widehat{A}(TX,\nabla^{TX})\mathrm{ch}(T_{\mathbf{C}}X,\nabla^{T_{\mathbf{C}}X})-32\widehat{A}(TX,\nabla^{TX})-24\widehat{A}(TX,\nabla^{TX})(e^c+e^{-c}-2)\right]\cosh\left(\frac{c}{2}\right)\right\}^{(12)}, \tag{0.6}$$

which extends the Alvarez-Gaumé and Witten "miraculous cancellation" formula (0.1) in dimension 12.

Note that (0.2) and (0.5) only hold under the condition $p_1(TX,\nabla^{TX})=p_1(W,\nabla^W)$. In this paper, we study what happens if we remove this condition. We find that the difference between the left hand sides and the right hand sides in (0.2) and (0.5) can actually be written in the form

$$(p_1(TX,\nabla^{TX})-p_1(W,\nabla^W))\cdot\mathcal{R},$$

where $\mathcal{R}$ is some characteristic form canonically determined by (TX,∇^{TX}), (W,∇^W) and (ξ,∇^ξ). For example, we find that in dimension 12, the following identity holds (for simplicity, we drop the connections):

$$\left\{\widehat{A}(TX)\det{}^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)\right\}^{(12)}-2^{l-3}\left\{\widehat{A}(TX)\mathrm{ch}(W_{\mathbf{C}})\right\}^{(12)}+2^{l-2}(l-4)\left\{\widehat{A}(TX)\right\}^{(12)}$$
$$=(p_1(TX)-p_1(W))\left\{\frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))}-1}{p_1(TX)-p_1(W)}\right.$$
$$\left.\times\left[\widehat{A}(TX)\left(2^{l-3}\mathrm{ch}(W_{\mathbf{C}})-2^{l-2}(l-4)\right)-\widehat{A}(TX)\det{}^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)\right]\right\}^{(8)}. \tag{0.7}$$

We will give similar general results for $8m+4$ and $8m$ dimensions in Theorem 1.1 and discuss various special cases in Corollaries 1.2–1.5. We obtain our generalized anomaly cancellation formulas still by developing modular invariance of characteristic forms.

In obtaining our cancellation formulas, we were also inspired by the Green–Schwarz mechanism. In [8], Green and Schwarz discovered that the anomaly in type I string theory with the gauge group SO(32) cancels because of an extra "classical" contribution from a 2-form field. One key step is that when the gauge group is 496 dimensional, the anomaly I_{12} can be written as (cf. [9])

$$I_{12}=(p_1(Z)-p_1(F))I_8. \tag{0.8}$$

Our cancellation formulas in Theorem 1.1 and its corollaries are of the same pattern. We hope that they will find applications in physics.

1. Results

The purpose of this section is to state our main results. We first recall the definitions of some characteristic forms to be used in Section 1.1 and then present our generalized anomaly cancellation formulas in Section 1.2.

1.1. Some characteristic forms

Let X be a $4k$-dimensional Riemannian manifold. Let ∇^{TX} be the associated Levi-Civita connection and $R^{TX} = \nabla^{TX,\,2}$ be the curvature of ∇^{TX}. Let $\widehat{A}(TX, \nabla^{TX})$ and $\widehat{L}(TX, \nabla^{TX})$ be the Hirzebruch characteristic forms defined respectively by (cf. [10])

$$\begin{aligned}\widehat{A}(TX, \nabla^{TX}) &= \det{}^{1/2}\left(\frac{\frac{\sqrt{-1}}{4\pi}R^{TX}}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^{TX}\right)}\right),\\ \widehat{L}(TX, \nabla^{TX}) &= \det{}^{1/2}\left(\frac{\frac{\sqrt{-1}}{2\pi}R^{TX}}{\tanh\left(\frac{\sqrt{-1}}{4\pi}R^{TX}\right)}\right).\end{aligned} \tag{1.1}$$

Let F, G be two Hermitian vector bundles over X carrying Hermitian connections ∇^F, ∇^G respectively. Let $R^F = \nabla^{F,\,2}$ (resp. $R^G = \nabla^{G,\,2}$) be the curvature of ∇^F (resp. ∇^G). If we set the formal difference $H = F - G$, then H carries an induced Hermitian connection ∇^H in an obvious sense. We define the associated Chern character form as (cf. [10])

$$\mathrm{ch}(H, \nabla^H) = \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^F\right)\right] - \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^G\right)\right]. \tag{1.2}$$

For any complex number t, let

$$\Lambda_t(F) = \mathbf{C}|_X + tF + t^2\Lambda^2(F) + \cdots, \qquad S_t(F) = \mathbf{C}|_X + tF + t^2S^2(F) + \cdots$$

denote respectively the total exterior and symmetric powers of F, which live in $K(X)[[t]]$. The following relations between these two operations [11, Chap. 3] hold:

$$S_t(F) = \frac{1}{\Lambda_{-t}(F)}, \qquad \Lambda_t(F - G) = \frac{\Lambda_t(F)}{\Lambda_t(G)}. \tag{1.3}$$

The connections ∇^F, ∇^G naturally induce connections on Λ_tF, S_tF etc. Moreover, if $\{\omega_i\}, \{\omega_j{}'\}$ are formal Chern roots for Hermitian vector bundles F, G respectively, then [12, Chap. 1]

$$\mathrm{ch}\left(\Lambda_t(F), \nabla^{\Lambda_t(F)}\right) = \prod_i(1 + e^{\omega_i}t). \tag{1.4}$$

We have the following formulas for Chern character forms:

$$\mathrm{ch}\left(S_t(F), \nabla^{S_t(F)}\right) = \frac{1}{\mathrm{ch}\left(\Lambda_{-t}(F), \nabla^{\Lambda_{-t}(F)}\right)} = \frac{1}{\prod_i(1 - e^{\omega_i}t)}, \tag{1.5}$$

$$\mathrm{ch}\left(\Lambda_t(F - G), \nabla^{\Lambda_t(F-G)}\right) = \frac{\mathrm{ch}\left(\Lambda_t(F), \nabla^{\Lambda_t(F)}\right)}{\mathrm{ch}\left(\Lambda_t(G), \nabla^{\Lambda_t(G)}\right)} = \frac{\prod_i(1 + e^{\omega_i}t)}{\prod_j(1 + e^{\omega_j{}'}t)}. \tag{1.6}$$

1.2. Statement of the results

We make the same assumptions and use the same notation as in Section 1.1.

Let W be a rank $2l$ real Euclidean vector bundle over X carrying a Euclidean connection ∇^W. Let $R^W = \nabla^{W,2}$ be the curvature of ∇^W. If W is the spin, let $\Delta(W) = S^+(W) \oplus S^-(W)$ be the spinor bundle of W with the induced connection $\nabla^{\Delta(W)}$. It is not hard to see that

$$\mathrm{ch}(\Delta(W), \nabla^{\Delta(W)}) = \det{}^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right).$$

Let $p_1(TX, \nabla^X)$ and $p_1(W, \nabla^W)$ be the first Pontryagin forms of (TX, ∇^{TX}) and (W, ∇^W) respectively.

Let ξ be a rank 2 real oriented Euclidean vector bundle over X carrying a Euclidean connection ∇^{ξ}. Let $c = e(\xi, \nabla^{\xi})$ be the Euler form canonically associated with ∇^{ξ}.

For simplicity, from now on, when there is no ambiguity, we will write characteristic forms without specifying the connections.

In the following, we will define some virtual bundles with connections and some differential forms on X associated with (TX, ∇^{TX}), (W, ∇^{W}) and (ξ, ∇^{ξ}).

If E is a vector bundle (real or complex) over X, set $\widetilde{E} = E - \dim E$ in $KO(X)$ or $K(X)$.

If E is a real Euclidean vector bundle over X carrying a Euclidean connection ∇^{E}, then its complexification $E_{\mathbf{C}} = E \otimes \mathbf{C}$ is a complex vector bundle over X carrying a canonically induced Hermitian metric derived from that of E, as well as a Hermitian connection $\nabla^{E_{\mathbf{C}}}$ induced from ∇^{E}.

If ω is a differential form, denote the degree j component of ω by $\omega^{(j)}$.

Let $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in \mathbf{H}$, the upper half of the complex plane. Let $T_{\mathbf{C}}X$ be the complexification of TX.

Set the Witten bundle (cf. [6,7,13])

$$\Theta_2(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}) = \bigotimes_{u=1}^{\infty} S_{q^u}(\widetilde{T_{\mathbf{C}}X}) \otimes \bigotimes_{v=1}^{\infty} \Lambda_{-q^{v-\frac{1}{2}}}(\widetilde{W_{\mathbf{C}}} - 2\widetilde{\xi_{\mathbf{C}}}) \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_{\mathbf{C}}}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{q^s}(\widetilde{\xi_{\mathbf{C}}}), \tag{1.7}$$

which is an element in $K(X)[[q^{\frac{1}{2}}]]$.

Clearly, $\Theta_2(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})$ admits a formal Fourier expansion in $q^{1/2}$ as

$$\Theta_2(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}) = B_0(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}) + B_1(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})q^{1/2} + \cdots, \tag{1.8}$$

where the B_j are elements in the semi-group formally generated by complex vector bundles over X. Moreover, they carry canonically induced connections denoted by ∇^{B_j} and we let ∇^{Θ_2} be the induced connections with $q^{1/2}$-coefficients on Θ_2 from the ∇^{B_j}.

Consider the following q-series:

$$\delta_1(\tau) = \frac{1}{4} + 6\sum_{n=1}^{\infty} \sum_{\substack{d|n \\ d \text{ odd}}} dq^n = \frac{1}{4} + 6q + 6q^2 + \cdots, \tag{1.9}$$

$$\varepsilon_1(\tau) = \frac{1}{16} + \sum_{n=1}^{\infty} \sum_{d|n} (-1)^d d^3 q^n = \frac{1}{16} - q + 7q^2 + \cdots, \tag{1.10}$$

$$\delta_2(\tau) = -\frac{1}{8} - 3\sum_{n=1}^{\infty} \sum_{\substack{d|n \\ d \text{ odd}}} dq^{n/2} = -\frac{1}{8} - 3q^{1/2} - 3q - \cdots, \tag{1.11}$$

$$\varepsilon_2(\tau) = \sum_{n=1}^{\infty} \sum_{\substack{d|n \\ n/d \text{ odd}}} d^3 q^{n/2} = q^{1/2} + 8q + \cdots, \tag{1.12}$$

and the Eisenstein series

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\left(\sum_{d|n} d\right) q^n = 1 - 24q - 72q^2 - 96q^3 - \cdots.$$

Remark 1.1. δ_1 and ε_1 will only be used later in the proof of our results. We list them here for completeness.

Now we define the virtual bundles with connections and the differential forms on X associated with (TX, ∇^{TX}), (W, ∇^{W}) and (ξ, ∇^{ξ}). This will be done in two cases separately.

Case 1: $\dim X = 8m + 4$.

Define virtual complex vector bundles $b_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})$ on X, $0 \le r \le m$, via the equality

$$\Theta_2(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}) \equiv \sum_{r=0}^{m} b_r (8\delta_2)^{2m+1-2r} \varepsilon_2^r \bmod q^{\frac{m+1}{2}} \cdot K(X)[[q^{\frac{1}{2}}]]. \tag{1.13}$$

Explicitly, define $b_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})$ by comparing the q-coefficients of the following identity:

$$\begin{aligned} &B_0(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}) + B_1(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})q^{1/2} + \cdots \\ &\quad = b_0(-1 - 24q^{1/2} - 24q - \cdots)^{2m+1} + b_1(-1 - 24q^{1/2} - 24q - \cdots)^{2m-1}(q^{1/2} + 8q + \cdots) \\ &\qquad + b_2(-1 - 24q^{1/2} - 24q - \cdots)^{2m-3}(q^{1/2} + 8q + \cdots)^2 \\ &\qquad + \cdots + b_r(-1 - 24q^{1/2} - 24q - \cdots)^{2m+1-2r}(q^{1/2} + 8q + \cdots)^r + \cdots \\ &\qquad + b_m(-1 - 24q^{1/2} - 24q - \cdots)(q^{1/2} + 8q + \cdots)^m \bmod q^{\frac{m+1}{2}} \cdot K(X)[[q^{\frac{1}{2}}]]. \end{aligned}$$

Note that on the right hand side of the above equality, the coefficient of $q^{r/2}$, $1 \le r \le m$, must be of the form

$$(-1)^{2m+1-2r}b_r + a_{r,r-1}b_{r-1} + a_{r,r-2}b_{r-2} + \cdots + a_{r,0}b_0,$$

where the $a_{r,j}$, $1 \le j \le r-1$, are all integers. One has

$$\begin{aligned}
B_0 &= -b_0,\\
B_1 &= -b_1 - 24(2m+1)b_0,\\
&\cdots\\
B_r &= -b_r + a_{r,r-1}b_{r-1} + a_{r,r-2}b_{r-2} + \cdots + a_{r,0}b_0,\\
&\cdots\\
B_m &= -b_m + a_{m,m-1}b_{m-1} + a_{m,m-2}b_{m-2} + \cdots + a_{m,0}b_0.
\end{aligned}$$

One can see by induction that each b_r, $0 \le r \le m$, is a canonical integral linear combination of $B_j(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})$, $0 \le j \le r$. These b_r carry canonically induced metrics and connections.

By expanding the right hand side of (1.7), we have

$$B_0 = \mathbf{C}, \qquad B_1 = -W_{\mathbf{C}} + 3\xi_{\mathbf{C}} + \mathbf{C}^{2l-6}.$$

Therefore, one has

$$b_0 = -B_0 = -\mathbf{C}, \qquad b_1 = -B_1 - 24(m+1)b_0 = W_{\mathbf{C}} - 3\xi_{\mathbf{C}} + \mathbf{C}^{48m-2l+30}. \tag{1.14}$$

Define degree $8m$ differential forms $\beta_r(\nabla^{TX}, \nabla^W, \nabla^{\xi})$ on X, $0 \le r \le m$, via the equality

$$\left\{ \frac{e^{\frac{1}{24}E_2(\tau)(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \mathrm{ch}\left(\Theta_2(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})\right) \right\}^{(8m)}$$
$$\equiv \sum_{r=0}^{m} \beta_r (8\delta_2)^{2m+1-2r} \varepsilon_2^r \mod q^{\frac{m+1}{2}} \cdot \Omega^{8m}(X)[\![q^{\frac{1}{2}}]\!]. \tag{1.15}$$

Expanding $\frac{e^{\frac{1}{24}E_2(\tau)(p_1(TX)-p_1(W))}-1}{p_1(TX)-p_1(W)}$, we have

$$\begin{aligned}
\frac{e^{\frac{1}{24}E_2(\tau)(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} &= \frac{1}{24}E_2(\tau) + \frac{1}{2!}\left(\frac{1}{24}E_2(\tau)\right)^2 (p_1(TX) - p_1(W))\\
&\quad + \frac{1}{3!}\left(\frac{1}{24}E_2(\tau)\right)^3 (p_1(TX) - p_1(W))^2 + \cdots\\
&= \sum_{j=0}^{\infty} c_j q^j,
\end{aligned}$$

where the c_j, $j \ge 0$, are all polynomials of $(p_1(TX) - p_1(W))$ with rational coefficients. For instance,

$$\begin{aligned}
c_0 &= \frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)},\\
c_1 &= -1 - \frac{1}{2!} \times 2 \times \frac{1}{24}(p_1(TX) - p_1(W)) - \frac{1}{3!} \times 3 \times \frac{1}{24^2}(p_1(TX) - p_1(W))^2 - \cdots.
\end{aligned}$$

From (1.15), we see that the β_r, $0 \le r \le m$, are defined by

$$\begin{aligned}
&\left\{ \left(\sum_{j=0}^{\infty} c_j q^j\right) \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \mathrm{ch}\left(B_0 + B_1 q^{1/2} + B_2 q + \cdots\right) \right\}^{(8m)}\\
&\quad = \beta_0(-1 - 24q^{1/2} - 24q - \cdots)^{2m+1} + \beta_1(-1 - 24q^{1/2} - 24q - \cdots)^{2m-1}(q^{1/2} + 8q + \cdots)\\
&\qquad + \beta_2(-1 - 24q^{1/2} - 24q - \cdots)^{2m-3}(q^{1/2} + 8q + \cdots)^2\\
&\qquad + \cdots + \beta_r(-1 - 24q^{1/2} - 24q - \cdots)^{2m+1-2r}(q^{1/2} + 8q + \cdots)^r + \cdots\\
&\qquad + \beta_m(-1 - 24q^{1/2} - 24q - \cdots)(q^{1/2} + 8q + \cdots)^m\\
&\quad = \sum_{r=0}^{m} (-\beta_r + a_{r,r-1}\beta_{r-1} + a_{r,r-2}\beta_{r-2} + \cdots + a_{r,0}\beta_0) q^{r/2} \mod q^{\frac{m+1}{2}} \cdot K(X)[\![q^{\frac{1}{2}}]\!].
\end{aligned}$$

Since the $c_j, j \geq 0$, are all polynomials of $(p_1(TX) - p_1(W))$, one can see by induction that each β_r, $0 \leq r \leq m$, is a canonical linear combination of degree $8m$ forms of the type $\{c(p_1(TX) - p_1(W))^a \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \mathrm{ch}(B_j)\}^{(8m)}, \ 0 \leq j \leq r$.

From the above definitions of β_r, $0 \leq r \leq m$, we see that

$$-\beta_0 = \left\{c_0 \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \mathrm{ch}(B_0)\right\}^{(8m)},$$

$$-\beta_1 - 24(2m+1)\beta_0 = \left\{c_0 \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \mathrm{ch}(B_1)\right\}^{(8m)}.$$

Therefore by using B_0, B_1 calculated above, we have

$$\beta_0 = -\left\{\frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \widehat{A}(TX) \cosh\left(\frac{c}{2}\right)\right\}^{(8m)},$$

$$\begin{aligned} \beta_1 &= -24(2m+1)\beta_0 - \left\{c_0 \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \mathrm{ch}(B_1)\right\}^{(8m)} \\ &= \left\{\frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) (\mathrm{ch}(W_{\mathbf{C}} - 3\xi_{\mathbf{C}}) + 48m - 2l + 30)\right\}^{(8m)}. \end{aligned} \tag{1.16}$$

We would like to point out that although

$$\beta_0 = \left\{\frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \mathrm{ch}(b_0)\right\}^{(8m)}$$

and

$$\beta_1 = \left\{\frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \mathrm{ch}(b_1)\right\}^{(8m)},$$

generally,

$$\beta_r \neq \left\{\frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \mathrm{ch}(b_r)\right\}^{(8m)}, \quad r > 2.$$

Case 2: $\dim X = 8m$.

Define virtual complex vector bundles $z_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})$ on X, $0 \leq r \leq m$, via the equality

$$\Theta_2(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}) \equiv \sum_{r=0}^{m} z_r (8\delta_2)^{2m-2r} \varepsilon_2^r \bmod q^{\frac{m+1}{2}} \cdot K(M)[[q^{\frac{1}{2}}]]. \tag{1.17}$$

Explicitly, define $z_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})$ by comparing the q-coefficients of the following identity:

$$\begin{aligned} & B_0(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}) + B_1(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})q^{1/2} + \cdots \\ &\quad = z_0(-1 - 24q^{1/2} - 24q - \cdots)^{2m} + z_1(-1 - 24q^{1/2} - 24q - \cdots)^{2m-2}(q^{1/2} + 8q + \cdots) \\ &\qquad + z_2(-1 - 24q^{1/2} - 24q - \cdots)^{2m-4}(q^{1/2} + 8q + \cdots)^2 \\ &\qquad + \cdots + z_r(-1 - 24q^{1/2} - 24q - \cdots)^{2m-2r}(q^{1/2} + 8q + \cdots)^r + \cdots \\ &\qquad + z_m(q^{1/2} + 8q + \cdots)^m \bmod q^{\frac{m+1}{2}} \cdot K(X)[[q^{\frac{1}{2}}]]. \end{aligned}$$

Note that on the right hand side of the above equality, the coefficient of $q^{r/2}$, $1 \leq r \leq m$, must be of the form

$$(-1)^{2m-2r} z_r + d_{r,r-1} z_{r-1} + d_{r,r-2} z_{r-2} + \cdots + d_{r,0} z_0,$$

where the $d_{r,j}$, $1 \leq j \leq r-1$, are all integers. One has

$$\begin{aligned} & B_0 = z_0, \\ & B_1 = z_1 + 48m z_0, \\ & \cdots \\ & B_r = z_r + d_{r,r-1} z_{r-1} + d_{r,r-2} z_{r-2} + \cdots + d_{r,0} z_0, \\ & \cdots \\ & B_m = z_m + d_{m,m-1} z_{m-1} + d_{m,m-2} z_{m-2} + \cdots + d_{m,0} z_0. \end{aligned}$$

By induction, each $z_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})$, $0 \le r \le m$, is a canonical integral linear combination of the $B_j(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})$, $0 \le j \le r$. These z_r carry canonically induced metrics and connections.

We have

$$\begin{aligned} z_0 &= \mathbf{C}, \\ z_1 &= B_1 - 48mz_0 = -W_{\mathbf{C}} + 3\xi_{\mathbf{C}} - \mathbf{C}^{48m-2l+6}. \end{aligned} \tag{1.18}$$

Define degree $8m-4$ differential forms $\zeta_r(\nabla^{TX}, \nabla^W, \nabla^\xi)$ on X, $0 \le r \le m$, via the equality

$$\begin{aligned} &\left\{ \frac{e^{\frac{1}{24}E_2(\tau)(p_1(TX)-p_1(W))}-1}{p_1(TX)-p_1(W)} \widehat{A}(TX)\cosh\left(\frac{c}{2}\right) \mathrm{ch}\left(\Theta_2(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})\right) \right\}^{(8m-4)} \\ &\quad \equiv \sum_{r=0}^{m} \zeta_r (8\delta_2)^{2m-2r} \varepsilon_2^r \bmod q^{\frac{m+1}{2}} \cdot \Omega^{8m-4}(X)[\![q^{\frac{1}{2}}]\!]. \end{aligned} \tag{1.19}$$

Explicitly, the ζ_r, $0 \le r \le m$, are defined via

$$\begin{aligned} &\left\{ \left(\sum_{j=0}^{\infty} c_j q^j\right) \widehat{A}(TX)\cosh\left(\frac{c}{2}\right) \mathrm{ch}\left(B_0 + B_1 q^{1/2} + B_2 q + \cdots\right) \right\}^{(8m-4)} \\ &\quad = \zeta_0(-1-24q^{1/2}-24q-\cdots)^{2m} + \zeta_1(-1-24q^{1/2}-24q-\cdots)^{2m-2}(q^{1/2}+8q+\cdots) \\ &\qquad + \zeta_2(-1-24q^{1/2}-24q-\cdots)^{2m-4}(q^{1/2}+8q+\cdots)^2 \\ &\qquad + \cdots + \zeta_r(-1-24q^{1/2}-24q-\cdots)^{2m-2r}(q^{1/2}+8q+\cdots)^r + \cdots + \zeta_m(q^{1/2}+8q+\cdots)^m \\ &\quad = \sum_{r=0}^{m} (\zeta_r + d_{r,r-1}\zeta_{r-1} + d_{r,r-2}\zeta_{r-2} + \cdots + d_{r,0}\zeta_0) q^{r/2} \bmod q^{\frac{m+1}{2}} \cdot K(X)[\![q^{\frac{1}{2}}]\!]. \end{aligned}$$

Like for β_r, by induction, we see that each ζ_r, $0 \le r \le m$, is a canonical linear combination of degree $8m-4$ forms of the type $\{c(p_1(TX)-p_1(W))^a \widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(B_j)\}^{(8m-4)}$, $0 \le j \le r$.

We can see that

$$\begin{aligned} \zeta_0 &= \left\{ c_0 \widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(B_0) \right\}^{(8m-4)}, \\ \zeta_1 + 48m\zeta_0 &= \left\{ c_0 \widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(B_1) \right\}^{(8m-4)}. \end{aligned}$$

So we have

$$\begin{aligned} \zeta_0 &= \left\{ \frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))}-1}{p_1(TX)-p_1(W)} \widehat{A}(TX)\cosh\left(\frac{c}{2}\right) \right\}^{(8m-4)}, \\ \zeta_1 &= \left\{ c_0 \widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(B_1) \right\}^{(8m-4)} - 48m\zeta_0 \\ &= -\left\{ \frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))}-1}{p_1(TX)-p_1(W)} \widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\left(\mathrm{ch}(W_{\mathbf{C}}-3\xi_{\mathbf{C}}) + 48m - 2l + 6\right) \right\}^{(8m-4)}. \end{aligned} \tag{1.20}$$

Like for β_r, generally,

$$\zeta_r \neq \left\{ \frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))}-1}{p_1(TX)-p_1(W)} \widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(z_r) \right\}^{(8m-4)}, \quad r > 2.$$

We can now state our main theorem as follows.

Theorem 1.1. (1) *When* $\dim X = 8m+4$, *one has*

$$\begin{aligned} &\left\{ \frac{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)}{\cosh^2\left(\frac{c}{2}\right)} \right\}^{(8m+4)} \\ &\quad - \sum_{r=0}^{m} 2^{l+2m+1-6r} \left\{ \widehat{A}(TX)\mathrm{ch}(b_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}))\cosh\left(\frac{c}{2}\right) \right\}^{(8m+4)} \\ &\quad = (p_1(TX)-p_1(W)) \cdot \mathfrak{B}(\nabla^{TX}, \nabla^W, \nabla^\xi), \end{aligned} \tag{1.21}$$

where

$$\mathfrak{B}(\nabla^{TX}, \nabla^{W}, \nabla^{\xi}) = \sum_{r=0}^{m} 2^{l+2m+1-6r} \beta_r(\nabla^{TX}, \nabla^{W}, \nabla^{\xi}) - \left\{ \frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX)-p_1(W)} \cdot \frac{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)}{\cosh^2\left(\frac{c}{2}\right)} \right\}^{(8m)}. \tag{1.22}$$

(2) *When* $\dim X = 8m$, *one has*

$$\left\{ \frac{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)}{\cosh^2\left(\frac{c}{2}\right)} \right\}^{(8m)} - \sum_{r=0}^{m} 2^{l+2m-6r} \left\{ \widehat{A}(TX)\mathrm{ch}(z_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}))\cosh\left(\frac{c}{2}\right) \right\}^{(8m)}$$
$$= (p_1(TX)-p_1(W)) \cdot \mathfrak{Z}(\nabla^{TX}, \nabla^{W}, \nabla^{\xi}), \tag{1.23}$$

where

$$\mathfrak{Z}(\nabla^{TX}, \nabla^{W}, \nabla^{\xi}) = \sum_{r=0}^{m} 2^{l+2m-6r} \zeta_r(\nabla^{TX}, \nabla^{W}, \nabla^{\xi}) - \left\{ \frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX)-p_1(W)} \cdot \frac{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)}{\cosh^2\left(\frac{c}{2}\right)} \right\}^{(8m-4)}. \tag{1.24}$$

We immediately obtain that

Corollary 1.2 (*Han and Zhang [7]*). *If* $p_1(TX, \nabla^{TX}) = p_1(W, \nabla^W)$, *then*
(1) *when* $\dim X = 8m+4$, *the following identity holds:*

$$\left\{ \frac{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)}{\cosh^2\left(\frac{c}{2}\right)} \right\}^{(8m+4)} = \sum_{r=0}^{m} 2^{l+2m+1-6r} \left\{ \widehat{A}(TX)\mathrm{ch}(b_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}))\cosh\left(\frac{c}{2}\right) \right\}^{(8m+4)}; \tag{1.25}$$

(2) *when* $\dim X = 8m$, *the following identity holds:*

$$\left\{ \frac{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)}{\cosh^2\left(\frac{c}{2}\right)} \right\}^{(8m)} = \sum_{r=0}^{m} 2^{l+2m-6r} \left\{ \widehat{A}(TX)\mathrm{ch}(z_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}))\cosh\left(\frac{c}{2}\right) \right\}^{(8m)}. \tag{1.26}$$

In Corollary 1.2, when $\dim X = 8m+4$ and $(W, \nabla^W) = (TX, \nabla^{TX})$, one has

$$\frac{1}{8}\left\{ \frac{\widehat{L}(TX)}{\cosh^2\left(\frac{c}{2}\right)} \right\}^{(8m+4)} = \sum_{r=0}^{m} 2^{6m-6r} \left\{ \widehat{A}(TX)\mathrm{ch}(b_r(T_{\mathbf{C}}X, T_{\mathbf{C}}X, \xi_{\mathbf{C}}))\cosh\left(\frac{c}{2}\right) \right\}^{(8m+4)}. \tag{1.27}$$

This formula is used in [7] to study higher dimensional Rokhlin type congruences.

When $(\xi, \nabla^\xi) = (\mathbf{R}^2, d)$, from Theorem 1.1, we obtain that

Corollary 1.3. (1) *When* $\dim X = 8m+4$, *one has*

$$\left\{ \widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right) \right\}^{(8m+4)} - \sum_{r=0}^{m} 2^{l+2m+1-6r} \left\{ \widehat{A}(TX)\mathrm{ch}(b_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \mathbf{C}^2)) \right\}^{(8m+4)}$$
$$= (p_1(TX)-p_1(W)) \cdot \mathfrak{B}(\nabla^{TX}, \nabla^{W}, d), \tag{1.28}$$

where

$$\mathfrak{B}(\nabla^{TX}, \nabla^{W}, d) = \sum_{r=0}^{m} 2^{l+2m+1-6r} \beta_r(\nabla^{TX}, \nabla^{W}, d) - \left\{ \frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX)-p_1(W)} \cdot \widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right) \right\}^{(8m)}. \tag{1.29}$$

(2) *When* $\dim X = 8m$, *one has*

$$\left\{\widehat{A}(TX)\det{}^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)\right\}^{(8m)} - \sum_{r=0}^{m} 2^{l+2m-6r}\left\{\widehat{A}(TX)\mathrm{ch}(z_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \mathbf{C}^2))\right\}^{(8m)}$$
$$= (p_1(TX) - p_1(W)) \cdot \mathfrak{Z}(\nabla^{TX}, \nabla^W, d), \tag{1.30}$$

where

$$\mathfrak{Z}(\nabla^{TX}, \nabla^W, d) = \sum_{r=0}^{m} 2^{l+2m-6r}\zeta_r(\nabla^{TX}, \nabla^W, d)$$
$$- \left\{\frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \cdot \widehat{A}(TX)\det{}^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)\right\}^{(8m-4)}. \tag{1.31}$$

It is interesting to notice that the above anomaly cancellation formulas also imply some integrality results.

From Corollary 1.3, we can see that if X is an $8m+4$-dimensional closed spin manifold and W is a $2l$-dimensional spin vector bundle over X, then by the Atiyah–Singer index theorem,

$$\int_X (p_1(TX) - p_1(W)) \cdot \mathfrak{B}(\nabla^{TX}, \nabla^W, d) = \mathrm{Ind}(D^X \otimes \Delta(W)) - \sum_{r=0}^{m} 2^{l+2m+1-6r}\mathrm{Ind}D^X \otimes b_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \mathbf{C}^2), \tag{1.32}$$

where D^X is the Dirac operator on X and $\Delta(W)$ is the spinor bundle of W. So when $l \geq 4m-1$, $\int_X (p_1(TX) - p_1(W)) \cdot \mathfrak{B}(\nabla^{TX}, \nabla^W, d)$ is an integer. Moreover, if X is a string manifold ($p_1(X) = 0$),

$$\int_X p_1(W) \cdot \mathfrak{B}(\nabla^{TX}, \nabla^W, d) \tag{1.33}$$

is an integer.

Similarly, we can see that if X is an $8m$-dimensional closed spin manifold and W is a $2l$-dimensional spin vector bundle over X, then by the Atiyah–Singer index theorem,

$$\int_X (p_1(TX) - p_1(W)) \cdot \mathfrak{Z}(\nabla^{TX}, \nabla^W, d) = \mathrm{Ind}(D^X \otimes \Delta(W)) - \sum_{r=0}^{m} 2^{l+2m-6r}\mathrm{Ind}D^X \otimes b_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \mathbf{C}^2). \tag{1.34}$$

So when $l \geq 4m$,

$$\int_X (p_1(TX) - p_1(W)) \cdot \mathfrak{Z}(\nabla^{TX}, \nabla^W, d) \tag{1.35}$$

is an integer. Moreover, if X is a string manifold, then

$$\int_X p_1(W) \cdot \mathfrak{Z}(\nabla^{TX}, \nabla^W, d) \tag{1.36}$$

is an integer.

From Corollary 1.3, we immediately obtain:

Corollary 1.4 (*Liu [2]*)**.** *If* $p_1(TX, \nabla^{TX}) = p_1(W, \nabla^W)$, *then*

(1) *When* $\dim X = 8m+4$, *the following identity holds:*

$$\left\{\widehat{A}(TX)\det{}^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)\right\}^{(8m+4)} = \sum_{r=0}^{m} 2^{l+2m+1-6r}\left\{\widehat{A}(TX)\mathrm{ch}(b_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \mathbf{C}^2))\right\}^{(8m+4)}; \tag{1.37}$$

(2) *When* $\dim X = 8m$, *one has*

$$\left\{\widehat{A}(TX)\det{}^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)\right\}^{(8m)} = \sum_{r=0}^{m} 2^{l+2m-6r}\left\{\widehat{A}(TX)\mathrm{ch}(z_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \mathbf{C}^2))\right\}^{(8m)}. \tag{1.38}$$

In Corollary 1.4, when $\dim X = 8m+4$ and $(W, \nabla^W) = (TX, \nabla^{TX})$, one has [2,3]

$$\frac{1}{8}\left\{\widehat{L}(TX)\right\}^{(8m+4)} = \sum_{r=0}^{m} 2^{6m-6r}\left\{\widehat{A}(TX)\mathrm{ch}(b_r(T_{\mathbf{C}}X, T_{\mathbf{C}}X, \mathbf{C}^2))\right\}^{(8m+4)}. \tag{1.39}$$

This formula implies the Ochanine divisibility [5], which asserts that the signature of an $8k+4$-dimensional smooth closed spin manifold is divisible by 16.

We give, as examples, the explicit formulas for when the dimension of X is 4, 8 and 12. Using (1.14), (1.16), (1.18) and (1.20), by direct computations, we have:

Corollary 1.5. (1) *When* $\dim X = 4$, *the following identities hold:*

$$\left\{\frac{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)}{\cosh^2\left(\frac{c}{2}\right)}\right\}^{(4)} + 2^{l+1}\left\{\widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\right\}^{(4)} = -2^{l-3}(p_1(TX)-p_1(W)), \tag{1.40}$$

$$\left\{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)\right\}^{(4)} + 2^{l+1}\left\{\widehat{A}(TX)\right\}^{(4)} = -2^{l-3}(p_1(TX)-p_1(W)); \tag{1.41}$$

(2) *when* $\dim X = 8$, *the following identities hold:*

$$\begin{aligned}
&\left\{\frac{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)}{\cosh^2\left(\frac{c}{2}\right)}\right\}^{(8)} \\
&\quad - \left\{\left[-2^{l-4}\widehat{A}(TX)\mathrm{ch}(W_{\mathbf{C}}) + 2^{l-3}(l+8)\widehat{A}(TX) + 3\cdot 2^{l-4}\widehat{A}(TX)(e^c+e^{-c}-2)\right]\cosh\left(\frac{c}{2}\right)\right\}^{(8)} \\
&= (p_1(TX)-p_1(W))\left\{\frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))}-1}{p_1(TX)-p_1(W)}\right. \\
&\quad \times\left[\widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\left(-2^{l-4}\mathrm{ch}(W_{\mathbf{C}}) + 2^{l-3}(l+8) + 3\cdot 2^{l-4}(e^c+e^{-c}-2)\right)\right. \\
&\quad \left.\left. - \frac{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)}{\cosh^2\left(\frac{c}{2}\right)}\right]\right\}^{(4)},
\end{aligned} \tag{1.42}$$

$$\begin{aligned}
&\left\{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)\right\}^{(8)} + 2^{l-4}\left\{\widehat{A}(TX)\mathrm{ch}(W_{\mathbf{C}})\right\}^{(8)} - 2^{l-3}(l+8)\left\{\widehat{A}(TX)\right\}^{(8)} \\
&= (p_1(TX)-p_1(W))\left\{\frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))}-1}{p_1(TX)-p_1(W)}\right. \\
&\quad \left.\times\left[\widehat{A}(TX)\left(-2^{l-4}\mathrm{ch}(W_{\mathbf{C}}) + 2^{l-3}(l+8)\right) - \widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)\right]\right\}^{(4)};
\end{aligned} \tag{1.43}$$

(3) *when* $\dim X = 12$, *the following identities hold:*

$$\begin{aligned}
&\left\{\frac{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)}{\cosh^2\left(\frac{c}{2}\right)}\right\}^{(12)} \\
&\quad - \left\{\left[2^{l-3}\widehat{A}(TX)\mathrm{ch}(W_{\mathbf{C}}) - 2^{l-2}(l-4)\widehat{A}(TX) - 3\cdot 2^{l-3}\widehat{A}(TX)(e^c+e^{-c}-2)\right]\cosh\left(\frac{c}{2}\right)\right\}^{(12)} \\
&= (p_1(TX)-p_1(W))\left\{\frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))}-1}{p_1(TX)-p_1(W)}\right.
\end{aligned}$$

$$\times\left[\widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\left(2^{l-3}\mathrm{ch}(W_{\mathbf{C}})-2^{l-2}(l-4)-3\cdot 2^{l-3}(e^{c}+e^{-c}-2)\right)\right.$$
$$\left.\left.-\frac{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)}{\cosh^2\left(\frac{c}{2}\right)}\right]\right\}^{(8)}, \tag{1.44}$$

$$\left\{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)\right\}^{(12)}-2^{l-3}\left\{\widehat{A}(TX)\mathrm{ch}(W_{\mathbf{C}})\right\}^{(12)}+2^{l-2}(l-4)\left\{\widehat{A}(TX)\right\}^{(12)}$$
$$=(p_1(TX)-p_1(W))\left\{\frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))}-1}{p_1(TX)-p_1(W)}\right.$$
$$\left.\times\left[\widehat{A}(TX)\left(2^{l-3}\mathrm{ch}(W_{\mathbf{C}})-2^{l-2}(l-4)\right)-\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)\right]\right\}^{(8)}. \tag{1.45}$$

Remark 1.2. It is not hard to see that (1.41)–(1.44) are respectively equivalent to the following identities:

$$\left\{e^{\frac{1}{24}(p_1(TX)-p_1(W))}\left[\frac{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)}{\cosh^2\left(\frac{c}{2}\right)}\right.\right.$$
$$\left.\left.-\widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\left(-2^{l-4}\mathrm{ch}(W_{\mathbf{C}})+2^{l-3}(l+8)+3\cdot 2^{l-4}(e^{c}+e^{-c}-2)\right)\right]\right\}^{(8)}=0, \tag{1.46}$$

$$\left\{e^{\frac{1}{24}(p_1(TX)-p_1(W))}\left[\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)-\widehat{A}(TX)\left(-2^{l-4}\mathrm{ch}(W_{\mathbf{C}})+2^{l-3}(l+8)\right)\right]\right\}^{(8)}=0; \tag{1.47}$$

$$\left\{e^{\frac{1}{24}(p_1(TX)-p_1(W))}\left[-\frac{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)}{\cosh^2\left(\frac{c}{2}\right)}\right.\right.$$
$$\left.\left.-\widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\left(2^{l-3}\mathrm{ch}(W_{\mathbf{C}})-2^{l-2}(l-4)-3\cdot 2^{l-3}(e^{c}+e^{-c}-2)\right)\right]\right\}^{(12)}=0, \tag{1.48}$$

$$\left\{e^{\frac{1}{24}(p_1(TX)-p_1(W))}\left[\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)-\widehat{A}(TX)\left(2^{l-3}\mathrm{ch}(W_{\mathbf{C}})-2^{l-2}(l-4)\right)\right]\right\}^{(12)}=0. \tag{1.49}$$

These formulas are simply in the form of products of $e^{\frac{1}{24}(p_1(TX)-p_1(W))}$ with the original anomaly cancellation formulas in dimensions 8 and 12 holding under the condition $p_1(X)=p_1(W)$.

However, as pointed out on page 6 and page 7, concerning the patterns of β_r and ζ_r for $r\geq 2$, we know that the anomaly cancellation formulas for higher (>12) dimensions are not as simple as the above lower anomaly cancellation formulas, i.e. they are not simply in the form of products of $e^{\frac{1}{24}(p_1(TX)-p_1(W))}$ with the original anomaly cancellation formulas holding under the condition $p_1(X)=p_1(W)$.

2. Proofs

In this section, we give the proof of Theorem 1.1. To prepare for the proof in Section 2.2, we will first recall some basic knowledge about the Jacobi theta functions, modular forms and Eisenstein series in Section 2.1.

2.1. Preliminaries

Let

$$SL_2(\mathbf{Z}):=\left\{\left.\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right| a,b,c,d\in\mathbf{Z},\ ad-bc=1\right\}$$

as usual be the modular group. Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

be the two generators of $SL_2(\mathbf{Z})$. Their actions on $\mathbf{H}$ are given by

$$S : \tau \to -\frac{1}{\tau}, \qquad T : \tau \to \tau + 1.$$

The four Jacobi theta functions are defined as follows (cf. [14]):

$$\theta(v,\tau) = 2q^{1/8}\sin(\pi v)\prod_{j=1}^{\infty}\left[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^j)(1-e^{-2\pi\sqrt{-1}v}q^j)\right],$$

$$\theta_1(v,\tau) = 2q^{1/8}\cos(\pi v)\prod_{j=1}^{\infty}\left[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^j)(1+e^{-2\pi\sqrt{-1}v}q^j)\right],$$

$$\theta_2(v,\tau) = \prod_{j=1}^{\infty}\left[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^{j-1/2})(1-e^{-2\pi\sqrt{-1}v}q^{j-1/2})\right],$$

$$\theta_3(v,\tau) = \prod_{j=1}^{\infty}\left[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^{j-1/2})(1+e^{-2\pi\sqrt{-1}v}q^{j-1/2})\right].$$

They are all holomorphic functions for $(v, \tau) \in \mathbf{C} \times \mathbf{H}$, where $\mathbf{C}$ is the complex plane and $\mathbf{H}$ is the upper half-plane.

If we act on the theta functions with S and T, the theta functions obey the following transformation laws (cf. [14]):

$$\theta(v,\tau+1) = e^{\frac{\pi\sqrt{-1}}{4}}\theta(v,\tau), \qquad \theta(v,-1/\tau) = \frac{1}{\sqrt{-1}}\left(\frac{\tau}{\sqrt{-1}}\right)^{1/2} e^{\pi\sqrt{-1}\tau v^2}\theta(\tau v,\tau); \tag{2.1}$$

$$\theta_1(v,\tau+1) = e^{\frac{\pi\sqrt{-1}}{4}}\theta_1(v,\tau), \qquad \theta_1(v,-1/\tau) = \left(\frac{\tau}{\sqrt{-1}}\right)^{1/2} e^{\pi\sqrt{-1}\tau v^2}\theta_2(\tau v,\tau); \tag{2.2}$$

$$\theta_2(v,\tau+1) = \theta_3(v,\tau), \qquad \theta_2(v,-1/\tau) = \left(\frac{\tau}{\sqrt{-1}}\right)^{1/2} e^{\pi\sqrt{-1}\tau v^2}\theta_1(\tau v,\tau); \tag{2.3}$$

$$\theta_3(v,\tau+1) = \theta_2(v,\tau), \qquad \theta_3(v,-1/\tau) = \left(\frac{\tau}{\sqrt{-1}}\right)^{1/2} e^{\pi\sqrt{-1}\tau v^2}\theta_3(\tau v,\tau). \tag{2.4}$$

Definition 2.1. Let Γ be a subgroup of $SL_2(\mathbf{Z})$. A modular form over Γ is a holomorphic function $f(\tau)$ on $\mathbf{H} \cup \{\infty\}$ such that for any

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

the following property holds:

$$f(g\tau) := f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(g)(c\tau+d)^l f(\tau),$$

where $\chi : \Gamma \to \mathbf{C}^*$ is a character of Γ and l is called the weight of f.

Let

$$E_{2k} = 1 - \frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\left(\sum_{d|n} d^{2k-1}\right)q^n \tag{2.5}$$

be the Eisenstein series, where B_{2k} is the $2k$th Bernoulli number.

When $k > 1$, E_{2k} is a modular form of weight $2k$ over $SL_2(\mathbf{Z})$. However, unlike for other Eisenstein theories, $E_2(\tau)$ is not a modular form over $SL(2, \mathbf{Z})$; instead $E_2(\tau)$ is a quasimodular form over $SL(2, \mathbf{Z})$, satisfying

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2E_2(\tau) - \frac{6\sqrt{-1}c(c\tau+d)}{\pi}. \tag{2.6}$$

In particular, we have

$$E_2(\tau+1)=E_2(\tau), \tag{2.7}$$

$$E_2\left(-\frac{1}{\tau}\right)=\tau^2E_2(\tau)-\frac{6\sqrt{-1}\tau}{\pi}. \tag{2.8}$$

For the precise definition of quasimodular forms, see [15].

In the following, let us review some level 2 modular forms.

Let

$$\Gamma_0(2)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})\,\middle|\, c\equiv 0\ (\mathrm{mod}\ 2)\right\},$$

$$\Gamma^0(2)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})\,\middle|\, b\equiv 0\ (\mathrm{mod}\ 2)\right\}$$

be the two modular subgroups of $SL_2(\mathbf{Z})$. It is known that the generators of $\Gamma_0(2)$ are T, ST^2ST and the generators of $\Gamma^0(2)$ are STS, T^2STS (cf. [14]).

Writing simply $\theta_j=\theta_j(0,\tau),\ 1\le j\le 3$, we have (cf. [16,17])

$$\delta_1(\tau)=\frac{1}{8}(\theta_2^4+\theta_3^4),\qquad \varepsilon_1(\tau)=\frac{1}{16}\theta_2^4\theta_3^4,$$

$$\delta_2(\tau)=-\frac{1}{8}(\theta_1^4+\theta_3^4),\qquad \varepsilon_2(\tau)=\frac{1}{16}\theta_1^4\theta_3^4.$$

If Γ is a modular subgroup, let $M_{\mathbf{R}}(\Gamma)$ denote the ring of modular forms over Γ with real Fourier coefficients.

Lemma 2.1 (*Cf. [2]*). *One has that $\delta_1(\tau)$ (resp. $\varepsilon_1(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma_0(2)$, $\delta_2(\tau)$ (resp. $\varepsilon_2(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma^0(2)$ and moreover $M_{\mathbf{R}}(\Gamma^0(2))=\mathbf{R}[\delta_2(\tau),\varepsilon_2(\tau)]$. Moreover, we have the transformation laws*

$$\delta_2\left(-\frac{1}{\tau}\right)=\tau^2\delta_1(\tau),\qquad \varepsilon_2\left(-\frac{1}{\tau}\right)=\tau^4\varepsilon_1(\tau). \tag{2.9}$$

2.2. Proof of Theorem 1.1

Set

$$\Theta_1(T_{\mathbf{C}}X,W_{\mathbf{C}},\xi_{\mathbf{C}})=\bigotimes_{u=1}^{\infty}S_{q^u}(\widetilde{T_{\mathbf{C}}X})\otimes\bigotimes_{v=1}^{\infty}\Lambda_{q^v}(\widetilde{W_{\mathbf{C}}}-2\widetilde{\xi_{\mathbf{C}}})\otimes\bigotimes_{r=1}^{\infty}\Lambda_{q^{r-1/2}}(\widetilde{\xi_{\mathbf{C}}})\otimes\bigotimes_{s=1}^{\infty}\Lambda_{-q^{s-1/2}}(\widetilde{\xi_{\mathbf{C}}}). \tag{2.10}$$

$\Theta_1(T_{\mathbf{C}}X,W_{\mathbf{C}},\xi_{\mathbf{C}})$ admits a formal Fourier expansion in $q^{1/2}$ as

$$\Theta_1(T_{\mathbf{C}}X,W_{\mathbf{C}},\xi_{\mathbf{C}})=A_0(T_{\mathbf{C}}X,W_{\mathbf{C}},\xi_{\mathbf{C}})+A_1(T_{\mathbf{C}}X,W_{\mathbf{C}},\xi_{\mathbf{C}})q^{1/2}+\cdots, \tag{2.11}$$

where the A_j are elements in the semi-group formally generated by complex vector bundles over X. Moreover, they carry canonically induced connections denoted by ∇^{A_j}, and we let ∇^{Θ_1} be the induced connections with $q^{1/2}$-coefficients on Θ_1 from the ∇^{A_j}.

To prove part 1 of Theorem 1.1 (the $8m+4$-dimensional case), set

$$P_1(\tau):=\left\{e^{\frac{1}{24}E_2(\tau)(p_1(TX)-p_1(W))}\frac{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)}{\cosh^2\left(\frac{c}{2}\right)}\mathrm{ch}\left(\Theta_1(T_{\mathbf{C}}X,W_{\mathbf{C}},\xi_{\mathbf{C}})\right)\right\}^{(8m+4)}, \tag{2.12}$$

$$P_2(\tau):=\left\{\widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\mathrm{ch}\left(\Theta_2(T_{\mathbf{C}}X,W_{\mathbf{C}},\xi_{\mathbf{C}})\right)\right\}^{(8m+4)} \tag{2.13}$$

and

$$\Xi_2(\tau):=\left\{\frac{e^{\frac{1}{24}E_2(\tau)(p_1(TX)-p_1(W))}-1}{p_1(TX)-p_1(W)}\widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\mathrm{ch}\left(\Theta_2(T_{\mathbf{C}}X,W_{\mathbf{C}},\xi_{\mathbf{C}})\right)\right\}^{(8m)}. \tag{2.14}$$

We have

Proposition 2.1. *$P_1(\tau)$ is a modular form of weight $4m+2$ over $\Gamma_0(2)$ while $P_2(\tau)+(p_1(TX)-p_1(W))\Xi_2(\tau)$ is a modular form of weight $4m+2$ over $\Gamma^0(2)$. Moreover, the following identity holds:*

$$P_1\left(-\frac{1}{\tau}\right)=2^l\tau^{4m+2}(P_2(\tau)+(p_1(TX)-p_1(W))\Xi_2(\tau)). \tag{2.15}$$

Proof. Let $\{\pm 2\pi\sqrt{-1}y_k\}$ (resp. $\{\pm 2\pi\sqrt{-1}x_j\}$) be the formal Chern roots for $(W_{\mathbf{C}},\nabla^{W_{\mathbf{C}}})$ (resp. $(TM_{\mathbf{C}},\nabla^{TM_{\mathbf{C}}})$). Let $c=2\pi\sqrt{-1}u$.

By the Chern root algorithm, we have (cf. [6,7,18])

$$P_1(\tau)=2^l\left\{e^{\frac{1}{24}E_2(\tau)(p_1(TX)-p_1(W))}\left(\prod_{j=1}^{4m+2}\left(x_j\frac{\theta'(0,\tau)}{\theta(x_j,\tau)}\right)\right)\left(\prod_{k=1}^{l}\frac{\theta_1(y_k,\tau)}{\theta_1(0,\tau)}\right)\frac{\theta_1^2(0,\tau)}{\theta_1^2(u,\tau)}\frac{\theta_3(u,\tau)}{\theta_3(0,\tau)}\frac{\theta_2(u,\tau)}{\theta_2(0,\tau)}\right\}^{(8m+4)}, \tag{2.16}$$

and

$$\begin{aligned}&P_2(\tau)+(p_1(TX)-p_1(W))\Xi_2(\tau)\\&=\left\{e^{\frac{1}{24}E_2(\tau)(p_1(TX)-p_1(W))}\widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\mathrm{ch}\left(\Theta_2(T_{\mathbf{C}}X,W_{\mathbf{C}},\xi_{\mathbf{C}})\right)\right\}^{(8m+4)}\\&=\left\{e^{\frac{1}{24}E_2(\tau)(p_1(TX)-p_1(W))}\left(\prod_{j=1}^{4m+2}\left(x_j\frac{\theta'(0,\tau)}{\theta(x_j,\tau)}\right)\right)\left(\prod_{j=1}^{l}\frac{\theta_2(y_j,\tau)}{\theta_2(0,\tau)}\right)\frac{\theta_2^2(0,\tau)}{\theta_2^2(u,\tau)}\frac{\theta_3(u,\tau)}{\theta_3(0,\tau)}\frac{\theta_1(u,\tau)}{\theta_1(0,\tau)}\right\}^{(8m+4)}.\end{aligned} \tag{2.17}$$

Then we can apply the transformation laws (2.1)–(2.4) for theta functions as well as the transformation laws (2.7)–(2.8) to (2.16) and (2.17) to get the desired results. □

We can now proceed to prove part 1 of Theorem 1.1 as follows.

Combining Lemma 2.1 and Proposition 2.1, we can write

$$P_2(\tau)+(p_1(TX)-p_1(W))\Xi_2(\tau)=h_0(8\delta_2)^{2m+1}+h_1(8\delta_2)^{2m-1}\varepsilon_2+\cdots+h_m(8\delta_2)\varepsilon_2^m, \tag{2.18}$$

where $h_r\in\Omega^{8m+4}(X)$, $0\le r\le m$.

By the definitions of $b_r(T_{\mathbf{C}}X,W_{\mathbf{C}},\xi_{\mathbf{C}})$ and $\beta_r(\nabla^{TX},\nabla^W,\nabla^\xi)$, it is easy to see that for $0\le r\le m$,

$$h_r=\left\{\widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(b_r(T_{\mathbf{C}}X,W_{\mathbf{C}},\xi_{\mathbf{C}}))\right\}^{(8m+4)}+(p_1(TX)-p_1(W))\beta_r(\nabla^{TX},\nabla^W,\nabla^\xi). \tag{2.19}$$

Therefore (simply denoting $b_r(T_{\mathbf{C}}X,W_{\mathbf{C}},\xi_{\mathbf{C}})$ and $\beta_r(\nabla^{TX},\nabla^W,\nabla^\xi)$ by b_r and β_r),

$$P_2(\tau)+(p_1(TX)-p_1(W))\Xi_2(\tau)=\sum_{r=0}^{m}\left(\left\{\widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(b_r)\right\}^{(8m+4)}+(p_1(TX)-p_1(W))\beta_r\right)(8\delta_2)^{2m+1-r}\varepsilon_2^r. \tag{2.20}$$

By (2.9) and (2.15), we have

$$\begin{aligned}P_1(\tau)&=\frac{2^l}{\tau^{4m+2}}\left[P_2\left(-\frac{1}{\tau}\right)+(p_1(TX)-p_1(W))\Xi_2\left(-\frac{1}{\tau}\right)\right]\\&=\frac{2^l}{\tau^{4m+2}}\left[\left(\left(\left\{\widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(b_0)\right\}^{(8m+4)}+(p_1(TX)-p_1(W))\beta_0\right)\right.\right.\\&\quad\times\left(8\delta_2\left(-\frac{1}{\tau}\right)\right)^{2m+1}+\cdots+\left(\left\{\widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(b_m)\right\}^{(8m+4)}+(p_1(TX)-p_1(W))\beta_m\right)\\&\quad\left.\times\ 8\delta_2\left(-\frac{1}{\tau}\right)\left(\varepsilon_2\left(-\frac{1}{\tau}\right)\right)^m\right]\\&=2^l\left[\left(\left(\left\{\widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(b_0)\right\}^{(8m+4)}+(p_1(TX)-p_1(W))\beta_0\right)(8\delta_1)^{2m+1}+\cdots\right.\right.\\&\quad\left.+\left(\left\{\widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(b_m)\right\}^{(8m+4)}+(p_1(TX)-p_1(W))\beta_m\right)(8\delta_1)\varepsilon_1^m\right].\end{aligned} \tag{2.21}$$

Comparing the constant terms of the two sides of (2.21), one has

$$e^{\frac{1}{24}(p_1(TX)-p_1(W))}\frac{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)}{\cosh^2\left(\frac{c}{2}\right)}$$
$$=\sum_{r=0}^{m}2^{l+2m+1-6r}\left(\left\{\widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(b_r)\right\}^{(8m+4)}+(p_1(TX)-p_1(W))\beta_r\right). \tag{2.22}$$

So we have

$$\left\{\frac{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)}{\cosh^2\left(\frac{c}{2}\right)}\right\}^{(8m+4)}-\sum_{r=0}^{m}2^{l+2m+1-6r}\left\{\widehat{A}(TX)\mathrm{ch}(b_r(T_{\mathbf{C}}X,W_{\mathbf{C}},\xi_{\mathbf{C}}))\cosh\left(\frac{c}{2}\right)\right\}^{(8m+4)}$$
$$=(p_1(TX)-p_1(W))\cdot\mathfrak{B}(\nabla^{TX},\nabla^W,\nabla^\xi), \tag{2.23}$$

where

$$\mathfrak{B}(\nabla^{TX},\nabla^W,\nabla^\xi)=\sum_{r=0}^{m}2^{l+2m+1-6r}\beta_r(\nabla^{TX},\nabla^W,\nabla^\xi)$$
$$-\left\{\frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))}-1}{p_1(TX)-p_1(W)}\cdot\frac{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)}{\cosh^2\left(\frac{c}{2}\right)}\right\}^{(8m)}.$$

To prove part 2 of Theorem 1.1 (the $8m$-dimensional case), set

$$Q_1(\tau):=\left\{e^{\frac{1}{24}E_2(\tau)(p_1(TX)-p_1(W))}\frac{\widehat{A}(TX)\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^W\right)\right)}{\cosh^2\left(\frac{c}{2}\right)}\mathrm{ch}\left(\Theta_1(T_{\mathbf{C}}X,W_{\mathbf{C}},\xi_{\mathbf{C}})\right)\right\}^{(8m)}, \tag{2.24}$$

$$Q_2(\tau):=\left\{\widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\mathrm{ch}\left(\Theta_2(T_{\mathbf{C}}X,W_{\mathbf{C}},\xi_{\mathbf{C}})\right)\right\}^{(8m)} \tag{2.25}$$

and

$$\Pi_2(\tau):=\left\{\frac{e^{\frac{1}{24}E_2(\tau)(p_1(TX)-p_1(W))}-1}{p_1(TX)-p_1(W)}\widehat{A}(TX)\cosh\left(\frac{c}{2}\right)\mathrm{ch}\left(\Theta_2(T_{\mathbf{C}}X,W_{\mathbf{C}},\xi_{\mathbf{C}})\right)\right\}^{(8m-4)}. \tag{2.26}$$

Like for the $8m+4$-dimensional case, one has:

Proposition 2.2. *$Q_1(\tau)$ is a modular form of weight $4m$ over $\Gamma_0(2)$ while $Q_2(\tau)+(p_1(TX)-p_1(W))\Pi_2(\tau)$ is a modular form of weight $4m$ over $\Gamma^0(2)$. Moreover, the following identity holds:*

$$Q_1\left(-\frac{1}{\tau}\right)=2^l\tau^{4m}(Q_2(\tau)+(p_1(TX)-p_1(W))\Pi_2(\tau)). \tag{2.27}$$

Then one can prove part 2 of Theorem 1.1 by adopting an idea similar to that in the above proof of part 1 of Theorem 1.1.

Acknowledgments

The first author was partially supported by a start-up grant from the National University of Singapore. The second author was partially supported by NSF. The third author was partially supported by MOE and NNSFC.

References

[1] L. Alvarez-Gaumé, E. Witten, Gravitational anomalies, Nuclear Phys. B 234 (1983) 269–330.
[2] K. Liu, Modular invariance and characteristic numbers, Comm. Math. Phys. 174 (1995) 29–42.

[3] P.S. Landweber, Elliptic cohomology and modular forms, in: P.S. Landweber (Ed.), Elliptic Curves and Modular Forms in Algebraic Topology, in: Lecture Notes in Mathematics, vol. 1326, Springer-Verlag, 1988, pp. 55–68.
[4] M.F. Atiyah, F. Hirzebruch, Riemann–Roch for differentiable manifolds, Bull. Amer. Math. Soc. 65 (1959) 276–281.
[5] S. Ochanine, Signature modulo 16, invariants de Kervaire géneralisés et nombre caractéristiques dans la K-théorie reelle, Mém. Soc. Math. Fr. Tom. 109 (1987) 1–141.
[6] F. Han, W. Zhang, Spin^c-manifold and elliptic genera, C. R. Acad. Sci. Paris, Sér. I 336 (2003) 1011–1014.
[7] F. Han, W. Zhang, Modular invariance, characteristic numbers and η invariants, J. Differential Geom. 67 (2004) 257–288.
[8] M.B. Green, J.H. Schwarz, Anomaly cancellations in supersymmetric $D = 10$ gauge theory and superstring theory, Phys. Lett. B 149 (1984) 117–122.
[9] J.H. Schwarz, Anomaly cancellation: a retrospective from a modern perspective, arXiv:hep-th/0107059.
[10] W. Zhang, Lectures on Chern–Weil Theory and Witten Deformations, in: Nankai Tracts in Mathematics, vol. 4, World Scientific, Singapore, 2001.
[11] M.F. Atiyah, K-Theory, Benjamin, New York, 1967.
[12] F. Hirzebruch, Topological Methods in Algebraic Geometry, Springer-Verlag, 1966.
[13] E. Witten, The index of the Dirac operator in loop space, in: P.S. Landweber (Ed.), Elliptic Curves and Modular Forms in Algebraic Topology, Proceedings, Princeton 1986, in: Lecture Notes in Math., vol. 1326, Springer, 1988, pp. 161–181. MR 0970288, Zbl 0679.58045.
[14] K. Chandrasekharan, Elliptic Functions, Springer-Verlag, 1985.
[15] M. Kaneko, D. Zagier, A generalized Jacobi theta function and quasimodular forms, in: R. Dijkgraaf, C. Faber, G. van der Geer (Eds.), The Moduli Space of Curves, Birkhäuser, Boston, 1995, pp. 165–172.
[16] F. Hirzebruch, T. Berger, R. Jung, Manifolds and Modular Forms, in: Aspects of Mathematics, vol. E20, Vieweg, Braunschweig, 1992.
[17] K. Liu, Modular forms and topology, in: Moonshine, the Monster, and Related Topics, South Hadley, MA, 1994, in: Contemp. Math., vol. 193, Amer. Math. Soc., Providence, RI, 1996, pp. 237–262.
[18] K. Liu, On elliptic genera and theta-functions, Topology 35 (1996) 617–640.