GAFA, Geom. funct. anal.
Vol. 8 (1998) 149 – 178
1016-443X/98/010149-30 $ 1.50+0.20/0

GAFA Geometric And Functional Analysis

EQUIVARIANT HOLOMORPHIC MORSE INEQUALITIES III: NON-ISOLATED FIXED POINTS

Siye Wu and Weiping Zhang

Abstract

We prove the equivariant holomorphic Morse inequalities for a holomorphic torus action on a holomorphic vector bundle over a compact Kähler manifold when the fixed-point set is not necessarily discrete. Such inequalities bound the twisted Dolbeault cohomologies of the Kähler manifold in terms of those of the fixed-point set. We apply the inequalities to obtain relations of Hodge numbers of the connected components of the fixed-point set and the whole manifold. We also investigate the consequences in geometric quantization, especially in the context of symplectic cutting.

1 Introduction

In [MWu] an analytic proof of equivariant holomorphic Morse inequalities of Witten [W2] was found based on the heat kernel approach. The result proved holds for circle actions on holomorphic vector bundles over compact Kähler manifolds with isolated fixed points. In [Wu], this result is extended to torus and non-Abelian group actions and it was shown that the Kähler condition is necessary. The purpose of this paper is to prove the general case when the fixed points can be non-isolated.

Recall that in his heat kernel proof of the Bott-Morse inequalities for the de Rham complex, Bismut [B] used a further rescaling near the critical set of a Morse function in the sense of Bott, besides the deformation of Witten [W1]. In our treatment of the holomorphic analog with non-isolated fixed points, instead of using [B], we adapt the methods and techniques in [BL],

Part of the work of S.W. was done in MSRI, Berkeley, supported by NSF grant DMS-90-22140, and in IAS, Princeton. S.W. thanks Nankai Institute of Mathematics, where this work was completed, for their hospitality. Part of the work of W.Z. was done in Courant Institute, New York. W.Z. is supported in part by the Chinese National Science Foundation, the State Education Commission of China and the Qiu Shi Foundation. The authors thank the referee for helpful comments.

where a general and direct localization procedure is developed which applies to a wide range of problems on localization in index theory. For example, to prove the Bott-Morse inequalities using this method, no rescaling near the critical set is needed.

As in [Wu], the equivariant holomorphic Morse inequalities for circle actions imply those for torus actions; we derive the latter by considering various circle subgroups of the torus. For circle actions, we deform the Dolbeault operator as in [W2]. The corresponding Laplacian is roughly of the form $\Delta + |V|^2 + L_V$, where Δ is the standard Laplacian, V is the vector field that generates the circle action, and L_V, the Lie derivative. L_V is a first order differential operator that commutes with all other operators. The crucial step is to restrict the problem to the eigenspaces of L_V, on each of which L_V is a constant and therefore the techniques of [BL] can be applied. The restriction to various eigenspaces of L_V leads naturally to an equivariant Morse theory.

The equivariant holomorphic Morse inequalities relate the Dolbeault cohomologies (twisted by a holomorphic vector bundle) of the whole manifold to those of the fixed point set. We apply our result to two situations. First, considering the exterior power of the holomorphic tangent bundle, we obtain relations of Hodge numbers of the fixed submanifolds, including some results of Carrell-Sommese [CaSo] and Carrell-Lieberman [CaLi]. Second, we apply the inequalities to symplectic cutting and geometric quantization. We obtain a gluing formula for the Poincaré-Hodge polynomials under the symplectic cutting process of Lerman [Le]. Applying our result to the pre-quantum line bundles, we recover and/or strengthen a few results in [DGMeWu], [MeSj], [TZ] when the symplectic manifold is Kähler.

Throughout this paper, $\mathbb{N}, \mathbb{R}, \mathbb{R}^{\pm}, \mathbb{C}$ and $\mathbb{C}^{\times}$ denote the sets of non-negative integers, real numbers, positive (negative) real numbers, complex numbers and non-zero complex numbers, respectively.

2 Main Results

In this section, we state the equivariant holomorphic Morse inequalities for holomorphic torus action on a holomorphic vector bundle over a compact Kähler manifold when the fixed-point set need not be isolated. We then show that the result for torus action can be deduced from that of circle action. The latter will be proved in section 3.

2.1 Equivariant holomorphic Morse inequalities for torus actions. We first recall from [Wu] a few notations of the Lie algebra and the formal character ring of the torus group. Let T be a torus group and $\mathfrak{t} = \sqrt{-1}\,\mathrm{Lie}(T)$, where $\mathrm{Lie}(T)$ is the Lie algebra of T. Let $\mathcal{L}$ be the integral lattice in $\mathfrak{t}$; the dual lattice $\mathcal{L}^*$ in $\mathfrak{t}^*$ is the weight lattice.

Definition 2.1. *Let $\mathbb{C}[\mathcal{L}^*]$ be the formal character ring of T consisting of elements $q = \sum_{\xi\in\mathcal{L}^*} q_\xi e^\xi$ ($q_\xi \in \mathbb{Z}$). Then we say $q \geq 0$ if $q_\xi \geq 0$ for all $\xi \in \mathcal{L}^*$. Let $Q(t) = \sum_{k=0}^n q_k t^k \in \mathbb{C}[\mathcal{L}^*][t]$ be a polynomial of degree n with coefficients in $\mathbb{C}[\mathcal{L}^*]$. Then we say $Q(t) \geq 0$ if $q_k \geq 0$ in $\mathbb{C}[\mathcal{L}^*]$ for all k. For two such polynomials $P(t)$ and $Q(t)$, we say $P(t) \leq Q(t)$ if $Q(t) - P(t) \geq 0$.*

If W is a finite dimensional representation of T, We denote by $\mathrm{mult}_\xi(W)$ the multiplicity of the weight $\xi \in \mathcal{L}^*$ in W. The character of W is $\mathrm{char}(W) = \sum_{\xi\in\mathcal{L}^*} \mathrm{mult}_\xi(W)\, e^\xi \geq 0$ in $\mathbb{C}[\mathcal{L}^*]$. Let the *support* of W be $\mathrm{supp}(W) = \{\xi \in \mathcal{L}^* \mid \mathrm{mult}_\xi(W) \neq 0\}$. For any $\theta \in \mathfrak{t}^*$, there is a homomorphism $\mathbb{C}[\mathcal{L}^*] \to \mathbb{C}$ given by $e^\xi \mapsto e^{\sqrt{-1}\langle\xi,\theta\rangle}$. For instance, $\mathrm{char}(W) \mapsto \mathrm{tr}_W\, e^{\sqrt{-1}\theta}$ under this homomorphism.

Now let (M,ω) be a compact Kähler manifold of complex dimension n. Suppose that T acts on M effectively preserving the complex structure and the Kähler form. If the fixed-point set $F \subset M$ of the T-action is non-empty, then the action is Hamiltonian [F]. F is a finite union of connected compact Kähler submanifolds $F_1, \dots, F_m$; let $n_1, \dots, n_m$ be their complex dimensions, respectively. For each connected component F_r ($1 \leq r \leq m$), the complexification of the normal bundle $N_r \to F_r$ in M has the decomposition $N_r^{\mathbb{C}} = N_r^{1,0} \oplus N_r^{0,1}$, and $N_r^{1,0}$ is a holomorphic vector bundle over F_r of rank $n - n_r$. The torus T acts on N_r preserving the base points in F_r. Moreover, the weights of the isotropy representation on the normal fiber remains unchanged within any connected component of F. Let $\lambda_{r,k}$ ($1 \leq k \leq n - n_r$) be the isotropy weights on N_r. N_r splits, and $N_r^{1,0}$ splits holomorphically, into the direct sum of various sub-bundles, each with a given weight. The hyperplanes $(\lambda_{r,k})^\perp \subset \mathfrak{t}$ cut $\mathfrak{t}$ into open polyhedral cones called *action chambers*, as in [GLeSt], [PWu], [Wu]. We fix a *positive* action chamber C. Let $\lambda_{r,k}^C = \pm\lambda_{r,k}$ be the *polarized weights*, with the sign chosen so that $\lambda_{r,k}^C \in C^*$. (Here C^* is the dual cone in $\mathfrak{t}^*$ defined by $C^* = \{\xi \in \mathfrak{t}^* \mid \langle\xi, C\rangle > 0\}$.) We define the *polarizing index* ν_r^C of the component F_r with respect to C as the number of weights $\lambda_{r,k} \in -C^*$. Let N_r^C be the direct sum of the sub-bundles corresponding to the weights $\lambda_{r,k} \in C^*$. ν_r^C is the rank of the holomorphic vector bundle $N_r^{-C,(1,0)}$; that of $N_r^{C,(1,0)}$ is $\nu_r^{-C} = n - n_r - \nu_r^C$. Finally, we define the *polarized symmetric tensor product* (with respect to C) of the holomorphic normal bundle $N_r^{1,0}$ by

$$K^C(N_r) = S((N_r^{C,(1,0)})^*) \otimes S(N_r^{-C,(1,0)}) \otimes \bigwedge^{\nu_r^C}(N_r^{-C,(1,0)})\,. \tag{2.1}$$

This is the bundle of eigenspaces of small eigenvalues in [W2]. It also appeared in [CG], which generalizes [GLeSt] to cases with non-isolated fixed points.

If E is a holomorphic vector bundle over M on which the T-action lifts holomorphically, then the fiber E_p over each fixed point $p \in F$ is a representation of T, and $\mathrm{char}(E_p)$ is also constant within any connected component of F. Consider an infinite dimensional holomorphic bundle $K^C(N_r) \otimes E|_{F_r}$. The torus T acts on the total space while preserving the base points in F_r. It is easy to see that each sub-bundle of any given weight is a holomorphic vector bundle of finite rank, i.e., $K^C(N_r) \otimes E|_{F_r} = \oplus_{\xi \in \mathcal{L}} E^C_{r,\xi}$, where $E^C_{r,\xi}$ is a T-invariant sub-bundle of finite rank on which the torus acts with weight ξ. The cohomology groups $H^k(F_r, \mathcal{O}(K^C(N_r) \otimes E|_{F_r}))$ are the sum of those with coefficients in $E^C_{r,\xi}$, each equipped with an induced T-action. Therefore $\mathrm{char}\, H^k(F_r, \mathcal{O}(K^C(N_r) \otimes E|_{F_r})) = \sum_{\xi \in \mathcal{L}} \dim_{\mathbb{C}} H^k(F_r, \mathcal{O}(E^C_{r,\xi}))\, e^{\xi}$ is a well-defined element in $\mathbb{C}[\mathcal{L}^*]$. Moreover, $\mathrm{supp}\, H^k(F_r, \mathcal{O}(K^C(N_r) \otimes E|_{F_r}))$ is contained in a suitably shifted cone $-C^*$ in $\mathfrak{t}^*$.

Our main result is the following

Theorem 2.2. *For each choice of the positive action chamber C, we have the strong equivariant holomorphic Morse inequalities*

$$\begin{aligned}\sum_{r=1}^{m} t^{\nu_r^C} \sum_{k=0}^{n_r} t^k\, \mathrm{char}\, H^k\big(F_r, \mathcal{O}(K^C(N_r) \otimes E|_{F_r})\big) \\ = \sum_{k=0}^{n} t^k\, \mathrm{char}\, H^k(M, \mathcal{O}(E)) + (1+t) Q^C(t)\end{aligned} \tag{2.2}$$

for some $Q^C(t) \geq 0$ in $\mathbb{C}[\mathcal{L}^*][t]$.

Remark 2.3. 1. Formula (2.2) clearly implies the corresponding weak inequalities

$$\mathrm{char}\, H^k(M, \mathcal{O}(E)) \leq \sum_{r=1}^{m} \mathrm{char}\, H^{k-\nu_r^C}\big(F_r, \mathcal{O}(K^C(N_r) \otimes E|_{F_r})\big)\,. \tag{2.3}$$

2. It is easy to see that for any choice of C, (2.2) reduces to the Atiyah-Bott-Segal-Singer fixed-point theorem [ABo], [AS], [ASi] after setting

$t=-1$. In fact,

$$\sum_{k=0}^{n}(-1)^k \operatorname{char} H^k(M,\mathcal{O}(E)) = \sum_{r=1}^{m}\int_{F_r} \mathrm{ch}_T\big(K^C(N_r)\otimes E|_{F_r}\big)\,\mathrm{td}(F_r)$$
$$= \sum_{r=1}^{m}\int_{F_r} \mathrm{ch}_T\left(\frac{E|_{F_r}}{\det(1-N_r^{*(1,0)})}\right)\mathrm{td}(F_r)\,, \tag{2.4}$$

where ch_T and td stand for the equivariant Chern character and the Todd class, respectively.

3. If $p\in F$ is an isolated fixed point, then the normal bundle $N_p=T_pM$ and

$$\operatorname{char} H^0(\{p\},\mathcal{O}\big(K^C(N_p)\otimes E_p)\big)$$
$$= \operatorname{char}(E_p)\prod_{\lambda^C_{p,k}\in C^*}\frac{1}{1-e^{-\lambda_{p,k}}}\prod_{\lambda^C_{p,k}\in -C^*}\frac{e^{-\lambda^C_{p,k}}}{1-e^{-\lambda^C_{p,k}}}\,. \tag{2.5}$$

Therefore, (2.2) reduces to the result in [Wu] when F is discrete.

4. Though the Kähler assumption is not necessary in the fixed-point formula (2.4), it is essential for the strong inequalities (2.2) even when all the fixed points are isolated and when T is a circle group [Wu].

As in [Wu], (2.3) or (2.2) is a set of inequalities for each choice of the action chamber C. These inequalities provide bounds, along various directions in $\mathfrak{t}^*$ given by C, the multiplicities of weights in the Dolbeault cohomology group $H^k(M,\mathcal{O}(E))$ $(0\le k\le n)$ in terms of the fixed-point data, which includes F_r and the bundles $N_r, E|_{F_r}\to F_r$ with T-actions. The applications will be given in section 4.

When the group acting on M is a non-Abelian group G, we can apply (2.2) to the maximal torus T of G. There is in addition an action of the Weyl group W on the fixed-point set F of T. Each $w\in W$ induces an action on the set of connected components $\{F_1,\ldots,F_m\}$ of F. The sum over the connected components on the left-hand side of (2.2) can be rearranged into sums over W (after incorporating the character of the isotropy representation of T on the bundle $E|_F$) and over its orbits in the set of connected components. Thus we can obtain the non-Abelian version of the holomorphic Morse inequalities like the case when F is isolated [Wu]. The details are left to the interested reader.

2.2 Reduction to the case of circle actions. If the torus T is the circle group S^1, then $\mathfrak{t}=\mathbb{R}$. There are two action chambers $\mathbb{R}^\pm$, labeled by $\pm$ for simplicity. The weights of isotropy representation of S^1 on N_r

$(1 \le r \le m)$ are $\lambda_{r,k} \in \mathbb{Z}\backslash\{0\}$ $(1 \le k \le n_r)$. As before, $N_r^{\pm}$ is the direct sum of the sub-bundles corresponding to the positive and negative weights, respectively, and $K^{\pm}(N_r) = S((N_r^{\pm,(1,0)})^*) \otimes S(N_r^{\mp,(1,0)}) \otimes \bigwedge^{\mathrm{top}}(N_r^{\mp,(1,0)}) \otimes E|_{F_r}$. Let ν_r be the rank of $N_r^{-,(1,0)}$; that of $N_r^{+,(1,0)}$ is then $n - n_r - \nu_r$.

Then we have the following result for the S^1-case.

Theorem 2.4. *Under the above assumptions, we have the strong equivariant holomorphic Morse inequalities for circle actions*

$$\sum_{r=1}^{m} t^{\nu_r} \sum_{k=0}^{n_r} t^k \operatorname{char} H^k(F_r, \mathcal{O}(K^+(N_r) \otimes E|_{F_r})) = \sum_{k=0}^{n} t^k \operatorname{char} H^k(M, \mathcal{O}(E)) + (1+t)Q^+(t)\,, \tag{2.6}$$

$$\sum_{r=1}^{m} t^{n-n_r-\nu_r} \sum_{k=0}^{n_r} t^k \operatorname{char} H^k(F_r, \mathcal{O}(K^-(N_r) \otimes E|_{F_r})) = \sum_{k=0}^{n} t^k \operatorname{char} H^k(M, \mathcal{O}(E)) + (1+t)Q^-(t)\,, \tag{2.7}$$

where $Q^{\pm}(t) \ge 0$.

Clearly (2.6) follows from (2.7) by reversing the S^1-action. Section 3 will be devoted to proving (2.7). To show that the strong inequalities (2.2) of the torus case follows from (2.6) or (2.7), we proceed slightly differently from [Wu].

Recall that for a general torus group T, a cone C in $\mathfrak{t}$ is *proper* if there is $\xi \in \mathfrak{t}^*$ such that $\langle \xi, C\backslash\{0\}\rangle > 0$. If C is an open proper cone, so is C^* in $\mathfrak{t}^*$. For example, if T acts on M effectively, then all action chambers are open proper cones. (See [PWu] for the geometry of cones in the context of Hamiltonian torus actions.)

Lemma 2.5. *If C is an open proper cone in $\mathfrak{t}$, then for any $\xi \in C^* \cap \mathcal{L}^*$, there is an element $v \in C \cap \mathcal{L}$ such that the hyperplane $\xi + v^{\perp} = \{\lambda \in \mathfrak{t}^* \mid \langle \lambda - \xi, v\rangle = 0\}$ contains no points in $C^* \cap \mathcal{L}^*$ other than ξ.*

Proof. Choose an open proper subcone $D \subset C$ such that $\overline{D} \subset C \cup \{0\}$. If $v \in D \cap \mathcal{L}$, then the intersection $(\xi + v^{\perp}) \cap (C^* \cap \mathcal{L}^*)$ is contained in $(C^*\backslash(\xi+\overline{D^*}))\cap\mathcal{L}^*$; the latter is a finite set since $\overline{C^*} \subset D^* \cup \{0\}$. For each $\lambda \in (C^*\backslash(\xi + \overline{D^*})) \cap \mathcal{L}^*$, consider the hyperplane $H_\lambda = \{v \in \mathfrak{t} \mid \langle \lambda - \xi, v\rangle = 0\}$ in $\mathfrak{t}$. Let $\pi\colon \mathfrak{t}\backslash\{0\} \to P(\mathfrak{t})$ be the canonical projection to the real projective space $P(\mathfrak{t})$. Since the images $\pi(H_\lambda) \subset P(\mathfrak{t})$ are of codimension 1, $\pi(\mathcal{L})$

is dense in $P(V)$, and $\pi(D)$ is open in $P(\mathfrak{t})$, there is an element $v \in \mathfrak{t}$ such that $\pi(v) \in \pi(D \cap \mathcal{L})\backslash\pi(H_\lambda)$ for any $\lambda \in (C^*\backslash(\mu + \overline{D^*})) \cap \mathcal{L}^*$. By multiplying with a non-zero real number, we may choose $v \in D \cap \mathcal{L}$. The condition $n \notin H_\lambda$ means that $\lambda + v^\perp$ does not contain the point λ. The result follows. □

Reduction of Theorem 2.2 to Theorem 2.4. For any action chamber C, there is $\lambda_0 \in \mathcal{L}^*$ such that

$$\operatorname{supp} H^k\big(F_r, \mathcal{O}(K^C(N_r) \otimes E|_{F_r})\big), \operatorname{supp} H^k(M, \mathcal{O}(E)) \subset (\lambda_0 - C^*) \cap \mathcal{L}^* \tag{2.8}$$

for all $1 \le r \le m$, $0 \le k \le n$. (2.2) is equivalent to

$$\sum_{r=1}^{m} t^{\nu_r^C} \sum_{k=0}^{n_r} t^k \operatorname{mult}_\xi H^k\big(F_r, \mathcal{O}(K^C(N_r) \otimes E|_{F_r})\big) = \sum_{k=0}^{n} t^k \operatorname{mult}_\xi H^k(M, \mathcal{O}(E)) + (1+t) Q_\xi^C(t)\,, \tag{2.9}$$

where $Q_\xi^C(t) \ge 0$ for all $\xi \in (\lambda_0 - C^*) \cap \mathcal{L}^*$. For any $v \in C \cap \mathcal{L}$, let S^1 be the circle subgroup generated by v. Consider this circle action on M and E. The isotropy weights on F_r $(1 \le r \le m)$ are $\langle \lambda_{r,k}, v\rangle$ $(1 \le k \le n_r)$. Since $\langle \lambda_{r,k}^C, v\rangle > 0$, $K^C(N_r) = K^+(N_r)$ for all $1 \le r \le m$. By the above lemma, for any $\xi \in (\lambda_0 - C^*) \cap \mathcal{L}^*$ there is an element $v \in C \cap \mathcal{L}$ such that $(\xi + v^\perp) \cap ((\lambda_0 - C^*) \cap \mathcal{L}^*) = \{\xi\}$. It follows that if W is a representation of T such that $\operatorname{supp}(W) \subset (\lambda_0 - C^*) \cap \mathcal{L}^*$ and the multiplicities of weights are finite, then $\operatorname{mult}_\xi(W) = \operatorname{mult}_{\langle \xi, v\rangle}(W)$. Therefore (2.9) follows from (2.6). □

3 The Case of Holomorphic Circle Actions

The entire section is devoted to proving (2.7) in Theorem 2.4. The proof is modeled on the paper of Bismut and Lebeau [BL]. We first give an outline.

In subsection 3.1, we study Witten's deformation [W2] of the Dolbeault operator $\bar\partial$ on a compact Kähler manifold M as well as on flat spaces. The deformation $\bar\partial_h$ is defined as the conjugation of $\bar\partial$ (twisted by a holomorphic vector bundle E on M) by the exponential of a moment map h (formula (3.2)). Using the notation of the Clifford algebra, the sum of the Dolbeault operator $\bar\partial$ and its (formal) adjoint $\bar\partial^*$ can be identified as a spin$^{\mathbb{C}}$-Dirac operator D^M on M (also twisted by E). Similarly, the sum of the deformed operators $\bar\partial_h$ and $\bar\partial_h^*$ is $D^M + \sqrt{-1}\, c(V)$, where $c(V)$ is the Clifford multiplication by V, the vector field on M that generates the S^1-action. We show that up to a zeroth-order operator, the Laplacian of $\bar\partial_h$ is equal to the sum of the Hamiltonian operator of a harmonic oscillator and the Lie derivative L_V (Lemma 3.1); this will play a crucial role in subsection 3.3. Finally, we study the problem on a complex vector space with a linear S^1-action. Though the kernel and the cokernel of $\bar\partial_h$ are infinite dimensional, the subspaces of a given weight of S^1 are finite dimensional (Proposition 3.2).

In subsection 3.2, we establish a Taylor expansion of $D^M + \sqrt{-1}\,c(V)$ near F. We first introduce a coordinate system near F which identifies neighborhoods of F in M and in the normal bundle $N \to M$. We show that in the appropriate limit of scaling in the normal directions, $D^M + \sqrt{-1}\,c(V)$ splits to two operators near F (Proposition 3.3). The first operator acts along the normal directions. On each fiber, it is the deformed Dolbeault operator with the linearized S^1-action. The kernels, collectively, form a holomorphic vector bundle $K^-(N)$ over F. The space of forms on F valued in $K^-(N) \otimes E|_F$ is the subspace of small eigenvalues of the operator $D^M + \sqrt{-1}\,c(V)$ near F; let D^F be the (twisted) spin$^{\mathbb{C}}$-Dirac operator corresponding to the Dolbeault operator on F twisted by $K^-(N) \otimes E|_F$. The second operator from $D^M + \sqrt{-1}\,c(V)$, denoted by D^H, acts along the horizontal directions of $N \to F$. We show that when restricted to the subspace of small eigenvalues, D^H is precisely D^F under the natural identifications given by the coordinates near F (Proposition 3.4).

In subsection 3.3, we study the operator $D^M + \sqrt{-1}\,c(V)$ on the whole manifold M. We decompose the space of forms valued in E into a subspace corresponding to small eigenvalues of and its orthogonal complement; this also decomposes the operator $D^M + \sqrt{-1}\,c(V)$ into a 2×2 block-matrix. Using the results of previous subsections and the techniques of [BL, Section 9], we establish estimations for operators in various blocks (Proposition 3.5). The off-diagonal blocks are bounded under appropriate Sobolev norms. The diagonal block acting on the subspace orthogonal to the small-eigenvalue subspace is bounded from below by a positive definite operator whose norm goes to infinity under the scaling limit; it is crucial in this estimate that we restrict the operator to a given eigenspace of L_V. These results allow us to localize the problem to F. Furthermore, D^F is the leading behavior of the diagonal block acting on the subspace of small eigenvalues. It follows that on each eigenspace of L_V, the spectrums of the resolvants of $D^M + \sqrt{-1}\,c(V)$ and D^F are sufficiently close in the scaling limit (Proposition 3.7).

In subsection 3.4, we prove Theorem 2.4. The last result of subsection 3.3 implies that upon restriction to an eigenspace of L_V, the spaces of small eigenvalues of $\bar{\partial}_h$ or $D^M + \sqrt{-1}\, c(V)$ can be identified with the kernel of D^F. By Hodge theory, the dimensions of the former are related to the dimensions of the Dolbeault cohomology groups by Morse-type inequalities. Therefore we obtain Morse-type inequalities relating the Dolbeault cohomologies of F and those of M. These are precisely the holomorphic Morse inequalities in (2.7). Since we restrict the problem to each eigenspace of L_V, the Morse inequalities are equivariant.

3.1 Witten's deformation of the Dolbeault operator. We consider a holomorphic S^1-action on a Kähler manifold M which preserves the Kähler structure. Let ω, g^{TM} and J be the Kähler form, the Kähler metric and the complex structure on M, respectively. We assume that the fixed point set F is non-empty. In this case, there is a moment map $h\colon M \to \mathbb{R}$ satisfying $i_V\omega = dh$, where V is the vector field on M that generates the S^1-action [F]. We further assume that the S^1-action can be lifted holomorphically to a holomorphic vector bundle E over M. We can choose an S^1-invariant Hermitian form on E. Then the Hermitian connection ∇^E is also S^1-invariant. ∇^E induces an (S^1-equivariant) operator, also denoted by ∇^E, on the space of E-valued differential forms $\Omega^*(M, E)$. The group elements of S^1 acts on the sections of E. Let L_V be the Lie derivative of E-valued forms along V. (The fibers of E over different points on the integral curve of V are related by the lifted S^1-action.) Then $-L_V$ is the infinitesimal generator of the S^1-action on $\Omega^*(M, E)$. Let $\overline{L}_V = \{i_V, \nabla^E\}$. Then the operator

$$r_V = \overline{L}_V - L_V \tag{3.1}$$

is an element of $\Gamma(M, \mathrm{End}(E))$. Over the fiber of a fixed point $p \in F$, $r_V(p)$ is simply the representation of $\mathrm{Lie}(S^1)$ on E_p; this is independent of the choice of the connections on E.

Let $\bar{\partial} = \bar{\partial}^E$ be the twisted Dolbeault operator of the complex $\Omega^{0,*}(M, E) = \Gamma(\wedge^*(T^{*(0,1)}M) \otimes E)$ and $\bar{\partial}^*$, its formal adjoint. Consider Witten's deformation of the (twisted) Dolbeault operator

$$\bar{\partial}_h = e^{-h}\bar{\partial}e^h\,, \qquad \bar{\partial}_h^* = e^h\bar{\partial}^*e^{-h}\,. \tag{3.2}$$

We define the Clifford action. For $X \in \Gamma(T^{\mathbb{C}}M)$, let $X = X^{1,0} + X^{0,1}$ such that $X^{1,0} \in \Gamma(T^{1,0}M)$ and $X^{0,1} \in \Gamma(T^{0,1}M)$. Set

$$c(X^{1,0}) = \sqrt{2}(X^{1,0})^*\,, \qquad c(X^{0,1}) = -\sqrt{2}\, i_{X^{0,1}}\,, \tag{3.3}$$

where $(X^{1,0})^* \in \Gamma(T^{*(0,1)}M)$ corresponds to $X^{1,0}$ via the metric g^{TM}. It is easy to verify that for $X, Y \in \Gamma(T^{\mathbb{C}}M)$, the anti-commutation relation

$\{c(X), c(Y)\} = -2g^{TM}(X,Y)$. Let $\{e_i, i=1,\dots,2n\}$ be a (local) orthonormal frame and $D^M = \sum_{i=1}^{2n} c(e_i)\nabla_{e_i}$, the spin$^{\mathbb{C}}$-Dirac operator acting on $\Omega^{0,*}(M,E)$.

Lemma 3.1. 1.

$$D^M = \sqrt{2}(\bar\partial + \bar\partial^*)\,; \tag{3.4}$$

2.

$$D^M + \sqrt{-1}\,c(V) = \sqrt{2}(\bar\partial_h + \bar\partial_h^*)\,; \tag{3.5}$$

3.

$$\begin{aligned}\big(D^M + \sqrt{-1}\,c(V)\big)^2 &= (D^M)^2 + |V|^2 - 2\sqrt{-1}L_V \\ &\quad + \frac{1}{2}\sqrt{-1}\sum_{i=1}^{2n} c(e_i)c(\nabla_{e_i}V) + \sqrt{-1}\ \mathrm{tr}\ \nabla.V|_{T^{0,1}M} - 2\sqrt{-1}\,r_V\,. \end{aligned}\tag{3.6}$$

Proof. Parts 1 and 2 follow from

$$\bar\partial = \frac{1}{2\sqrt{2}}\sum_{i=1}^{2n} c\big(e_i - \sqrt{-1}Je_i\big)\nabla_{e_i}\,, \quad \bar\partial^* = \frac{1}{2\sqrt{2}}\sum_{i=1}^{2n} c\big(e_i + \sqrt{-1}Je_i\big)\nabla_{e_i}\,, \tag{3.7}$$

$$\bar\partial_h = \bar\partial + \frac{1}{2\sqrt{2}}\sum_{i=1}^{2n} c\big(e_i - \sqrt{-1}Je_i\big)h_{,i}\,,\ \bar\partial_h^* = \bar\partial^* - \frac{1}{2\sqrt{2}}\sum_{i=1}^{2n} c\big(e_i + \sqrt{-1}Je_i\big)h_{,i}\,, \tag{3.8}$$

and that $J\,\mathrm{grad}\,h = -V$. To show part 3, we compute

$$\big(D^M + \sqrt{-1}\,c(V)\big)^2 = (D^M)^2 + \sqrt{-1}\big\{D^M, c(V)\big\} + |V|^2 \tag{3.9}$$

and

$$\begin{aligned}\{D^M, c(V)\} &= \sum_{i=1}^{2n}\big(\{c(e_i), c(V)\}\nabla_{e_i} + c(e_i)c(\nabla_{e_i}V)\big) \\ &= -2\nabla_V + \sum_{i=1}^{2n} c(e_i)c(\nabla_{e_i}V)\,. \end{aligned}\tag{3.10}$$

Further, since $L_V = \nabla_V - \nabla.V$ on vector fields, the induced relation on $\Omega^{0,*}(M,E)$ is

$$\begin{aligned}\nabla_V &= \overline{L}_V + \frac{1}{4}\sum_{i,j=1}^{2n} g(\nabla_{e_i}V, e_j)c(e_i)c(e_j) + \frac{1}{2}\,\mathrm{tr}\ \nabla.V|_{T^{*(0,1)}M} \\ &= \overline{L}_V + \frac{1}{4}\sum_{i=1}^{2n} c(e_i)c(\nabla_{e_i}V) - \frac{1}{2}\,\mathrm{tr}\ \nabla.V|_{T^{0,1}M}\,. \end{aligned}\tag{3.11}$$

(3.6) follows from (3.9), (3.10), (3.11) and (3.1). $\square$

A formula equivalent to (3.6) was derived in [MWu] without using the Clifford algebra.

Next we study Witten's deformation on flat spaces [W2]. Let W be a complex vector space of (complex) dimension n with an Hermitian form. Let ρ be a unitary representation of the circle group S^1 on W such that all the weights are non-zero. Suppose $W^{\pm}$ are the subspaces of W corresponding to positive (negative) weights, respectively, and $\dim_{\mathbb{C}} W^- = \nu$, $\dim_{\mathbb{C}} W^+ = n-\nu$. Let $z = \{z^k\}$ be the complex linear coordinates on W such that the Hermitian structure on W takes the standard form and such that ρ is diagonal with weights $\lambda_k \in \mathbb{Z}\backslash\{0\}$ $(1 \le k \le n)$. The Lie algebra action is given by the vector field $V = \sqrt{-1}\sum_{k=1}^n \lambda_k\big(z^k\frac{\partial}{\partial z^k} - \bar z^k\frac{\partial}{\partial \bar z^k}\big)$ on W. With respect to the standard Kähler form $\omega = \frac{\sqrt{-1}}{2}\sum_{k=1}^n dz^k \wedge d\bar z^k$, ρ is a Hamiltonian action with moment map $h(z) = -\frac{1}{2}\sum_{k=1}^n \lambda_k|z^k|^2$. As in the compact case, we have the deformed operators $\bar\partial_h$ and $\bar\partial_h^*$ in (3.2) and $D+\sqrt{-1}c(V) = \sqrt{2}(\bar\partial_h + \bar\partial_h^*)$, where D is the Dirac operator with the same Clifford action (3.3). We set $K^{\pm}(W) = S((W^{\pm})^*) \otimes S(W^{\mp}) \otimes \bigwedge^{\mathrm{top}}(W^{\mp})$. Let E be a finite dimensional complex vector space with an Hermitian form and suppose E carries a unitary representation of S^1.

Proposition 3.2. 1. *The space of L^2-solutions of a given weight of $D+\sqrt{-1}\,c(V)$ on the space of $(0,*)$-forms on W with values in E is finite dimensional. The direct sum of these weight spaces is isomorphic to $K^-(W)\otimes E$ as representations of S^1.*

2. *When restricted to an eigenspace of L_V, the operator $D+\sqrt{-1}\,c(V)$ has discrete eigenvalues.*

Proof. It suffices to prove the case when $E \cong \mathbb{C}$ is a trivial representation.

1. Since W decomposes into a direct sum of 1-dimensional representations and since the solution space on the direct sum corresponds to a tensor product, it suffices to prove the result when $\dim_{\mathbb{C}} W = 1$ with weight $\lambda \in \mathbb{Z}\backslash\{0\}$. Let $\varphi \in \Omega^{0,*}(\mathbb{C})$ solve $(D+\sqrt{-1}c(V))\varphi = 0$, i.e., $\bar\partial_h\varphi = \bar\partial_h^*\varphi = 0$. For $\lambda < 0$, an L^2-solution of a fixed weight are proportional to

$$\varphi_k(z) = \sqrt{\frac{(-\lambda)^{k+1}}{\pi k!}}\, z^k e^{\frac{\lambda}{2}|z|^2} \qquad (k \in \mathbb{N}). \tag{3.12}$$

φ_k has unit norm and weight $-k\lambda$. Since $z \in W^* = (W^-)^*$, the direct sum of $\mathbb{C}\varphi_k$ $(k\in\mathbb{N})$ is $S((W^-)^*)$. If $\lambda > 0$, then $W = W^+$, and a solution of a given weight is proportional to

$$\varphi_{\bar k} = \sqrt{\frac{\lambda^k}{\pi k!}}\, \bar z^k e^{-\frac{\lambda}{2}|z|^2} d\bar z \qquad (k \in \mathbb{N}), \tag{3.13}$$

of unit norm and weight $(k+1)\lambda$. The direct sum of $\mathbb{C}\varphi_{\bar{k}}$ $(k \in \mathbb{N})$ is $S(\overline{W}^*) \otimes \overline{W}^* \cong S(W^+) \otimes W^+$; the last isomorphism is induced by the Hermitian form on $W \cong \mathbb{C}$.

2. This follows from (3.6) after setting L_V to a constant and from the standard properties of harmonic oscillators. $\square$

3.2 A Taylor expansion of the operator near the fixed-point set. Since the fixed point set F is the zero set of the Killing vector field V, F is a totally geodesic compact Kähler submanifold of M. We denote by n_F and ν_F the complex dimension and the polarizing index of F, respectively, which are locally constant. Let g^{TF} be the induced metric and dv_F, the volume element on F. Let $\tilde{\pi}\colon N \to F$ be the normal bundle of F in M. We identify N as the orthogonal complement of TF in $TM|_F$, i.e., $TM|_F = TF \oplus N$ and $g^{TM} = g^{TF} \oplus g^N$, where g^N is the induced inner product on N. Following [BL, Section 8e)], we now describe a coordinate system on M near F.

If $y \in F$, $Z \in N_y$, let $t \in \mathbb{R} \mapsto x_t = \exp_y^M(tZ) \in M$ be the geodesic in M with $x_0 = y$, $\frac{dx}{dt}|_{t=0} = Z$. For $\epsilon > 0$, set $B_\epsilon = \{Z \in N; |Z| < \epsilon\}$. Since M and F are compact, there exists $\epsilon_0 > 0$ such that for $0 \le \epsilon < \epsilon_0$, the map $(y,Z) \in N \mapsto \exp_y^M(Z) \in M$ is a diffeomorphism from B_ϵ onto a tubular neighborhood U_ϵ of F in M. From now on, we identify B_ϵ with U_ϵ and use the notation $x = (y,Z)$ instead of $x = \exp_y^M(Z)$. Finally, we identify $y \in F$ with $(y,0) \in N$.

Let dv_N denote the volume element of the fibres in N. Then $dv_F(y)dv_{N_y}(Z)$ is a natural volume element on the total space of N. Let $k(y,Z)$ be the smooth positive function defined on B_{ϵ_0} by the equation $dv_M(y,Z) = k(y,Z)dv_F(y)dv_{N_y}(Z)$. The function k has a positive lower bound on $U_{\epsilon_0/2}$. Also, $k(y) = 1$ and $\frac{\partial k}{\partial Z}(y) = 0$ for $y \in F$; the latter follows from [BL, Proposition 8.9] and the fact that F is totally geodesic in M.

As in [BL, Section 8g)], for $x = (y,Z) \in U_{\epsilon_0}$, we will identify E_x with E_y and $\bigwedge(T_x^{*(0,1)}M)$ with $\bigwedge(T_y^{*(0,1)}M)$ by the parallel transport with respect to the S^1-invariant connections ∇^E and ∇^{TM}, respectively, along the geodesic $t \mapsto (y,tZ)$. The induced identification of $(\bigwedge(T^{*(0,1)}M)\otimes E)_x$ with $(\bigwedge(T^{*(0,1)}M) \otimes E)_y$ preserves the metric, the $\mathbb{Z}$-grading of the Dolbeault complex, and is moreover S^1-equivariant. Consequently, D^M can be considered as an operator acting on the sections of the bundle $\tilde{\pi}^*((\bigwedge(T^{*(0,1)}M) \otimes E)|_F)$ over B_{ϵ_0}. It still commutes with the S^1-action.

For $\epsilon > 0$, let $\mathbf{H}(B_\epsilon)$ (resp. $\mathbf{H}(N)$) be the set of smooth sections of $\tilde{\pi}^*(\bigwedge(T^{*(0,1)}M) \otimes E)$ on B_ϵ (resp. on the total space of N). If $f, g \in \mathbf{H}(N)$

have compact support, set

$$\langle f,g\rangle=\left(\frac{1}{2\pi}\right)^n\int_F\Big(\int_N\langle f,g\rangle(y,Z)dv_{N_y}(Z)\Big)dv_F(y)\,. \tag{3.14}$$

Notice that the identification of elements in $\mathbf{H}(N)$ which have compact support in B_{ϵ_0} with those in $\Gamma(\bigwedge(T^{*(0,1)}M)\otimes E)$ with support in U_{ϵ_0} is not unitary with respect to the Hermitian product (3.14). Consequently D^M as an operator on the sections of $\tilde{\pi}^*((\bigwedge(T^{*(0,1)}M)\otimes E)|_F)$ over B_{ϵ_0} is not in general self-adjoint with respect to (3.14). However $k^{1/2}D^Mk^{-1/2}$ is a (formal) self-adjoint operator on $\mathbf{H}(B_{\epsilon_0})$.

The connection ∇^N on N induces a splitting $TN=N\oplus T^HN$, where T^HN is the horizontal part of TN with respect to ∇^N. Moreover, since F is totally geodesic, this splitting, when restricted to F, is preserved by the connection $\nabla^{TM|_F}$ on $TM|_F$. If we denote by p^{TF},p^N the orthogonal projections from $TM|_F$ to TF,N, respectively, then $\nabla^{TF}=p^{TF}\nabla^{TM|_F}$ and $\nabla^N=p^N\nabla^{TM|_F}$. Let $\tilde{\nabla}^F$ be the connection on $(\bigwedge(T^{*(0,1)}M)\otimes E)|_F$ induced by the restrictions of ∇^{TM} and ∇^E to F. The connection $\tilde{\nabla}^F$ lifts to one on $\tilde{\pi}^*((\bigwedge(T^{*(0,1)}M)\otimes E)|_F)$, which we still denote by $\tilde{\nabla}^F$.

We choose a local orthonormal frame such that $e_1,\dots,e_{2n_F}$ form a basis of TF, and $e_{2n_F+1},\cdots,e_{2n}$, that of N. Denote the horizontal lift of e_i $(1\le i\le 2n_F)$ to T^HN by e_i^H. As in [BL, Definition 8.16], we define

$$D^H=\sum_{i=1}^{2n_F}c(e_i)\tilde{\nabla}^F_{e_i^H}\,,\qquad D^N=\sum_{i=2n_F+1}^{2n}c(e_i)\tilde{\nabla}^F_{e_i}\,. \tag{3.15}$$

Clearly, D^N acts along the fibers of N. Let $\bar{\partial}^N$ be the $\bar{\partial}$-operator along the fibers of N, and let $(\bar{\partial}^N)^*$ be its formal adjoint under (3.14). It is easy to see that $D^N=\sqrt{2}(\bar{\partial}^N+(\bar{\partial}^N)^*)$. Both D^N and D^H are self-adjoint with respect to (3.14).

For $T>0$, we define a scaling $f\in\mathbf{H}(B_{\epsilon_0})\mapsto S_Tf\in\mathbf{H}(B_{\epsilon_0\sqrt{T}})$ by

$$S_Tf(y,Z)=f\left(y,\frac{Z}{\sqrt{T}}\right)\,,\qquad (y,Z)\in B_{\epsilon_0T}\,. \tag{3.16}$$

For a first order differential operator

$$Q_T=\sum_{i=1}^{2n_F}a_T^i(y,Z)\tilde{\nabla}^F_{e_i^H}+\sum_{i=2n_F+1}^{2n}b_T^i(y,Z)\tilde{\nabla}^F_{e_i}+c_T(y,Z) \tag{3.17}$$

acting on $\mathbf{H}(B_{\epsilon_0\sqrt{T}})$, where a_T^i, b_T^i and c_T are endomorphisms of $\tilde{\pi}^*((\bigwedge(T^{*(0,1)}M)\otimes E)|_F)$, we write

$$Q_T=O\big(|Z|^2\partial^N+|Z|\partial^H+|Z|+|Z|^p\big)\qquad (p\in\mathbb{N})\,, \tag{3.18}$$

if there is a constant $C > 0$ such that for any $T \geq 1$, $(y, Z) \in B_{\epsilon_0\sqrt{T}}$, we have

$$\begin{aligned}
&|a_T^i(y,Z)| \leq C|Z| \qquad (1 \leq i \leq 2n_F)\,,\\
&|b_T^i(y,Z)| \leq C|Z|^2 \qquad (2n_F + 1 \leq i \leq 2n)\,,\\
&|c_T(y,Z)| \leq C(|Z| + |Z|^p)\,.
\end{aligned} \tag{3.19}$$

Let J_V be the representation of $\mathrm{Lie}(S^1)$ on N. Then $Z \mapsto J_V Z$ is a Killing vector field on N. We have the following analog of [BL, Theorem 8.18].

Proposition 3.3. *As $T \to \infty$,*

$$\begin{aligned}
S_T\, k^{1/2}\big(D^M + \sqrt{-1}\,Tc(V)\big)\,k^{-1/2}S_T^{-1}&\\
= \sqrt{T}\big(D^N + \sqrt{-1}\,c(J_V Z)\big) + D^H&\\
+ \frac{1}{\sqrt{T}}O\big(|Z|^2\partial^N + |Z|\partial^H + |Z| + |Z|^3\big)\,.&
\end{aligned} \tag{3.20}$$

Proof. Following the proof of [BL, Theorem 8.18], we get

$$S_T\, k^{1/2}D^M k^{-1/2}S_T^{-1} = \sqrt{T}D^N + D^H + \frac{1}{\sqrt{T}}O\big(|Z|^2\partial^N + |Z|\partial^H + |Z|\big)\,. \tag{3.21}$$

In fact the proof is much easier here because F is totally geodesic, hence the second fundamental form in [BL, (8.18)] vanishes. Next, we observe that $V = 0$ on F and that the vector field $J_V Z$ on N is the linear approximation of V on M near F. Further, since the actions of S^1 on N and M commute with the exponential map, $V(y,Z)$ is odd in Z, and hence the second order terms vanish. Therefore

$$S_T\, c(V)S_T^{-1} = \frac{1}{\sqrt{T}}c(J_V Z) + \frac{1}{\sqrt{T^3}}O(|Z|^3)\,. \tag{3.22}$$

The result follows. □

By Proposition 3.2, the solution space of the operator $D^N + \sqrt{-1}\,c(J_V Z)$ along the fiber N_y ($y \in F$) is (the L^2-completion of) $K^-(N_y) \otimes E_y$. These form an (infinite dimensional) Hermitian holomorphic bundle $K^-(N) \otimes E|_F$ over F, whose Hermitian connection ∇^F is induced from those in the bundles $N, E|_F \to F$. Let $\bar\partial^F$ be the corresponding operator of the twisted Dolbeault complex $\Omega^{0,*}(F, K^-(N) \otimes E|_F)$. Set $D^F = \sum_{i=1}^{2n_F} c(e_i)\nabla^F_{e_i} = \sqrt{2}(\bar\partial^F + (\bar\partial^F)^*)$.

Let $\mathbf{H}^0(F)$ be the Hilbert space of square-integrable sections of $\bigwedge(T^{*(0,1)}F) \otimes K^-(N) \otimes E|_F$, and $\mathbf{H}^0(N)$, that of the bundle $\tilde\pi^*((T^{*(0,1)}M) \otimes E)|_F)$, equipped with the Hermitian form (3.14). We define an embedding

$\psi\colon \mathbf{H}^0(F) \to \mathbf{H}^0(N)$ by

$$\psi\colon \alpha \otimes \beta \in \mathbf{H}^0(F) \longmapsto \tilde{\pi}^*\alpha \wedge \tau(\beta) \in \mathbf{H}^0(N)\,. \tag{3.23}$$

Here $\alpha \in \Omega^{0,*}(F)$, $\beta \in L^2(K^-(N) \otimes E|_F)$ and τ is the isometry from $L^2(K^-(N) \otimes E|_F)$ to $L^2(\tilde{\pi}^*(\bigwedge(N^{*(0,1)}) \otimes E|_F))$ given by Proposition 3.2. Let the image of ψ be $\mathbf{H}'^{,0} = \psi(\mathbf{H}^0(F)) \subset \mathbf{H}^0(N)$. Clearly, ψ is an isometry onto $\mathbf{H}'^{,0}$. Let $p\colon \mathbf{H}^0(N) \to \mathbf{H}'^{,0}$ be the orthogonal projection. Then we have the following analog of [BL, Theorem 8.21].

Proposition 3.4.

$$\psi^{-1} p\, D^H p\, \psi = D^F\,. \tag{3.24}$$

Proof. For simplicity, we prove in the case when E is a trivial line bundle and N is of rank 1 with isotropy weight $\lambda \in \mathbb{Z}\backslash\{0\}$. By the definitions of D^H and D^F, we need to prove

$$\tilde{\nabla}^F_{X^H}(\tau(\beta)) = \tau(\nabla^F_X \beta) \tag{3.25}$$

for any $\beta \in L^2(K^-(N))$ and any vector field X on F. On a small neighborhood $U \subset F$, choose a unitary trivialization $N|_U \cong U \times \mathbb{C} = \{(y,z) \mid y \in U, z \in \mathbb{C}\}$. Let $A \in \Omega^1(U)$ be the connection 1-form of the Hermitian connection ∇^N in $N \to F$. The horizontal lift of X is $X^H = X - i_X A\big(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\big)$. Notice that if $\lambda < 0$, then $\big(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\big)\varphi_k = k\varphi_k$, where φ_k ($k \in \mathbb{N}$) is given by (3.12). Therefore for $\beta(y,z) = f(y)\varphi_k(z)$, we have

$$\tilde{\nabla}^F_{X^H}\tau(\beta)(y,z) = X^H\big(f(y)\varphi_k(z)\big) = (Xf - k\, i_X A)(y)\varphi_k(z)\,; \tag{3.26}$$

this is the connection on $(N^*)^{\otimes k}$. On the other hand if $\lambda > 0$, then $\big(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\big)\varphi_{\bar{k}} = -k\varphi_{\bar{k}}$, where $\varphi_{\bar{k}}$ ($k \in \mathbb{N}$) is given by (3.13). Therefore for $\beta(y,z) = f(y)\varphi_{\bar{k}}(z)$, we have

$$\begin{aligned}\tilde{\nabla}^F_{X^H}\tau(\beta)(y,z) &= \big(\nabla^N_X(f(y)d\bar{z}) + k\, i_X A(y)\big)\varphi_{\bar{k}}(z)\\ &= \big(Xf + (k+1)\, i_X A\big)(y)\varphi_{\bar{k}}(z)\,;\end{aligned} \tag{3.27}$$

this is the connection on $N^{\otimes(k+1)}$. The result is proved. □

3.3 Estimates of the operator and resolvent as $T \to \infty$. For $p \geq 0$, let $\mathbf{H}^p(M), \mathbf{H}^p(N)$ and $\mathbf{H}^p(F)$ be the p-th Sobolev spaces of the sections of the bundles $\bigwedge(T^{*(0,1)}M) \otimes E \to M$, $\tilde{\pi}^*((\bigwedge(T^{*(0,1)}M) \otimes E)|_F) \to N$ and $\bigwedge(T^{*(0,1)}F) \otimes K^-(N) \otimes E|_F \to F$, respectively. The circle group S^1 acts on all these spaces; let $\mathbf{H}^p_\xi(M), \mathbf{H}^p_\xi(N)$ and $\mathbf{H}^p_\xi(F)$ be the corresponding weight spaces of weight $\xi \in \mathbb{Z}$. Recall the constant $\epsilon_0 > 0$ defined in the previous subsection. We now take $\epsilon \in \big(0, \frac{\epsilon_0}{2}\big]$, which is small enough for each eigenvalue of L_V we will consider, but otherwise can be assumed fixed. Let $\rho\colon \mathbb{R} \to [0,1)$ be a smooth function such that

$$\rho(a) = \begin{cases} 1, & \text{if } a \le 1/2\,, \\ 0, & \text{if } a \ge 1\,. \end{cases} \tag{3.28}$$

For $Z \in N$, set $\rho_\epsilon(Z) = \rho(|Z|/\epsilon)$. For $\beta \in L^2(K^-(N) \otimes E|_F)$, denote the component of weight $\xi \in \mathbb{Z}$ by $\beta_\xi \in L^2(K^-(N) \otimes E|_F)_\xi$. Let $\alpha \in L^2(\bigwedge(T^{*(0,1)}F))$. We define a linear map $I_{T,\xi}$ by

$$\sigma = \alpha \otimes \beta \in \mathbf{H}^0(F) \longmapsto I_{T,\xi}\sigma = \frac{\rho_\epsilon ||\beta||_0}{||\rho_\epsilon S_T^{-1}(\tau(\beta_\xi))||_0}\, \alpha \wedge S_T^{-1}(\tau(\beta_\xi))\,. \tag{3.29}$$

Let the image of $I_{T,\xi}$ from $\mathbf{H}^p(F)$ be $\mathbf{H}^p_{T,\xi}(N) = I_{T,\xi}\mathbf{H}^p(F) \subset \mathbf{H}^p_\xi(N)$. Denote the orthogonal complement of $\mathbf{H}^0_{T,\xi}(N)$ in $\mathbf{H}^0_\xi(N)$ by $\mathbf{H}^{0,\perp}_{T,\xi}(N)$, and let $\mathbf{H}^{p,\perp}_{T,\xi}(N) = \mathbf{H}^p_\xi(N) \cap \mathbf{H}^{0,\perp}_{T,\xi}(N)$. Let $p_{T,\xi}$ and $p^\perp_{T,\xi}$ be the orthogonal projections from $\mathbf{H}^0_\xi(N)$ onto $\mathbf{H}^0_{T,\xi}(N)$ and $\mathbf{H}^{0,\perp}_{T,\xi}(N)$, respectively.

Moreover, since the bundle $\bigwedge(T^{*(0,1)}M) \otimes E$ over U_{ϵ_0} is identified with $\tilde{\pi}^*((\bigwedge(T^{*(0,1)}M) \otimes E)|_F)$ over B_{ϵ_0}, we can consider $k^{-1/2}I_{T,\xi}\sigma$ as an element of $\mathbf{H}^p_\xi(M)$ for $\sigma \in \mathbf{H}^p(F)$. Define the linear map $J_{T,\xi}$ by

$$\sigma \in \mathbf{H}^p(F) \longmapsto J_{T,\xi}\sigma = k^{-1/2} I_{T,\xi}\sigma \in \mathbf{H}^p(M)\,. \tag{3.30}$$

$J_{T,\xi}$ is an isometry onto its image. Let $\mathbf{H}^p_{T,\xi}(M) = J_{T,\xi}\mathbf{H}^p(F)$. Denote the orthogonal complement of $\mathbf{H}^0_{T,\xi}(M)$ in $\mathbf{H}^0_\xi(M)$ by $\mathbf{H}^{0,\perp}_{T,\xi}(M)$, and let $\mathbf{H}^{p,\perp}_{T,\xi}(M) = \mathbf{H}^p_\xi(M) \cap \mathbf{H}^{0,\perp}_{T,\xi}(M)$. Let $\bar{p}_{T,\xi}$ and $\bar{p}^\perp_{T,\xi}$ be the orthogonal projections from $\mathbf{H}^0_\xi(M)$ onto $\mathbf{H}^0_{T,\xi}(M)$ and $\mathbf{H}^{0,\perp}_{T,\xi}(M)$, respectively. It is clear that $\bar{p}_{T,\xi} = k^{-1/2} p_{T,\xi} k^{1/2}$.

For any (possibly unbounded) operator A on $\mathbf{H}^0_\xi(M)$, write

$$A = \begin{pmatrix} A^{(1)} & A^{(2)} \\ A^{(3)} & A^{(4)} \end{pmatrix} \tag{3.31}$$

according to the decomposition $\mathbf{H}^0_\xi(M) = \mathbf{H}^0_{T,\xi}(M) \oplus \mathbf{H}^{0,\perp}_{T,\xi}(M)$, i.e., $A^{(1)} = \bar{p}_{T,\xi} A\, \bar{p}_{T,\xi}$, $A^{(2)} = \bar{p}_{T,\xi} A\, \bar{p}^\perp_{T,\xi}$ $A^{(3)} = \bar{p}^\perp_{T,\xi} A\, \bar{p}_{T,\xi}$, and $A^{(4)} = \bar{p}^\perp_{T,\xi} A\, \bar{p}^\perp_{T,\xi}$.

Let $D_T = D^M + \sqrt{-1}\, Tc(V)$. Let $D_{T,\xi}$ and D^F_ξ be the restrictions of the operators D_T and D^F on the subspaces $\mathbf{H}^0_\xi(M)$ and $\mathbf{H}^0_\xi(F)$, respectively, of weight $\xi \in \mathbb{Z}$.

PROPOSITION 3.5. 1. *As* $T \to \infty$,

$$J^{-1}_{T,\xi} D^{(1)}_{T,\xi} J_{T,\xi} = D^F_\xi + O\left(\frac{1}{\sqrt{T}}\right), \tag{3.32}$$

where $O(1/\sqrt{T})$ denotes a first order differential operator whose coefficients are dominated by $C/\sqrt{T}$ $(C>0)$.

2. *For each $\xi \in \mathbb{Z}$, there exists $C>0$ such that for any $T \geq 1$, $\sigma \in \mathbf{H}^{1,\perp}_{T,\xi}(M)$, $\sigma' \in \mathbf{H}^{1}_{T,\xi}(M)$, we have*

$$||D^{(2)}_{T,\xi}\sigma||_0 \leq C\left(\frac{||\sigma||_1}{\sqrt{T}} + ||\sigma||_0\right), \tag{3.33}$$

$$||D^{(3)}_{T,\xi}\sigma'||_0 \leq C\left(\frac{||\sigma'||_1}{\sqrt{T}} + ||\sigma'||_0\right). \tag{3.34}$$

3. *For each $\xi \in \mathbb{Z}$, there exist $\epsilon \in \left(0, \frac{\epsilon_0}{2}\right]$, $T_0 > 0$, $C>0$ such that for any $T \geq T_0$, $\sigma \in \mathbf{H}^{1,\perp}_{T,\xi}(M)$, we have*

$$||D^{(4)}_{T,\xi}\sigma||_0 \geq C\big(||\sigma||_1 + \sqrt{T}||\sigma||_0\big). \tag{3.35}$$

Proof. 1. This is the analogue of [BL, Theorem 9.8] and can be proved using the same method with little modification. In fact, by part 1 of Proposition 3.2, we have

$$\big(D^N + \sqrt{-1}\,c(J_V Z)\big)\tau(\beta_\xi) = 0 \tag{3.36}$$

for any $\beta_\xi \in L^2(K^-(N)\otimes E|_F)_\xi$; this is the analog of [BL, (9.23)]. Also, by a simple degree counting, we have, for any S^1-invariant $\zeta \in L^2(\tilde{\pi}^* N)$ and any $\beta_\xi \in L^2(K^-(N)\otimes E|_F)_\xi$,

$$c(\zeta)\tau(\beta_\xi) \perp \tau\big(L^2(K^-(N)\otimes E|_F)_\xi\big); \tag{3.37}$$

this is the analog of [BL, (9.25)]. Finally, from (3.20) we obtain easily the analog of [BL, (9.30)]. By these and by Proposition 3.4, which is the analog of [BL, Theorem 8.21], we obtain (3.32) immediately by following exactly the same arguments in [BL, pp. 104-105].

2. This is the analog of [BL, Theorem 9.10] and can be proved in the same way as in [BL, pp. 106-108] by using Proposition 3.3 and (3.36), (3.37). In fact, the situation is simpler here due to the absence of the terms corresponding to "$\tilde{\nabla}_Z\tilde{\nabla}_Z V(y)$" and "$V^+$" in [BL].

3. This is the analogue of [BL, Theorem 9.14] and, as in [BL, p. 117], follows from part 2 and the following proposition, which is the analog of [BL, Theorem 9.11]. □

PROPOSITION 3.6. *There exist $\epsilon \in \left(0, \frac{\epsilon_0}{2}\right)$, $T_0>0$, $C>0$, $b>0$ such that for any $T \geq T_0$, $\sigma \in \mathbf{H}^{1,\perp}_{T,\xi}(M)$, we have*

$$||D_{T,\xi}\sigma||_0^2 \geq C(||\sigma||_1^2 + (T-b)||\sigma||_0^2). \tag{3.38}$$

Proof. Just as in [BL, pp. 109-117], the proof here will consist of three steps:

(i) We show that for $\epsilon \in \left(0, \frac{\epsilon_0}{2}\right)$ small enough, if the support of $\sigma \in \mathbf{H}^{1,\perp}_{T,\xi}(M)$ is included in $U_{2\epsilon}$, then (3.38) holds;

(ii) $\epsilon \in \left(0, \frac{\epsilon_0}{2}\right)$ being now fixed, we show that if $\sigma \in \mathbf{H}^1_\xi(M)$ vanishes on $U_{\epsilon/2}$, then (3.38) still holds;

(iii) Using partition of unity, one finally proves (3.38) in full generality.

We remark that all the arguments in [BL] regarding these three steps can be made equivariant with respect to the S^1-action.

Step (i) can be proved in the same way as in [BL, pp. 111-114]. The only difference is that we should replace [BL, Theorem 7.4] by Proposition 3.2. Thus we encounter the kernel of $D^N + \sqrt{-1}\, c(J_V Z)$ on each normal space. Notice that any subspace of this kernel with a given weight ξ of S^1 is finite dimensional, whereas the corresponding kernel in [BL] is one dimensional. However this difference does not cause any trouble in following the arguments in [BL, pp. 111-114].

For step (ii), recall equation (3.6). For any $\sigma \in \mathbf{H}^1_\xi(M)$, σ satisfies $L_V\sigma = \sqrt{-1}\xi\sigma$. Therefore

$$\left|\left\langle\left(-2\sqrt{-1}L_V + \frac{1}{2}\sqrt{-1}\sum_{i=1}^{2n} c(e_i)c(\nabla_{e_i}V) + \sqrt{-1}\,\mathrm{tr}\,\nabla.V|_{T^{0,1}M} \right.\right.\right.$$
$$\left.\left.\left. - 2\sqrt{-1}\, r_V\right)\sigma, \sigma\right\rangle\right| \leq C'|\sigma||_0^2 \quad (3.39)$$

for some constant $C' > 0$. This is the analog of [BL, (9.91)]. Furthermore, $c(V)$ is invertible on $M \backslash U_{\epsilon/4}$. (3.38) then follows in exactly the same way as in [BL, p. 115].

After completing steps (i) and (ii), step (iii) follows in exactly the same way as in [BL, pp. 115-117].

The proof of Proposition 3.6, and thus part 3 of Proposition 3.5, is complete. □

For two bounded operators $A \in \mathcal{L}(\mathbf{H}^0_\xi(M))$, $B \in \mathcal{L}(\mathbf{H}^0_\xi(F))$, set

$$d(A, B) = \sum_{j=2}^{4} ||A^{(j)}||_1 + ||J^{-1}_{T,\xi} A^{(1)} J_{T,\xi} - B||_1 \quad (3.40)$$

where the operator norm is given by $||A||_1 = \mathrm{Tr}\,(A^*A)^{1/2}$. We fix a constant $c_0 \in (0, 1]$ such that

$$\mathrm{Spec}(D^F_\xi) \cap [-2c_0, 2c_0] \subset \{0\}\,, \quad (3.41)$$

where Spec denotes the spectrum of an operator. Then we have the following analog of [BL, Theorem 9.23].

PROPOSITION 3.7. *For any $\xi \in \mathbb{Z}$, there exists $T_0 \geq 1$, such that for any $T \geq T_0$, $\lambda \in \mathbb{C}$ with $|\lambda| = c_0$, $\lambda - D_{T,\xi}$ is invertible on $\mathbf{H}^0(M)$. Moreover, for any integer $p \geq 2n+2$, there exists $c_p > 0$ such that for any $T \geq T_0$, $\lambda \in \mathbb{C}$ with $|\lambda| = c_0$, we have*

$$d\big((\lambda - D_{T,\xi})^{-p}, (\lambda - D_\xi^F)^{-p}\big) \leq c_p/\sqrt{T}\,. \tag{3.42}$$

Proof. In view of Proposition 3.5, the result can be proved using exactly the same formal arguments in [BL, Sections 9c)-9e)]. □

3.4 Proof of Theorem 2.4. Let γ be the circle in $\mathbb{C}$ of center 0 and radius c_0, oriented counterclockwise when necessary. By Proposition 3.7, $\gamma \cap \mathrm{Spec}(D_{T,\xi}) = \emptyset$ for T large enough. Let $K_{T,\xi}^{c_0}$ be the direct sum of eigenspaces of $D_{T,\xi}$ associated to the eigenvalues λ such that $|\lambda| \leq c_0$. For T large enough,

$$P_{T,\xi}^{c_0} = \frac{1}{2\pi\sqrt{-1}} \int_\gamma (\lambda - D_{T,\xi})^{-1} d\lambda \tag{3.43}$$

is the orthogonal projection from $\mathbf{H}_\xi^0(M)$ onto $K_{T,\xi}^{c_0}$. Integrating by parts in (3.43), we get for any $p \in \mathbb{N}$,

$$P_{T,\xi}^{c_0} = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \lambda^{p-1} (\lambda - D_{T,\xi})^{-p} d\lambda\,. \tag{3.44}$$

Using Proposition 3.7, we obtain, for some $C > 0$,

$$d(P_{T,\xi}^{c_0}, P_\xi^F) \leq C/\sqrt{T}\,, \tag{3.45}$$

where P_ξ^F is the orthogonal projection from $\mathbf{H}_\xi^0(F)$ onto $K_\xi = \ker D_\xi^F$. Therefore for T large enough,

$$\dim_{\mathbb{C}} K_{T,\xi}^{c_0} = \dim_{\mathbb{C}} K_\xi = \sum_{k=0}^{n_F} \dim_{\mathbb{C}} H^k\big(F, \mathcal{O}(K^-(N) \otimes E|_F)_\xi\big)\,. \tag{3.46}$$

In fact, (3.46) holds for each degree in the Dolbeault complexes. By Hodge theorem and Proposition 3.2, we have for any $0 \leq k \leq n_F$,

$$J_{T,\xi}\, H^k\big(F, \mathcal{O}(K^-(N) \otimes E|_F)_\xi\big) \subset \mathbf{H}_\xi^0(M) \cap L^2\big(\textstyle\bigwedge^{k+(n-n_F-\nu_F)}(T^{*(0,1)}M) \otimes E\big)\,. \tag{3.47}$$

Let m_ξ^k be the sum of the multiplicities of all eigenvalues λ of $D_{T,\xi}$ on $(0,k)$-forms such that $|\lambda| < c_0$. Then

$$m_\xi^k = \dim_{\mathbb{C}} H^{k-(n-n_F-\nu_F)}\big(F, \mathcal{O}(K^-(N) \otimes E|_F)_\xi\big)\,. \tag{3.48}$$

On the other hand, an obvious application of (the equivariant version of) the Hodge theorem yields that m_ξ^k $(0 \leq k \leq n)$ satisfy the Morse-type inequalities

$$\sum_{k=0}^{n} t^k m_\xi^k = \sum_{k=0}^{n} t^k \operatorname{mult}_\xi H^k(M, \mathcal{O}(E)) + (1+t) Q_\xi(t)\,, \tag{3.49}$$

where $Q_\xi(t) \geq 0$. Combining (3.48) and (3.49), we get

$$\begin{aligned}\sum_{k=0}^{n} t^k \operatorname{mult}_\xi H^{k-(n-n_F-\nu_F)}\big(F, \mathcal{O}(K^-(N) \otimes E|_F)\big) \\ = \sum_{k=0}^{n} t^k \operatorname{mult}_\xi H^k(M, \mathcal{O}(E)) + (1+t) Q_\xi(t)\,. \end{aligned} \tag{3.50}$$

This is exactly (2.7). □

4 Applications

We consider various applications of the equivariant holomorphic Morse inequalities. In subsection 4.1, we apply the inequalities (2.2) to exterior powers of holomorphic cotangent bundles. The results are relations of the Hodge numbers of compact Kähler manifolds and the fixed submanifolds of torus actions. In subsection 4.2, we obtain a gluing formula of the Poincaré-Hodge polynomials in the context of symplectic cutting. We also study geometric quantization by applying the inequalities to the prequantum line bundles.

4.1 Relations among Hodge numbers of the fixed submanifolds. Consider a compact Kähler manifold (M, ω) of complex dimension n. Let $H^{k,l}(M) = H^l(M, \mathcal{E}^k(M))$ $(k, l = 0, 1, \ldots, n)$ be the Dolbeault cohomology groups of M, where $\mathcal{E}^k(M) = \mathcal{O}(\bigwedge^k(T^{*(1,0)}M))$ is the sheaf of holomorphic sections in the k-th exterior power of the holomorphic cotangent bundle of M. Let $h^{k,l}(M) = \dim_{\mathbb{C}} H^{k,l}(M)$ be the *Hodge numbers* of M and $P(M; s, t) = \sum_{k,l=0}^{n} s^k t^l h^{k,l}(M)$, the *Poincaré-Hodge polynomial.* (The Poincaré polynomial of M is $P(M; t) = P(M; t, t)$.) The *Todd genus* (or *arithmetic genus*) of M is given by $\tau(M) = P(M; 0, -1)$. We also have the well-known relations $h^{k,l}(M) = h^{l,k}(M) = h^{n-k,n-l}(M)$ for any $0 \leq k, l \leq n$, or $P(M; s, t) = P(M; t, s) = (st)^n P(M; s^{-1}, t^{-1})$ for compact Kähler manifolds.

Suppose that there is a holomorphic action of the torus group T on M preserving the Kähler form. As before, if the fixed-point set F is non-empty, it is a disjoint union of connected compact Kähler submanifolds $F_1, \ldots, F_m$, of (complex) dimensions $n_1, \ldots, n_m$, respectively. Recall that the weights $\lambda_{r,k}$ $(1 \le r \le m,\ 1 \le k \le n)$ of isotropy representations of T on the normal bundles of F_r cut the Lie algebra $\mathfrak{t}$ into action chambers.

Theorem 4.1. 1. *All the cohomology groups $H^{k,l}(M)$ $(k, l = 0, 1, \ldots, n)$ are trivial representations of T.*

2. *For any choice of action chamber C,*

$$\sum_{r=1}^{m} (st)^{\nu_r^C} P(F_r; s, t) = P(M; s, t)\,. \tag{4.1}$$

In particular, if all the fixed points are isolated, then for any choice of chamber C, we have

$$h^{k,k}(M) = \#\{p \in F \mid \nu_p^C = k\} \qquad (0 \le k \le n) \tag{4.2}$$

and $h^{k,l}(M) = 0$ for $k \ne l$.

Proof. The T-action can be lifted holomorphically to the bundles $\bigwedge^k (T^{*(1,0)} M)$ $(0 \le k \le n)$. Applying (2.2), we obtain, for some $Q^{k,C}(t) \ge 0$,

$$\sum_{r=1}^{m} t^{\nu_r^C} \sum_{l=0}^{n_r} t^l \operatorname{char} H^l(F_r, \mathcal{O}(T_r^{k,C})) = \sum_{l=0}^{n} t^l \operatorname{char} H^{k,l}(M) + (1+t) Q^{k,C}(t)\,, \tag{4.3}$$

where

$$T_r^{k,C} = S((N_r^{C,(1,0)})^*) \otimes S(N_r^{-C,(1,0)}) \otimes \textstyle\bigwedge^{\nu_r^C}(N_r^{-C,(1,0)}) \otimes \bigwedge^k (N_r^{*(1,0)} \oplus T^{*(1,0)} F_r)\,. \tag{4.4}$$

If $\lambda_{r,l}$ $(1 \le l \le n - n_r)$ are the weights of the T-action on the fiber of $N^{1,0}$, then the weights in T_r^k are of the form

$$-\sum_{l=1}^{n-n_r} m_l \lambda_{r,l}^C + \sum_{\lambda_{r,l} \in -C^*} \lambda_{r,l} - \sum_{l \in I} \lambda_{r,l}\,, \tag{4.5}$$

where $m_l \ge 0$ and $I \subset \{1, \ldots, n\}$ with $|I| \le k$. (4.5) is an element in the closed cone $-\overline{C^*}$; it is 0 if and only if all $m_l = 0$ and $I = \{l \mid \lambda_{r,l} \in -C^*\}$ (hence $|I| = \nu_r^C \le k$), corresponding to the sub-bundle $\bigwedge^{k-\nu_r^C}(T^{*(1,0)} F_r)$ (if $k \ge \nu_r^C$) of $T_r^{k,C}$. By the weak inequalities (2.3), we conclude that $\operatorname{supp} H^{k,l}(M) \subset -\overline{C^*} \cap \mathcal{L}^*$. Had we chosen the opposite chamber $-C$, we would get $\operatorname{supp} H^{k,l}(M) \subset \overline{C^*} \cap \mathcal{L}^*$. Therefore $\operatorname{supp} H^{k,l}(M) \subset \{0\}$, and hence $H^{k,l}(M)$ are trivial representations of T. Restricting (4.3) to $0 \in \mathcal{L}^*$, we get

$$\sum_{r=1}^{m} t^{\nu_r^C} \sum_{l=0}^{n_r} t^l h^{k-\nu_r^C, l}(F_r) = \sum_{l=0}^{n} t^l h^{k,l}(M) + (1+t) Q_0^{k,C}(t) \tag{4.6}$$

for some polynomial $Q_0^{k,C}(t) \geq 0$ in $\mathbb{Z}[t]$. By the symmetry of Hodge numbers, we obtain

$$\sum_{r=1}^{m} (st)^{\nu_r^C} P(F_r; s,t) = P(M; s,t) + (1+s)(1+t) Q_0^C(s,t)\,, \tag{4.7}$$

where $Q_0^C(s,t) \geq 0$ in $\mathbb{Z}[s,t]$ is defined by the relation $\sum_{k=0}^{n} s^k Q_0^{k,C}(t) = (1+s)Q_0^C(s,t)$. Now let $s = t$ in (4.7). Since the moment map (after projecting to any direction in $\pm C$) is a perfect Morse function in the sense of Bott (see for example [F]), we get $Q_0^C(t,t) = 0$. Therefore $Q_0^C(s,t) = 0$ and (4.1) follows. □

Remark 4.2. 1. Part 1 of Theorem 4.1 is in fact true for any connected group acting holomorphically and isometrically: Since the action of any group element is homotopic to the identity map, its actions on the de Rham cohomology groups $H^k(M)$ $(0 \leq k \leq 2n)$ are trivial. Moreover, since the action preserves the complex and Kähler structures, it preserves the Hodge decomposition. Hence the result.

2. Part 2 of the theorem in the case of $\mathbb{C}^\times$-actions was obtained by Carrell-Sommese [CaSo, Theorem 2]. For torus actions, that formula (4.1) is true for any choice of action chamber gives various constraints on the fixed-point data. For example, when the fixed points are isolated, the number of fixed points $p \in F$ with polarizing index $\nu_p^C = k$ is independent of the choice of the chamber C. In general, consider two action chambers $\pm C$. We have

$$\sum_{r=1}^{m} (st)^{\nu_r^{-C}} P(F_r; s,t) = \sum_{r=1}^{m} (st)^{\nu_r^C} P(F_r; s,t)\,. \tag{4.8}$$

In fact, using $\nu_r^{-C} = n - n_r - \nu_r^C$ and $P(F_r; s,t) = (st)^{n_r} P(F_r; s^{-1}, t^{-1})$, it is not hard to see that (4.8) is equivalent to $P(M; s,t) = (st)^n P(M; s^{-1}, t^{-1})$.

Corollary 4.3. *For any choice of action chamber C, there is only one component F_{r_C} with $\nu_{r_C}^C = 0$. Moreover, $h^{0,k}(M) = h^{0,k}(F_{r_C})$ for all $0 \leq k \leq n$. In particular, $\tau(M) = \tau(F_{r_C})$.*

Proof. The first part is well-known and is included here for completeness; in fact, F_{r_C} is the inverse image of a vertex of the moment polytope $\mu(M)$, which is a convex polytope in $\mathfrak{t}^*$ [A], [GSt1]. Taking $s = 0$ in (4.1), we get $P(M; 0,t) = P(F_{r_C}; 0,t)$. The rest follows easily. □

Example 4.4. Hamiltonian S^1-actions on symplectic 4-manifolds have been classified up to S^1-equivariant diffeomorphisms [Au], and subsequently, up to S^1-equivariant symplectomorphisms [K]. Moreover it was shown that all such manifolds are Kähler [K]. Let M be such a manifold with a Hamiltonian S^1-action and let $h\colon M \to \mathbb{R}$ be a moment map. Suppose $\Sigma_\pm$ are the critical components on which h reaches its maximum, minimum, respectively. For dimensional reasons, other critical points of h have Morse indices 2 and are isolated; let m_2 be the number of such. The Hodge-Poincaré polynomial of M is

$$\begin{aligned} P(M;s,t) &= P(\Sigma_-;s,t) + stP(\Sigma_+;s,t) + m_2 st \\ &= P(\Sigma_+;s,t) + stP(\Sigma_-;s,t) + m_2 st \end{aligned} \tag{4.9}$$

It follows that there are three possibilities:

I: Both $\Sigma_\pm$ are isolated points, i.e., $P(\Sigma_\pm;s,t) = 1$. In this case, $h^{0,0}(M) = h^{2,2}(M) = 1$, $h^{1,1}(M) = m_2 = b_2(M)$, others $= 0$.

II: One of them is reached at an isolated point, the other at a sphere, i.e., $P(\Sigma_\pm;s,t) = 1$ and st, respectively. In this case, $h^{0,0}(M) = h^{2,2}(M) = 1$, $h^{1,1}(M) = m_2 + 1 = b_2(M)$, others $= 0$.

III: Both $\Sigma_\pm$ are Riemann surfaces of the same genus, i.e., $P(\Sigma_\pm;s,t) = 1 + g(s+t) + st$, where g is the genus. In this case, $h^{0,0}(M) = h^{2,2}(M) = 1$, $h^{0,1}(M) = h^{1,0}(M) = h^{1,3}(M) = h^{3,1}(M) = g = \frac{1}{2}b_1(M)$, $h^{1,1}(M) = m_2 + 2 = b_2(M)$, others $= 0$.

In all cases, $b_2^+(M) = 1$.

If V is a holomorphic Killing vector field on (M,ω), let $Z_1,\dots,Z_m$ be the connected components of the zero-set of V, of (complex) dimensions $n_1,\dots,n_m$, respectively. We have the following special case of the theorem of Carrell-Lieberman [CaLi], [CaSo], which holds for a more general (not necessarily Killing) holomorphic vector field.

Corollary 4.5. *$h^{k,l}(M) = 0$ if $|k-l| > \max\{n_1,\dots,n_m\}$.*

Proof. The 1-parameter group generated by V is a subgroup of the isometry group of M. Therefore its closure is a torus T, whose fixed-point set is precisely Z. We choose any action chamber C (for example, the one which contains the generator V). Let $s = t^{-1}$ in (4.1). We get

$$\sum_{r=1}^{m} \sum_{k,l=0}^{n_r} h^{k,l}(Z_r) t^{l-k} = \sum_{k,l=0}^{n} h^{k,l}(M) t^{l-k}\,. \tag{4.10}$$

The result follows easily. □

When the fixed-point set F is discrete, this result was deduced from holomorphic Morse inequalities in [W2].

4.2 Symplectic quotients, symplectic cuts and quantization. Let (M,ω) be a symplectic manifold of (real) dimension $2n$. If the circle group S^1 acts Hamiltonianly on (M,ω), let V be the vector field on M that generates the action and $h\colon M\to\mathbb{R}$, the moment map such that $i_V\omega=dh$. If 0 is a regular value of h, then S^1 acts locally freely on $h^{-1}(0)$ and the symplectic quotient $M_0=h^{-1}(0)/S^1$ is an orbifold. Let $i\colon h^{-1}(0)\to M$ be the inclusion and $\pi\colon h^{-1}(0)\to M_0$, the projection. There is a canonical symplectic form ω_0 on M_0 such that $\pi^*\omega_0=i^*\omega$. To avoid orbifold singularities, we further assume that S^1 acts freely on $h^{-1}(0)$. In this case, M_0 is a smooth manifold of dimension $2n-2$.

We now recall the notion of symplectic cutting [Le]. Let the complex plane $\mathbb{C}$ be equipped with the standard Kähler form $\omega=\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}$. Consider two actions of the circle group S^1 on $\mathbb{C}$ with weights ±1. Both actions are Hamiltonian with the moment maps $\mp\frac{1}{2}|z|^2$, respectively. The diagonal actions of S^1 on $M\times\mathbb{C}$ are again Hamiltonian and the moment maps are $h\mp\frac{1}{2}|z|^2$, of which 0 is still a regular value. Let $M_\pm$ be the symplectic quotients of $M\times\mathbb{C}$ at level 0 by the two S^1-actions defined above. $(M_\pm,\omega_\pm)$ are smooth symplectic manifolds with Hamiltonian S^1-actions and M_0 is embedded as one of the components (still denoted by M_0) fixed by S^1; the compliments $M_\pm\backslash M_0$ are S^1-equivariantly symplectomorphic to $h^{-1}(\mathbb{R}^\pm)\subset M$, respectively. Therefore the fixed-point set of $M_\pm$ is the union of M_0 and all the components F_r $(1\le r\le m)$ such that $h(F_r)\in\mathbb{R}^\pm$. Moreover the circle bundle of the normal bundle N_0 of M_0 in M_+ is isomorphic to the bundle $h^{-1}(0)\to M_0$, with weight -1, while the normal bundle of M_0 in M_- is isomorphic to N_0^*, with weight 1.

When (M,ω) is a Kähler manifold with a Hamiltonian S^1-action preserving the complex structure, the symplectic quotient (M_0,ω_0) is also Kähler. In fact, such an S^1-action on M induces a holomorphic $\mathbb{C}^\times$-action, and as a complex manifold, $M_0=M^{\rm ss}/\mathbb{C}^\times$, where $M^{\rm ss}=\{x\in M\mid$ there exists $u\in\mathbb{C}^\times$ such that $ux\in\mu^{-1}(0)\}$ is called the *semi-stable* part of M [GSt2]. The two symplectic cuts $(M_\pm,\omega_\pm)$ are also Kähler because they are symplectic quotients of $M\times\mathbb{C}$ under the diagonal S^1-actions. The Kähler structures on $M_\pm$ are invariant under the remaining S^1-action on $M_\pm$. The fixed-point sets in M, $M_\pm$ are Kähler submanifolds.

LEMMA 4.6. *The Kähler structure of M_0 as a submanifold of $M_\pm$ is the same as that of M_0 as the symplectic quotient of M. The Kähler structure of F_r $(1\le r\le m)$ as a submanifold of $M_\pm$ is the same as that of F_r as a submanifold of M.*

Proof. We only need to check the matching of complex structures on these manifolds. As complex manifolds, $M_\pm = (M \times \mathbb{C})^{\mathrm{ss}}/\mathbb{C}^\times$, where $\mathbb{C}^\times$ acts on $\mathbb{C}$ with weights ± 1, respectively. Therefore the embedding image of M_0 in M_+ or M_- is $(M^{\mathrm{ss}} \times \{0\})/\mathbb{C}^\times$; this has the same complex structure as the symplectic quotient. Similarly, if $F_r \subset M$ $(1 \le r \le m)$ is a connected component of the fixed-point set in M, the corresponding set in M_+ or M_- is $(F_r \times \mathbb{C}^\times)/\mathbb{C}^\times$; this has the same complex structure as F_r itself. □

We now establish a gluing formula for the Poincaré-Hodge polynomials.

PROPOSITION 4.7. *Let (M, ω) be a compact Kähler manifold with a Hamiltonian S^1-action. If 0 is a regular value of the moment map h and S^1 acts freely on $h^{-1}(0)$, then*

$$P(M_+; s,t) + P(M_-; s,t) = P(M; s,t) + (1 + st)P(M_0; s,t)\,. \tag{4.11}$$

Proof. If we choose the positive chamber $C = \mathbb{R}^+$, then M_0 is embedded in $M_\pm$ with polarizing indices 1 and 0, respectively. Applying (4.1) to $M_\pm$ while taking into account of Lemma 4.6, we get

$$P(M_+; s,t) = \sum_{F_r \subset h^{-1}(\mathbb{R}^+)} (st)^{\nu_r} P(F_r; s,t) + st P(M_0; s,t) \tag{4.12}$$

and

$$P(M_-; s,t) = \sum_{F_r \subset h^{-1}(\mathbb{R}^-)} (st)^{\nu_r} P(F_r; s,t) + P(M_0; s,t)\,. \tag{4.13}$$

(4.11) follows from the above two equalities and (4.1). □

As a consequence, we have the following interesting

COROLLARY 4.8. *Under the above assumptions, $h^{0,k}(M) = h^{0,k}(M_0)$ for all $0 \le k \le n$. Hence $\tau(M) = \tau(M_0)$.*

Proof. From Corollary 4.3 we get $P(M_+; 0,t) = P(M_-; 0,t) = P(M_0; 0,t)$. This, together with (4.11) when $s = 0$, implies that $P(M; 0,t) = P(M_0; 0,t)$. □

By induction, these results are true for Hamiltonian torus actions. In fact, the second part is true when M is a general symplectic (hence almost complex) manifold with a (possibly) non-Abelian group action [MeSj], [TZ]. The first part is the refinement of this result when M is Kähler.

Recall that a symplectic manifold (M, ω) is *quantizable* if the de Rham class $[\omega/2\pi] \in H^2(M, \mathbb{Z})$. In this case, there is a complex line bundle, called the *pre-quantum line bundle*, with a connection ∇ whose curvature is $\omega/\sqrt{-1}$. We have the following

LEMMA 4.9. *Suppose that a symplectic manifold (M, ω) is quantizable and is equipped with a Hamiltonian S^1-action with moment map h. If one of $h(F_r) \in \mathbb{Z}$, then all $h(F_r) \in \mathbb{Z}$ and the S^1-action can be lifted to the pre-quantum line bundle L. In this case, $M_\pm$, M_0 are quantizable; let $L_\pm, L_0$ be their pre-quantum line bundles. There are following isomorphisms of line bundles with connections: $L_\pm|_{M_0} \cong L_0$, $L_\pm|_{M_\pm \setminus M_0} \cong L|_{h^{-1}(\mathbb{R}^\pm)}$.*

Proof. The generator of $\mathrm{Lie}(S^1)$ acts on the space of sections of L by $-\nabla_V + \sqrt{-1}h$; this gives an $\mathbb{R}$-action on L preserving ∇. If $h(F_r) \in \mathbb{Z}$, then the $\mathbb{R}$-action is an S^1-action on $L|_{F_r}$. Since ∇ is $\mathbb{R}$-invariant, the parallel transport commutes with the $\mathbb{R}$-action. Therefore the $\mathbb{R}$-action factorizes through S^1 on the total space of L. In particular, $h(F_r) \in \mathbb{Z}$ on any fixed component F_r. The line bundle $L_0 = i^*L/S^1 \to M_0$ has a connection ∇^0 such that $\pi^*\nabla^0 = i^*\nabla$ so that its curvature is $\omega_0/\sqrt{-1}$ [GSt2]. Following the construction of symplectic cuts, there are S^1-invariant pre-quantum line bundles $L_\pm$ on $M_\pm$ whose curvature is $\omega_\pm/\sqrt{-1}$. The isomorphisms in the last part were proved in [DGMeWu, Me2]. □

If in addition (M, ω) is Kähler, the pre-quantum line bundle L can be made into an S^1-invariant holomorphic Hermitian line bundle. Therefore the line bundle L_0 over M_0 is also holomorphic and Hermitian, and so are $L_\pm$. Furthermore, the actions of S^1 on $L_\pm$ preserve holomorphic structures. By *quantization* on (M, ω) we mean to associate to (M, ω) the virtual vector space

$$\mathcal{H}(M) = \bigoplus_{k=0}^{n} (-1)^k H^k(M, \mathcal{O}(L))\,. \tag{4.14}$$

When M is not a complex manifold, each individual cohomology group in (4.14) does not make sense, however the alternating sum can be defined as the index of a spin$^{\mathbb{C}}$-Dirac operator using only an almost complex structure. In [DGMeWu], it was proved that under this more general setting, we have the following relation on quantization, symplectic cutting and reduction

$$\dim_{\mathbb{C}} \mathcal{H}(M_\pm)^{S^1} = \dim_{\mathbb{C}} \mathcal{H}(M_0) = \dim_{\mathbb{C}} \mathcal{H}(M)^{S^1}\,, \tag{4.15}$$

and the gluing formula

$$\mathrm{char}\,\mathcal{H}(M) = \mathrm{char}\,\mathcal{H}(M_+) + \mathrm{char}\,\mathcal{H}(M_-) - \dim_{\mathbb{C}} \mathcal{H}(M_0)\,. \tag{4.16}$$

The last equality in (4.15) was the S^1-case of a conjecture by Guillemin and Sternberg [GSt2]; the cases with higher rank torus and non-Abelian group actions were proved by Meinrenken [Me1,2], Jeffrey and Kirwan [JKi], Vergne [V] and others under various generalities using localization techniques, and by Tian and Zhang [TZ] using a direct analytic approach. Moreover, when M is a compact Kähler manifold with a Hamiltonian action of a (possibly) non-Abelian group G, there are Morse-type inequalities which bound the invariant cohomologies of M in terms of those of the symplectic quotient M_0 (of complex dimension n_0) [TZ]. That is,

$$\sum_{k=0}^{n_0} t^k \dim_{\mathbb{C}} H^k(M_0, \mathcal{O}(L_0)) = \sum_{k=0}^{n} t^k \dim_{\mathbb{C}} H^k(M, \mathcal{O}(L))^G + (1+t)Q_0(t) \tag{4.17}$$

for some $Q_0(t) \geq 0$. We prove similar Morse-type inequalities relating quantizations on M_0 and $M_\pm$, which will strengthen the first equality of (4.15) when M is Kähler and the group is S^1.

Proposition 4.10. *If M is Kähler, then there exist polynomials $Q_0^\pm(t) \geq 0$ such that*

$$\sum_{k=0}^{n-1} t^k \dim_{\mathbb{C}} H^k(M_0, \mathcal{O}(L_0)) = \sum_{k=0}^{n} t^k \dim_{\mathbb{C}} H^k(M_\pm, \mathcal{O}(L_\pm))^{S^1} + (1+t)Q_0^\pm(t). \tag{4.18}$$

Proof. Consider the S^1-action on M_+, whose fixed-point set consists of M_0 and F_r with $h(F_r) > 0$. The weights of the S^1-action on the fibers of L_+ over M_0 and F_r are 0 and $h(F_r)$, respectively. Applying (2.7) to M_+, we obtain

$$\begin{aligned} \sum_{k=0}^{n-1} t^k \operatorname{char} H^k\big(M_0, \mathcal{O}(S(N_0^*) \otimes L_0)\big) \\ + \sum_{F_r \subset h^{-1}(\mathbb{R}^+)} t^k \operatorname{char} H^k\big(F_r, \mathcal{O}(K^-(N_r) \otimes L|_{F_r})\big) \\ = \sum_{k=0}^{n} t^k \operatorname{char} H^k(M_+, \mathcal{O}(L_+)) + (1+t)Q^+(t)\,, \end{aligned} \tag{4.19}$$

for $Q^+(t) \geq 0$. Notice that all the weights on $S(N_0^*) \otimes L_0$ and $K^-(N_r) \otimes L|_{F_r}$ are non-negative, and the zero weight comes only from the sub-bundle L_0 of the former. By restricting (4.19) to the zero weight, we obtain (4.18) for M_+. □

Remark 4.11. In the light of (4.16), we conjecture that there is an S^1-equivariant Mayer-Vietoris-type long exact sequence

$$\begin{aligned} \cdots \to H^k(M, \mathcal{O}(L)) \to H^k(M_+, \mathcal{O}(L_+)) \oplus H^k(M_-, \mathcal{O}(L_-)) \\ \to H^k(M_0, \mathcal{O}(L_0)) \to H^{k+1}(M, \mathcal{O}(L)) \to \cdots \end{aligned} \tag{4.20}$$

when M is Kähler. If so, then there is a polynomial $Q(t) \geq 0$ such that

$$\sum_{k=0}^{n} t^k \operatorname{char} H^k(M, \mathcal{O}(L)) + \sum_{k=0}^{n-1} t^k \dim_{\mathbb{C}} H^k(M_0, \mathcal{O}(L_0))$$
$$= \sum_{k=0}^{n} t^k \operatorname{char} H^k(M_+, \mathcal{O}(L_+))$$
$$+ \sum_{k=0}^{n} t^k \operatorname{char} H^k(M_-, \mathcal{O}(L_-)) + (1+t)Q(t) \,. \quad (4.21)$$

In fact, the polynomial $Q(t) \leq \sum_{k=0}^{n-1} t^k \operatorname{char} H^k(M_0, \mathcal{O}(L_0))$. Therefore (4.21), if correct, also implies

$$\sum_{k=0}^{n} t^k \operatorname{char} H^k(M_+, \mathcal{O}(L_+)) + \sum_{k=0}^{n} t^k \operatorname{char} H^k(M_-, \mathcal{O}(L_-))$$
$$+ \sum_{k=0}^{n-1} t^{k+1} \operatorname{char} H^k(M_0, \mathcal{O}(L_0))$$
$$= \sum_{k=0}^{n} t^k \operatorname{char} H^k(M, \mathcal{O}(L)) + (1+t)Q'(t)$$

for some $Q'(t) \geq 0$.

References

[A] M.F. Atiyah, Convexity and commuting Hamiltonians, Bull. London Math. Soc. 14 (1982), 1-15.

[ABo] M.F. Atiyah, R. Bott, A Lefschetz fixed point formula for elliptic complexes, Part I, Ann. Math. 86 (1967), 374-407; Part II, Ann. Math. 87 (1968), 451-491.

[AS] M.F. Atiyah, G. Segal, The index of elliptic operators. II, Ann. Math. 87 (1968), 531-545.

[ASi] M.F. Atiyah, I.M. Singer, The index of elliptic operators. III, Ann. Math. 87 (1968), 546-604.

[Au] M. Audin, Hamiltoniens périodiques sur les variétés symplectiques compactes de dimension 4, Géométrie symplectique et méchanique, Proceedings 1988 (C. Albert, ed.), Lecture Notes in Mathematics 1416, Springer, Berlin, Heidelberg, New York (1990), 1–25; The topology of torus action on symplectic manifolds, Progress in Mathematics, Vol. 93, Birkhäuser, Basel, Boston, Berlin, 1991, Ch. IV.

[B] J.-M. Bismut, The Witten complex and the degenerate Morse inequalities, J. Diff. Geom. 23 (1986), 207-240.

[BL] J.-M. Bismut, G. Lebeau, Complex immersions and Quillen metrics, Inst. Hautes Etudes Sci. Publ. Math. 74 (1991), 1-297.

[CG] A. Canas da Silva, V. Guillemin, On the Kostant multiplicity formula for group actions with non-isolated fixed points, Adv. Math. 123 (1996), 1-15.

[CaLi] J.B. Carrell, D. Lieberman, Holomorphic vector fields and Kaehler manifolds, Invent. Math. 23 (1973), 303-309.

[CaSo] J.B. Carrell, A.J. Sommese, Some topological aspects of $\mathbf{C}^*$ actions on compact Kaehler manifolds, Comment. Math. Helvetica 54 (1979), 567-582.

[DGMeWu] H. Duistermaat, V. Guillemin, E. Meinrenken, S. Wu, Symplectic reduction and Riemann-Roch for circle actions, Math. Res. Lett. 2 (1995), 259-266.

[F] T. Frankel, Fixed points and torsion on Kähler manifold, Ann. Math. 70 (1959), 1-8.

[GLeSt] V. Guillemin, E. Lerman, S. Sternberg, On the Kostant multiplicity formula, J. Geom. Phys. 5 (1988), 721-750.

[GSt1] V. Guillemin, S. Sternberg, Convexity properties of the moment mapping, Invent. Math. 67 (1982), 491-513.

[GSt2] V. Guillemin, S. Sternberg, Geometric quantization and multiplicities of group representations, Invent. Math. 67 (1982), 515-538.

[JKi] L. C. Jeffrey, F. C. Kirwan, Localization and the quantization conjecture, Topology 36 (1997), 647-693.

[K] Y. Karshon, Periodic Hamiltonian flows on four dimensional manifolds, preprint `dg-ga/9510004` (1995).

[Le] E. Lerman, Symplectic cuts, Math. Res. Lett. 2 (1995), 247-258.

[MWu] V. Mathai, S. Wu, Equivariant holomorphic Morse inequalities I: a heat kernel proof, J. Diff. Geom. 46 (1997), 78-98.

[Me1] E. Meinrenken, On Riemann-Roch formulas for multiplicities, J. Amer. Math. Soc. 9 (1996), 373-389

[Me2] E. Meinrenken, Symplectic surgery and the spin^c-Dirac operator, MIT math. preprint (1995), to appear in Adv. Math.

[MeSj] E. Meinrenken, R. Sjamaar, Singular reduction and quantization, preprint `dg-ga/9707023` (1997).

[PWu] E. Prato, S. Wu, Duistermaat-Heckman measures in a non-compact setting, Comp. Math. 94 (1994), 113-128.

[TZ] Y. Tian, W. Zhang, Symplectic reduction and quantization, C. R. Acad. Sci. Paris, Série I 324 (1997), 433-438; An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg, preprint (1996), to appear in Invent. Math.

[V] M. Vergne, Multiplicities formula for geometric quantization. I, II, Duke Math. J. 82 (1996), 143-179, 181-194.

[W1] E. Witten, Supersymmetry and Morse theory, J. Diff. Geom. 17 (1982), 661-692.

[W2] E. Witten, Holomorphic Morse inequalities, Algebraic and differential topology, Teubner-Texte Math., 70, ed. G. Rassias, Teubner, Leipzig, (1984), pp. 318-333.

[Wu] S. Wu, Equivariant holomorphic Morse inequalities II: torus and non-Abelian group actions, MSRI preprint No. 1996-013, dg-ga/9602008 (1996).

Siye Wu
Department of Pure Mathematics
The University of Adelaide
Adelaide, South Australia 5005
Australia
E-mail: swu@maths.adelaide.edu.au

Weiping Zhang
Nankai Institute of Mathematics
Tianjin, 300071
P. R. China
E-mail: weiping@sun.nankai.edu.cn

Received: February 1997
Revised version: August 1997