Geometric quantization on manifolds with boundary (joint with Xiaonan Ma)

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- $S(TM) = S_+(TM) \oplus S_-(TM)$ the bundle of spinors associated to (TM, g^{TM})

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- $D^E_{\partial M} : \Gamma\left((S(TM)\otimes E)|_{\partial M}\right) \to \Gamma\left((S(TM)\otimes E)|_{\partial M}\right)$ the induced Dirac operator on the boundary ∂M

- $D^E_{\pm,\partial M} : \Gamma\left((S_\pm(TM) \otimes E)|_{\partial M}\right) \rightarrow \Gamma\left((S_\pm(TM) \otimes E)|_{\partial M}\right)$ are elliptic and formally self-adjoint.

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- Atiyah-Patodi-Singer index theorem :

$$\mathrm{index}\left(D^E_{+,\mathrm{APS}}\right) = \int_M \widehat{A}(TM)\mathrm{ch}(E) - \overline{\eta}\left(D^E_{+,\partial M}\right)$$

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- Well defined $D^E_{+,\mathrm{APS}}$ as well as its index
- Can be extended to Dirac type operators which can be written as

$$D = D^E + A$$

with A an odd endomorphism of $S(TM) \otimes E$

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- Homotopy invariance :

$$\operatorname{ind}\left(D_{0,+,\mathrm{APS}}\right) = \operatorname{ind}\left(D_{1,+,\mathrm{APS}}\right)$$

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- Let $X^M \in \Gamma(TM)$ be defined by

$$X^M(m) = \left(X(m)^M\right)(m)$$

Equivariant APS index I

- We consider the Dirac type operator

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- Purpose : study the G-decomposition of

$$\ker\left(D_{T,+,\mathrm{APS}}^E\right) - \ker\left(D_{T,+,\mathrm{APS}}^E\right)^*$$

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- Finite sum on the right hand side (ellipticity).

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- Proposition. There exists $T_0 \geq 0$ such that for any $T \geq T_0$, $D^E_{T,+,\partial M}(\xi)$ is invertible

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$$\left(D^E_{T,+,\partial M}\right)^2 = \left(D^E_{+,\partial M}\right)^2 + TA - 2\sqrt{-1}T \sum_{i=1}^{\dim G} f_i \nabla_{V_i} + T^2 \left|X^M\right|^2$$

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- Thus, when $T >> 0$, $\left(D^E_{T,+,\partial M}\right)^2 > 0$. Q.E.D.

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- Definition (Possibly infinite sum)

$$Q_{\mathrm{APS}}\left(E|_M, X^M\right) = \bigoplus_{\xi\in\Lambda_+} m_\xi\left(E|_M, X^M\right)V^\xi.$$

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- Possibly infinite number of nonzero $\widehat{m}_\xi\left(E|_M, X^M\right) \in \mathbf{Z}$

Equivariant APS and transversal indices

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- Theorem (Ma-Zhang) For any $\xi \in \Lambda_+$, one has

$$m_\xi \left(E|_M, X^M \right) = \widehat{m}_\xi \left(E|_M, X^M \right).$$

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- Definition

$$Q_{\mathrm{red}}(L) = \bigoplus_{\xi \in \Lambda_+} m\left(L_\xi\right) V^\xi$$

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- Vergne conjecture (ICM 2006, plenary lecture) : If $\mu : M \to \mathfrak{g}^*$ is proper and $\mathrm{zero}(\mu^M) \subset M$ is compact, then

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  i.e., for any $\xi \in \Lambda_+$,

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- Reformulation of the Vergne conjecture

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- The deformation (used in defining $Q_{\mathrm{APS}}(E|_{\widehat{M}}, \mu^{\widehat{M}})$),

$$D_T^L = D^L + \sqrt{-1}Tc\left(\mu^M\right)$$

appeared first in Tian-Zhang's analytic proof of the Guillemin-Sternberg conjecture

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- Resolution (Ma-Zhang) : establish a general quantization formula without using the Vergne condition

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- Paradan later gave a different proof of our results

Thanks !