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SUB-SIGNATURE OPERATORS, η-INVARIANTS AND A RIEMANN-ROCH THEOREM FOR FLAT VECTOR BUNDLES**

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Abstract

The author presents an extension of the Atiyah-Patodi-Singer invariant for unitary representations [2, 3] to the non-unitary case, as well as to the case where the base manifold admits certain finer structures. In particular, when the base manifold has a fibration structure, a Riemann-Roch theorem for these invariants is established by computing the adiabatic limits of the associated η-invariants.

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§ 0. Introduction

Let M be an odd dimensional closed oriented manifold. Let ρ be a unitary representation of the fundamental group of M. By using their η-invariant, Atiyah-Patodi-Singer [2, 3] introduced an $\mathbf{R}/\mathbf{Z}$-valued smooth invariant associated to ρ. One of the purposes of this paper is to propose an extension of this invariant to the case of non-unitary representations, as well as to the case where the base manifold may admit certain finer structures.

To simplify the presentation, we work in the real category. Thus, let M be an odd dimensional closed oriented Riemannian manifold with Riemannian metric g^{TM}. Let F be a real flat vector bundle over M with flat connection ∇^F. Let g^F be an Euclidean metric on F. Then F admits canonically an Euclidean connection $\nabla^{F,e}$ (cf. [10, (4.3)]).

Let E be an even dimensional oriented sub-bundle of the tangent vector bundle TM. Then TM/E carries an induced orientation.

With these data in hand and by proceeding similarly as in [18, 19], we can construct a first order elliptic differential operator $D^F_{M/E,\mathrm{sig}}$ acting on $C^\infty(M;\wedge^{\mathrm{even}}(T^*M)\otimes F)$, which

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we call a (twisted) sub-signature operator associated to TM/E. In particular, when $E=0$, it is just the usual (real) signature operator twisted by the Euclidean vector bundle $(F, \nabla^{F,e})$. Furthermore, one verifies easily that $D^F_{M/E,\mathrm{sig}}$ is formally self-adjoint (resp. skew-adjoint) if $\dim(TM/E) = 4k+3$ (resp. $4k+1$) for some $k \in \mathbf{N}$.

In the case where $\dim(TM/E) \equiv 3 \bmod 4\mathbf{Z}$, we denote by

$$\overline{\eta}\left(D^F_{M/E,\mathrm{sig}}\right) = \frac{\dim\left(\ker D^F_{M/E,\mathrm{sig}}\right) + \eta\left(D^F_{M/E,\mathrm{sig}}\right)}{2} \tag{0.1}$$

the reduced η-invariant of $D^F_{M/E,\mathrm{sig}}$ in the sense of Atiyah-Patodi-Singer [1].

Let $\mathbf{R}_M$ denote the trivial line bundle over M with the trivial metric and the trivial flat connection. We will denote the corresponding operator $D^{\mathbf{R}_M}_{M/E,\mathrm{sig}}$ by $D_{M/E,\mathrm{sig}}$.

We can now state the following easy result.

Theorem 0.1. (a) *If* $\dim(TM/E) \equiv 1 \bmod 4\mathbf{Z}$, *then*

$$\phi(M/E, F) = \dim\left(\ker D^F_{M/E,\mathrm{sig}}\right) \in \mathbf{Z}/2\mathbf{Z}$$

does not depend on g^{TM} *and* g^F*;*

(b) *If* $\dim(TM/E) \equiv 3 \bmod 4\mathbf{Z}$, *then*

$$\phi(M/E, F) = \overline{\eta}\left(D^F_{M/E,\mathrm{sig}}\right) - (\dim F)\overline{\eta}(D_{M/E,\mathrm{sig}}) \in \mathbf{R}/\mathbf{Z}$$

does not depend on g^{TM} *and* g^F.

The smooth invariant in Theorem 0.1(a) may be viewed as an example of the Atiyah-Singer mod 2 index (cf. [4]) for skew-adjoint elliptic operators. On the other hand, when $E = 0$ and ∇^F preserves g^F, the smooth invariance of $\phi(M/E, F)$ in Theorem 0.1(b) was proved by Atiyah-Patodi-Singer in [2, 3]. Furthermore, one obtains in this case a refined $\mathbf{R}$-invariant (cf. [2]). However, for the general case where ∇^F may not preserve g^F, $\dim\left(\ker D^F_{M/E,\mathrm{sig}}\right)$ may well jump so that one only gets $\mathbf{R}/\mathbf{Z}$ invariants. From the other point of view, when ∇^F preserves g^F, the invariant $\phi(M/E, F)$ in Theorem 0.1(b) may be viewed as an example of the Atiyah-Patodi-Singer invariant defined in [3, Proposition 2.14].

Recall that Atiyah-Patodi-Singer [3] also have an extension of their invariants to $\mathbf{C}/\mathbf{Z}$-invariants for non-unitary flat vector bundles. Clearly, our generalization is not identical with theirs.

Now let $Z \to M \xrightarrow{\pi} B$ be a smooth fibration of even dimensional closed manifolds Z_b, $b \in B$, over an odd dimensional closed oriented manifold B. We make the assumption that the vertical tangent vector bundle TZ is oriented. Then TM carries a canonically induced orientation.

Let F be a flat vector bundle over M. Then F induces canonically a $\mathbf{Z}$-graded flat vector bundle $H^*(Z, F|_Z)$ over B (cf. [9, Section 3f)]).

One can then construct various smooth invariants as in Theorem 0.1. Our second main result establishes a relationship between these invariants.

Theorem 0.2. *The following identity holds,*

$$\phi(M/TZ,F)=\sum_{i=0}^{\dim Z}(-1)^i\phi(B,H^i(Z,F|_Z))-\mathrm{rk}(F)\sum_{i=0}^{\dim Z}(-1)^i\phi(B,H^i(Z,\mathbf{R}_Z)). \tag{0.2}$$

Theorem 0.2 may be thought of as a Riemann-Roch type theorem for these generalized Atiyah-Patodi-Singer invariants.

We prove Theorem 0.2 by using the methods and techniques of Bismut-Cheeger [6, 7] and Dai [13] to evaluate the adiabatic limits of the associated reduced η-invariants, under the procedure of enlarging the metric on B.

Theorems 0.1 and 0.2 can be extended easily to the case of complex flat vector bundles, with the obvious replacement of the mod 2 indices by the corresponding reduced η-invariants. Furthermore, when TM/E is spin we can construct smooth mod $\mathbf{Z}$ invariants by using the sub-Dirac operators studied in [16], and prove the corresponding Riemann-Roch type theorem for the fibration case. See Section 3 for more details.

This paper is organized as follows. In Section 1, we construct the sub-signature operators associated to sub-bundles of the tangent bundle of a manifold, as well as to flat vector bundles. We also construct the corresponding generalized Atiyah-Patodi-Singer invariants and prove Theorem 0.1 in this section. In Section 2, we apply the methods and techniques of Bismut-Cheeger [6, 7] and Dai [13] to prove Theorem 0.2. In Section 3, we discuss the extensions and analogues mentioned above for the case where TM/E is spin.

This paper was first written in 1998. We here submit in its original form. Some minor corrections suggested by the referee have been adopted.

§ 1. Sub-signature Operators and the Generalized Atiyah-Patodi-Singer Invariants for Flat Vector Bundles

The purpose of this section is to construct the objects involved in Theorem 0.1 and to prove Theorem 0.1. By this one gets a series of smooth invariants for flat vector bundles generalizing those of Atiyah-Patodi-Singer [2, 3].

This section is organized as follows. In (a), we recall some algebraic preliminaries. In (b), we construct the corresponding sub-signature operators. In (c), we prove certain Lichnerowicz type formulas needed for the proof of Theorem 0.1. In (d), we prove Theorem 0.1. There is also an appendix to this section in which we prove a regularity result for the η-function of sub-signature operators.

(a) Algebraic Preliminaries

Let V be an oriented Euclidean vector space of dimension n. If $e\in V$, let $e^*\in V^*$ correspond to e by the scalar product on V. If $e\in V$, set $c(e)=e^*\wedge -i_e$ (resp. $\widehat{c}(e)=e^*\wedge +i_e$), where $e^*\wedge$ and i_e are the standard notation for the exterior and interior multiplications acting on the (real) exterior algebra $\wedge^*(V^*)$. If $e,\ e'\in V$, the following identities hold,

$$\begin{aligned} c(e)c(e') + c(e')c(e) &= -2\langle e, e'\rangle, \\ \widehat{c}(e)\widehat{c}(e') + \widehat{c}(e')\widehat{c}(e) &= 2\langle e, e'\rangle, \\ c(e)\widehat{c}(e') + \widehat{c}(e')c(e) &= 0. \end{aligned} \tag{1.1}$$

If we view

$$\wedge^*(V^*) = \wedge^{\mathrm{even}}(V^*) \oplus \wedge^{\mathrm{odd}}(V^*)$$

as a $\mathbf{Z}_2$-graded space, then $c(e)$, $\widehat{c}(e)$ are odd elements of $\mathrm{End}(\wedge^*(V^*))$.

Let $e_1, \ldots, e_n$ be an oriented orthonormal basis of V.

Proposition 1.1. *Among the monomials in terms of $c(e_i)$'s and $\widehat{c}(e_i)$'s, only the term $c(e_1)\widehat{c}(e_1)\cdots c(e_n)\widehat{c}(e_n)$ has a nonzero supertrace. Moreover,*

$$\mathrm{Tr}_s\left[c(e_1)\widehat{c}(e_1)\cdots c(e_n)\widehat{c}(e_n)\right] = (-2)^n. \tag{1.2}$$

For a proof of (1.2), see [10, §4d)].

We now assume $n = 2m+1$.

Let $\widehat{c}^{\mathrm{odd}}(V)$ be the algebra generated by monomials of the form

$$c_{I,J} = c\left(e_{i_1}\right)\cdots c\left(e_{i_k}\right)\widehat{c}\left(e_{j_1}\right)\cdots\widehat{c}\left(e_{j_l}\right), \tag{1.3}$$

where both k and l are odd integers. Then $\widehat{c}^{\mathrm{odd}}(V)$ preserves $\wedge^{\mathrm{even}}(V^*)$ and $\wedge^{\mathrm{odd}}(V^*)$. We will view $\widehat{c}^{\mathrm{odd}}(V)$ as a subalgebra of $\mathrm{End}(\wedge^{\mathrm{even}}(V^*))$ (resp. $\mathrm{End}(\wedge^{\mathrm{odd}}(V^*))$).

Proposition 1.2. *Among the elements of the form* (1.3) *in $\widehat{c}^{\mathrm{odd}}(V)$, only the term $c(e_1)\widehat{c}(e_1)\cdots c(e_n)\widehat{c}(e_n)$ has a nonzero trace on $\wedge^{\mathrm{even}}(V^*)$ (resp. $\wedge^{\mathrm{odd}}(V^*)$). Moreover,*

$$\mathrm{Tr}^{\wedge^{\mathrm{even}}(V^*)}\left[c(e_1)\widehat{c}(e_1)\cdots c(e_n)\widehat{c}(e_n)\right] = -2^{2m} \tag{1.4}$$

$$(\mathit{resp.}\quad \mathrm{Tr}^{\wedge^{\mathrm{odd}}(V^*)}\left[c(e_1)\widehat{c}(e_1)\cdots c(e_n)\widehat{c}(e_n)\right] = 2^{2m}). \tag{1.5}$$

For a proof of (1.4), see [19, Section 1]. (1.5) follows from (1.2) and (1.4).

(b) Flat Vector Bundles and the Sub-signature Operators

Let M be an oriented closed manifold of dimension n. Let E be an oriented sub-bundle of the tangent vector bundle TM.

Let g^{TM} be a metric on TM. Let g^E be the induced metric on E. Let $E^\perp$ be the sub-bundle of TM orthogonal to E with respect to g^{TM}. Let $g^{E^\perp}$ be the metric on $E^\perp$ induced from g^{TM}. Then (TM, g^{TM}) has the following orthogonal splittings,

$$TM = E \oplus E^\perp, \qquad g^{TM} = g^E \oplus g^{E^\perp}. \tag{1.6}$$

Clearly, $E^\perp$ carries a canonically induced orientation. We identify the quotient bundle TM/E with $E^\perp$.

Let F be a (real) flat vector bundle over M. Let g^F be an Euclidean metric on F.

Let

$$\wedge^*(T^*M) = \bigoplus_{i=0}^{n} \wedge^i(T^*M)$$

be the (real) exterior algebra bundle of T^*M. Let

$$\Omega^*(M,F)=\bigoplus_{i=0}^{n}\Omega^i(M,F)=\bigoplus_{i=0}^{n}C^\infty(M;\wedge^i(T^*M)\otimes F)$$

be the set of smooth sections of $\wedge^*(T^*M)\otimes F$. Let $*$ be the Hodge star operator of g^{TM}. It extends on $\wedge^*(T^*M)\otimes F$ by acting on F as identity.[1] Then $\Omega^*(M,F)$ inherits the following standardly induced inner product

$$\langle\alpha,\beta\rangle=\int_M\langle\alpha\wedge*\beta\rangle_F,\qquad \alpha,\ \beta\in\Omega^*(M,F). \tag{1.7}$$

Denote $k=\dim E^\perp$. Let $f_1,\cdots,f_k$ be an oriented (local) orthonormal basis of $E^\perp$. Set

$$\widehat{c}\left(E^\perp,g^{E^\perp}\right)=\widehat{c}(f_1)\cdots\widehat{c}(f_k). \tag{1.8}$$

Clearly, $\widehat{c}\big(E^\perp,g^{E^\perp}\big)$ does not depend on the choice of the orthonormal basis.

Let

$$\epsilon=\mathrm{Id}_{\wedge^{\mathrm{even}}(T^*M)\otimes F}-\mathrm{Id}_{\wedge^{\mathrm{odd}}(T^*M)\otimes F}$$

be the $\mathbf{Z}_2$-grading operator of

$$\wedge^*(T^*M)\otimes F=\wedge^{\mathrm{even}}(T^*M)\otimes F\oplus\wedge^{\mathrm{odd}}(T^*M)\otimes F.$$

Set

$$\tau\left(M,g^{E^\perp}\right)=\epsilon\widehat{c}\left(E^\perp,g^{E^\perp}\right). \tag{1.9}$$

One verifies easily that

$$\tau\left(M,g^{E^\perp}\right)^2=(-1)^{\frac{k(k+1)}{2}}. \tag{1.10}$$

Thus, when $k\equiv 0 \bmod 4\mathbf{Z}$, $\tau\big(M,g^{E^\perp}\big)$ defines a $\mathbf{Z}_2$-grading of $\wedge^*(T^*M)\otimes F$. In this case, we will denote by

$$\wedge_\pm\left(T^*M,g^{E^\perp},F\right)=\left\{\omega\in\wedge^*(T^*M)\otimes F,\ \tau\left(M,g^{E^\perp}\right)\omega=\pm\omega\right\} \tag{1.11}$$

the (even/odd) eigen-bundles of $\tau\big(M,g^{E^\perp}\big)$ and by $\Omega_\pm\big(M,g^{E^\perp},F\big)$ the corresponding set of smooth sections.

Let ∇^F be the flat connection on F. Let d^F be the obvious extension of ∇^F on $\Omega^*(M,F)$. Let $\delta^F=d^{F*}$ be the formal adjoint operator of d^F with respect to the inner product (1.7).

Let $\widetilde{D}^F_{M/E}$ be the differential operator acting on $\Omega^*(M,F)$ defined by

$$\widetilde{D}^F_{M/E}=\frac{1}{2}\left(\widehat{c}\left(E^\perp,g^{E^\perp}\right)(d^F+\delta^F)+(-1)^k(d^F+\delta^F)\widehat{c}\left(E^\perp,g^{E^\perp}\right)\right). \tag{1.12}$$

[1]In what follows, whenever we extend an endomorphism of $\wedge^*(T^*M)$ to $\wedge^*(T^*M)\otimes F$, we mean that it is extended by acting on F as identity.

Then one verifies easily that

$$\widetilde{D}^F_{M/E}\tau\left(M,g^{E^\perp}\right)=-\tau\left(M,g^{E^\perp}\right)\widetilde{D}^F_{M/E},$$
$$\left(\widetilde{D}^F_{M/E}\right)^*=(-1)^{\frac{k(k+1)}{2}}\widetilde{D}^F_{M/E}, \tag{1.13}$$

where $\left(\widetilde{D}^F_{M/E}\right)^*$ is the formal adjoint of $\widetilde{D}^F_{M/E}$ with respect to the inner product (1.7).

Definition 1.1. *The sub-signature operator* $D^F_{M/E,\mathrm{sig}}$ *with respect to* (E,g^{TM},F) *is defined as follows:*

(a) *if* $\dim TM/E$ *is odd, then*

$$D^F_{M/E,\mathrm{sig}}=\widetilde{D}^F_{M/E}:\Omega^{\mathrm{even}}(M,F)\to\Omega^{\mathrm{even}}(M,F); \tag{1.14}$$

(b) *if* $\dim TM/E\equiv 0 \bmod 4\mathbf{Z}$*, then*

$$D^F_{M/E,\mathrm{sig}}=\widetilde{D}^F_{M/E}:\Omega_+\left(M,g^{E^\perp},F\right)\to\Omega_-\left(M,g^{E^\perp},F\right). \tag{1.15}$$

From (1.10)–(1.13), one sees that $D^F_{M/E,\mathrm{sig}}$ is well defined. In particular, if $\dim TM/E \equiv 1 \bmod 4\mathbf{Z}$, $D^F_{M/E,\mathrm{sig}}$ is formally skew-adjoint, while when $\dim TM/E\equiv 3 \bmod 4\mathbf{Z}$, $D^F_{M/E,\mathrm{sig}}$ is formally self-adjoint.

Remark 1.1. When both $\dim M$ and $\dim E$ are even integers and $F=\mathbf{R}_M$, the trivial line bundle over M with the trivial flat connection and the trivial flat metric, the sub-signature operators, in a complexified form, were constructed in [18]. While when $\dim TM/E=1$ and $F=\mathbf{R}_M$, the sub-signature operator has been constructed in [19].

Example 1.1. If one takes $E=TM$, then one has $D^F_{M/E,\mathrm{sig}}=d^F+\delta^F$, which is the de Rham-Hodge operator studied for example in [10]. While if $E=0$ and $F=\mathbf{R}_M$, one gets the usual (real) signature operator. Thus, in some sense, the sub-signature operator unifies the de Rham-Hodge operator and the signature operator.

(c) A Lichnerowicz Type Formula for Sub-signature Operators

We first recall some basic facts from [10, Section 4] concerning the flat vector bundle F.

Thus as in [10, (4.1)], set

$$\omega(F,g^F)=(g^F)^{-1}\nabla^F g^F, \tag{1.16}$$
$$\nabla^{F,e}=\nabla^F+\frac{1}{2}\omega(F,g^F). \tag{1.17}$$

Then $\nabla^{F,e}$ is an Euclidean connection on (F,g^F).

Let $\nabla^{\wedge^*(T^*M)}$ be the Euclidean connection on $\wedge^*(T^*M)$ induced canonically by the Levi-Civita connection ∇^{TM} of g^{TM}. Let ∇^e be the Euclidean connection on $\wedge^*(T^*M)\otimes F$ obtained from the tensor product of $\nabla^{\wedge^*(T^*M)}$ and $\nabla^{F,e}$.

Let $e_1,\cdots,e_{\dim M}$ be an oriented (local) orthonormal basis of TM. The following result was proved in [10, Proposition 4.12].

Proposition 1.3. *The following identity holds,*

$$d^F + \delta^F = \sum_{i=1}^{\dim M} c(e_i)\nabla^e_{e_i} - \frac{1}{2}\sum_{i=1}^{\dim M} \widehat{c}(e_i)\omega(F, g^F)(e_i). \tag{1.18}$$

Now recall that E is a sub-bundle of TM and that we have the orthogonal decomposition (1.6) of TM and the metric g^{TM}. Let P^E $\left(\text{resp. } P^{E^\perp}\right)$ be the orthogonal projection from TM to E (resp. $E^\perp$).

Set

$$\nabla^E = P^E \nabla^{TM} P^E, \quad \nabla^{E^\perp} = P^{E^\perp}\nabla^{TM} P^{E^\perp}. \tag{1.19}$$

Then ∇^E $\left(\text{resp. } \nabla^{E^\perp}\right)$ is an Euclidean connection on E $\left(\text{resp. } E^\perp\right)$. Let $\nabla^{\wedge^*(E^*)}$ (resp. $\nabla^{\wedge^*(E^{\perp,*})}$) be the Euclidean connection on $\wedge^*(E^*)$ (resp. $\wedge^*(E^{\perp,*})$) induced canonically from ∇^E $\left(\text{resp. } \nabla^{E^\perp}\right)$.

Let S be the tensor defined by

$$\nabla^{TM} = \nabla^E + \nabla^{E^\perp} + S. \tag{1.20}$$

Then S takes values in skew-adjoint endomorphisms of TM, and interchanges E and $E^\perp$.

Let $f_1, \cdots, f_k$ be an oriented (local) orthonormal basis of $E^\perp$. We will use the greek subscripts for the basis of $E^\perp$.

Definition 1.2. *Let $\nabla^{e,E^\perp}$ be the connection on $\wedge^*(T^*M)\otimes F$ defined by*

$$\nabla^{e,E^\perp}_X = \nabla^e_X - \frac{1}{2}\sum_{\alpha=1}^k \widehat{c}(S(X)f_\alpha)\widehat{c}(f_\alpha), \qquad X \in C^\infty(M;TM). \tag{1.21}$$

One verifies that $\nabla^{e,E^\perp}$ is still an Euclidean connection on $\wedge^*(T^*M)\otimes F$.

Set $l = \dim M - k$. Let $h_1, \cdots, h_l$ be an oriented (local) orthonormal basis of E.

Proposition 1.4. *The following identity holds,*

$$\widetilde{D}^F_{M/E} = \widehat{c}\left(E^\perp, g^{E^\perp}\right)\Big(\sum_{i=1}^{\dim M} c(e_i)\nabla^{e,E^\perp}_{e_i} - \frac{1}{2}\sum_{j=1}^{l}\widehat{c}(h_j)\omega(F,g^F)(h_j)\Big). \tag{1.22}$$

Proof. From (1.1), (1.12) and Proposition 1.3, one verifies directly that

$$\begin{aligned}\widetilde{D}^F_{M/E} = {} & \widehat{c}\left(E^\perp, g^{E^\perp}\right)\Big(\sum_{i=1}^{\dim M} c(e_i)\nabla^e_{e_i} - \frac{1}{2}\sum_{j=1}^{l}\widehat{c}(h_j)\omega(F,g^F)(h_j)\Big)\\ & + \frac{(-1)^k}{2}\sum_{i=1}^{\dim M} c(e_i)\left(\nabla^{\wedge^*(T^*M)}_{e_i}\widehat{c}\left(E^\perp, g^{E^\perp}\right)\right).\end{aligned} \tag{1.23}$$

Lemma 1.1. *For any $X \in C^\infty(M;TM)$, the following identity holds,*

$$\begin{aligned}&\nabla^{\wedge^*(T^*M)}_X \widehat{c}\left(E^\perp, g^{E^\perp}\right)\\ &= -\widehat{c}\left(E^\perp, g^{E^\perp}\right)\sum_{\alpha=1}^k \widehat{c}(S(X)f_\alpha)\widehat{c}(f_\alpha).\end{aligned} \tag{1.24}$$

Proof. By (1.20), one deduces that

$$\nabla_X^{\wedge^*(T^*M)}\widehat{c}\left(E^\perp, g^{E^\perp}\right)$$
$$= \nabla_X^{\wedge^*(E^{\perp,*})}\widehat{c}\left(E^\perp, g^{E^\perp}\right) + \sum_{\alpha=1}^{k}\widehat{c}(f_1)\cdots\widehat{c}(S(X)f_\alpha)\cdots\widehat{c}(f_k). \tag{1.25}$$

Now since

$$\widehat{c}\left(E^\perp, g^{E^\perp}\right)^2 = (-1)^{\frac{k(k-1)}{2}}, \tag{1.26}$$

one has

$$\left(\nabla_X^{\wedge^*(E^{\perp,*})}\widehat{c}\left(E^\perp, g^{E^\perp}\right)\right)\widehat{c}\left(E^\perp, g^{E^\perp}\right)$$
$$+\widehat{c}\left(E^\perp, g^{E^\perp}\right)\left(\nabla_X^{\wedge^*(E^{\perp,*})}\widehat{c}\left(E^\perp, g^{E^\perp}\right)\right) = 0. \tag{1.27}$$

From (1.27) and (1.1), one deduces easily that

$$\nabla_X^{\wedge^*(E^{\perp,*})}\widehat{c}\left(E^\perp, g^{E^\perp}\right) = 0. \tag{1.28}$$

(1.24) follows from (1.25) and (1.28).

(1.22) follows from (1.23), (1.24) and Definition 1.2.

Definition 1.3. *Let $D^F_{M/E}$ be the operator acting on $\wedge^*(T^*M)\otimes F$ defined by*

$$D^F_{M/E} = \sum_{i=1}^{\dim M} c(e_i)\nabla^{e,E^\perp}_{e_i} - \frac{1}{2}\sum_{j=1}^{l}\widehat{c}(h_j)\omega(F,g^F)(h_j). \tag{1.29}$$

Then one verifies that $D^F_{M/E}$ is a formally self-adjoint elliptic first order differential operator. Furthermore, by (1.22) the following identities hold,

$$\widetilde{D}^F_{M/E} = \widehat{c}\left(E^\perp, g^{E^\perp}\right)D^F_{M/E} = (-1)^k D^F_{M/E}\widehat{c}\left(E^\perp, g^{E^\perp}\right). \tag{1.30}$$

Let $\Delta^{e,E^\perp}$ be the Bochner Laplacian

$$\Delta^{e,E^\perp} = \sum_{i=1}^{\dim M}\left(\left(\nabla^{e,E^\perp}_{e_i}\right)^2 - \nabla^{e,E^\perp}_{\nabla^{TM}_{e_i}e_i}\right). \tag{1.31}$$

Let K be the scalar curvature of (M, g^{TM}). Let R^{TM}, R^E, $R^{E^\perp}$ be the curvatures of ∇^{TM}, ∇^E, $\nabla^{E^\perp}$ respectively. Now we can state the following Lichnerowicz type formula for $D^{F,2}_{M/E}$.

Theorem 1.1. *The following identity holds,*

$$D^{F,2}_{M/E} = -\Delta^{e,E^\perp} + \frac{K}{4} - \frac{1}{8}\sum_{i,j=1}^{\dim M} c(e_i)c(e_j)(\omega(F,g^F))^2(e_i,e_j)$$
$$+\frac{1}{8}\sum_{i,j=1}^{\dim M}\sum_{\alpha,\beta=1}^{k}\left\langle R^{E^\perp}(e_i,e_j)f_\beta,f_\alpha\right\rangle c(e_i)c(e_j)\widehat{c}(f_\alpha)\widehat{c}(f_\beta)$$
$$+\frac{1}{8}\sum_{i,j=1}^{\dim M}\sum_{s,t=1}^{l}\langle R^{E}(e_i,e_j)h_t,h_s\rangle c(e_i)c(e_j)\widehat{c}(h_s)\widehat{c}(h_t)$$
$$+\frac{1}{4}\sum_{j=1}^{l}(\omega(F,g^F)(h_j))^2+\frac{1}{8}\sum_{i,j=1}^{l}\widehat{c}(h_i)\widehat{c}(h_j)(\omega(F,g^F))^2(h_i,h_j)$$
$$-\frac{1}{4}\sum_{i=1}^{\dim M}\sum_{j=1}^{l}c(e_i)\widehat{c}(h_j)(\nabla^F_{e_i}\omega(F,g^F)(h_j)+\nabla^F_{h_j}\omega(F,g^F)(e_i)). \tag{1.32}$$

Proof. Set

$$D^0=\sum_{i=1}^{\dim M}c(e_i)\nabla^{e,E^\perp}_{e_i}. \tag{1.33}$$

From (1.20), one deduces that, for any $X\in C^\infty(M;TM)$,

$$\nabla_X^{\wedge^*(T^*M)}=\nabla_X^{\wedge^*(E^*)}$$
$$+\nabla_X^{\wedge^*(E^{\perp,*})}+\frac{1}{4}\sum_{i,j=1}^{\dim M}\langle S(X)e_i,e_j\rangle\left(c(e_i)c(e_j)-\widehat{c}(e_i)\widehat{c}(e_j)\right), \tag{1.34}$$

from which one has

$$\nabla_X^{\wedge^*(T^*M)}-\frac{1}{2}\sum_{\alpha=1}^{k}\widehat{c}(S(X)f_\alpha)\widehat{c}(f_\alpha)$$
$$=\nabla_X^{\wedge^*(E^*)}+\nabla_X^{\wedge^*(E^{\perp,*})}+\frac{1}{4}\sum_{i,j=1}^{\dim M}\langle S(X)e_i,e_j\rangle c(e_i)c(e_j). \tag{1.35}$$

Also, by [10, (4.6)] one has

$$(\nabla^{F,e})^2=-\frac{1}{4}(\omega(F,g^F))^2. \tag{1.36}$$

From (1.20), (1.21), (1.31), (1.35), (1.36) and the definition of ∇^e, one deduces easily the following extension of Lichnerowicz's formula (see [15]),

$$(D^0)^2=-\Delta^{e,E^\perp}+\frac{K}{4}-\frac{1}{8}\sum_{i,j=1}^{\dim M}c(e_i)c(e_j)(\omega(F,g^F))^2(e_i,e_j)$$
$$+\frac{1}{8}\sum_{i,j=1}^{\dim M}\sum_{\alpha,\beta=1}^{k}\left\langle R^{E^\perp}(e_i,e_j)f_\beta,f_\alpha\right\rangle c(e_i)c(e_j)\widehat{c}(f_\alpha)\widehat{c}(f_\beta)$$
$$+\frac{1}{8}\sum_{i,j=1}^{\dim M}\sum_{s,t=1}^{l}\langle R^{E}(e_i,e_j)h_t,h_s\rangle c(e_i)c(e_j)\widehat{c}(h_s)\widehat{c}(h_t). \tag{1.37}$$

From (1.1), (1.29), (1.33), (1.35), (1.37) and proceeding similarly as in [10, (4.33), (4.34)], one gets (1.32).

Remark 1.2. When $E = TM$, Theorem 1.1 was proved in [10, Theorem 4.13].

(d) The Generalized Atiyah-Patodi-Singer Invariants for Flat Vector Bundles

We assume from now on that M is of odd dimension.

We first restate Theorem 0.1 for convenience.

Theorem 1.2. (a) *If* $\dim(TM/E) \equiv 1 \bmod 4\mathbf{Z}$*, then* $\dim\left(\ker D^F_{M/E,\mathrm{sig}}\right) \bmod 2\mathbf{Z}$ *does not depend on* g^{TM} *and* g^F*;*

(b) *If* $\dim(TM/E) \equiv 3 \bmod 4\mathbf{Z}$*, then* $\overline{\eta}\left(D^F_{M/E,\mathrm{sig}}\right) - (\dim F)\overline{\eta}(D_{M/E,\mathrm{sig}}) \in \mathbf{R}/\mathbf{Z}$ *does not depend on* g^{TM} *and* g^F*.*

Proof. (a). From (1.13), (1.14), one knows that when $\dim(TM/E) \equiv 1 \bmod 4\mathbf{Z}$, $D^F_{M/E,\mathrm{sig}}$ is formally (real) skew-adjoint. Thus $\dim\left(\ker D^F_{M/E,\mathrm{sig}}\right) \bmod 2\mathbf{Z}$ is the mod 2 analytic index of $D^F_{M/E,\mathrm{sig}}$ in the sense of Atiyah-Singer [4], which clearly does not depend on the metrics g^{TM} and g^F used in its definition.

(b). We assume $k = 4m+3$. Then $l = \dim M - k = \dim E$ is even. Set $\widetilde{M} = M \times [0,1]$. Denote by $\pi : M \times [0,1] \to M$ the canonical projection. Then $(\pi^* F, \pi^* \nabla^F)$ is canonically a flat vector bundle over $\widetilde{M}$. At the same time, $\pi^* E$ is naturally a sub-bundle of $T\widetilde{M}$, the tangent vector bundle of $\widetilde{M}$.

For clarity, we use $M(i)$, $i = 0$ or 1, to denote the boundary copy of $\widetilde{M}$ at i. Thus one has the decomposition

$$\partial \widetilde{M} = M(0) \cup (-M(1)), \tag{1.38}$$

where $-M(1)$ means the inverse orientation with respect to that of $M(1)$.

Let $g^{TM}(i)$ (resp. $g^F(i)$), $i = 0$ or 1, be a Remannian metric on $TM(i)$ (resp. $F|_{M(i)}$). We denote by $D^F_{M/E,\mathrm{sig}}(i)$ the corresponding sub-signature operator on $M(i)$.

Let $\gamma : [0,1] \to [0,1]$ be a nonnegative function such that it equals 0 near 0 and equals 1 near 1. Then

$$g^{T\widetilde{M}} = dt^2 \oplus ((1-\gamma(t))\pi^* g^{TM}(0) + \gamma(t)\pi^* g^{TM}(1)) \tag{1.39}$$

$$(\textit{resp.} \quad g^{\pi^* F} = (1-\gamma(t))\pi^* g^F(0) + \gamma(t)\pi^* g^F(1)\ ) \tag{1.40}$$

defines a metric on $T\widetilde{M}$ (resp. $\pi^* F$), which is of product nature near the boundary of $\widetilde{M}$.

Now as $\dim(T\widetilde{M}/\pi^* E) = k+1 \equiv 0 \bmod 4\mathbf{Z}$, with the metrics $g^{T\widetilde{M}}$ and $g^{\pi^* F}$ one can construct the sub-signature operator on $\widetilde{M}$ as in Definition 1.1(b):

$$D^{\pi^* F}_{\widetilde{M}/\pi^* E,\mathrm{sig}} : \Omega_+\left(\widetilde{M}, g^{(\pi^* E)^\perp}, \pi^* F\right) \to \Omega_-\left(\widetilde{M}, g^{(\pi^* E)^\perp}, \pi^* F\right). \tag{1.41}$$

Furthermore, one can introduce the associated Atiyah-Patodi-Singer boundary condition to obtain an elliptic boundary value problem $\left(D^{\pi^* F}_{\widetilde{M}/\pi^* E,\mathrm{sig}}, P\right)$ (cf. [1]).

We are now going to apply the Atiyah-Patodi-Singer index theorem [1, Theorem 3.10] to compute the index of $\left(D^{\pi^* F}_{\widetilde{M}/\pi^* E,\mathrm{sig}}, P\right)$.

Let $\widetilde{M}'$ denote the double gluing of $\widetilde{M}$ which is now a closed Riemannian manifold with the Riemannian metric induced from $g^{T\widetilde{M}}$ obviously. We use the same notation $D^{\pi^*F}_{\widetilde{M}/\pi^*E,\mathrm{sig}}$ to denote its obvious extension on $\widetilde{M}'$. Denote by $P_t(x,y)$ (resp. $Q_t(x,y)$) the C^∞ kernel of

$$\exp\left(-t\left(D^{\pi^*F}_{\widetilde{M}/\pi^*E,\mathrm{sig}}\right)^* D^{\pi^*F}_{\widetilde{M}/\pi^*E,\mathrm{sig}}\right)$$
$$\left(\text{resp.}\quad \exp\left(-tD^{\pi^*F}_{\widetilde{M}/\pi^*E,\mathrm{sig}}\left(D^{\pi^*F}_{\widetilde{M}/\pi^*E,\mathrm{sig}}\right)^*\right)\right)$$

with respect to $dv_{\widetilde{M}'}$.

If $\omega\in\Omega^*(\widetilde{M})$ is a differential form over $\widetilde{M}$, denote by $\{\omega\}^{\max}\in\Omega^{\dim\widetilde{M}}(\widetilde{M})$ its top degree component. Also, we use the sign convention from [10] to define the Pfaffian of a skew-adjoint endomorphism.

Proposition 1.5. *The following convergence result holds uniformly for $x\in\widetilde{M}$,*

$$\begin{aligned}&\lim_{t\to 0}\left(\mathrm{tr}\left[P_t(x,x)\right]-\mathrm{tr}\left[Q_t(x,x)\right]\right)dv_{\widetilde{M}}(x)\\ &=2^{\frac{k+1}{2}}(\dim F)\Big\{\det{}^{1/2}\Big(\frac{R^{T\widetilde{M}}/4\pi}{\sinh(R^{T\widetilde{M}}/4\pi)}\Big)\\ &\quad\cdot\det{}^{1/2}\Big(\cosh\Big(\frac{R^{(\pi^*E)^\perp}}{4\pi}\Big)\Big)\\ &\quad\cdot\det{}^{1/2}\Big(\frac{\sinh(R^{\pi^*E}/4\pi)}{R^{\pi^*E}/4\pi}\Big)\mathrm{Pf}\Big(\frac{R^{\pi^*E}}{2\pi}\Big)\Big\}^{\max}(x)\end{aligned}\tag{1.42}$$

with the obvious notation for curvatures of connections on $T\widetilde{M}$ and its sub-bundles.

Proof. Let $D^{\pi^*F}_{\widetilde{M}/\pi^*E}$ be the operator acting on $\Omega^*(\widetilde{M},\pi^*F)$ defined as in Definition 1.3 with respect to the metrics $g^{T\widetilde{M}}$ and g^{π^*F}. We again use the same notation $D^{\pi^*F}_{\widetilde{M}/\pi^*E}$ to denote its extension on the double $\widetilde{M}'$. Then from (1.29), (1.30) and Definition 1.1(b), one verifies easily that

$$\left(D^{\pi^*F}_{\widetilde{M}/\pi^*E,\mathrm{sig}}\right)^* D^{\pi^*F}_{\widetilde{M}/\pi^*E,\mathrm{sig}}+D^{\pi^*F}_{\widetilde{M}/\pi^*E,\mathrm{sig}}\left(D^{\pi^*F}_{\widetilde{M}/\pi^*E,\mathrm{sig}}\right)^*=D^{\pi^*F,2}_{\widetilde{M}/\pi^*E}.\tag{1.43}$$

Let $T_t(x,y)$ denote the C^∞ kernel of $\exp\left(-tD^{\pi^*F,2}_{\widetilde{M}/\pi^*E}\right)$ with respect to $dv_{\widetilde{M}'}$.

From (1.43), (1.30), (1.13) and (1.11), one deduces that for any $x\in\widetilde{M}$,

$$\mathrm{tr}\left[P_t(x,x)\right]-\mathrm{tr}\left[Q_t(x,x)\right]=\mathrm{tr}_s\left[\widehat{c}\left((\pi^*E)^\perp,g^{(\pi^*E)^\perp}\right)T_t(x,x)\right],\tag{1.44}$$

where the supertrace tr_s is with respect to the natural even/odd $\mathbf{Z}_2$-grading of $\wedge^*(T^*\widetilde{M}\otimes\pi^*F)$.

Now consider the Lichnerowicz type formula (1.32) for $D^{\pi^*F,2}_{\widetilde{M}/\pi^*E}$, where we will use the obvious modified notation for curvatures in our situation.

Let $e_0=\frac{\partial}{\partial t},e_1,\cdots,e_{\dim M}$ be an oriented orthonormal basis of $T\widetilde{M}$ near x constructed from $e_j(0)=e_j(x)$, $0\le j\le\dim M$, through radial parallel transports. Let $(y_0,\cdots,y_{\dim M})$ be the associated normal coordinate system.

Let $\{e^j\}$ be the dual basis of $\{e_j\}$. Let ∂_j be the derivative in direction e_j.

Let $h_1, \cdots, h_l$ (resp. $f_0 = \frac{\partial}{\partial t}, f_1, \cdots, f_k$) be an orthonormal basis of $\pi^* E$ (resp. $(\pi^* E)^\perp$).

In view of Proposition 1.1 and Theorem 1.1, to compute the local index, it is convenient to use the rescaling

$$\partial_j \to \frac{1}{\sqrt{t}}\partial_j, \quad c(e_j) \to \frac{1}{\sqrt{t}} e^j \wedge -\sqrt{t} i_{e_j}, \quad \widehat{c}(e_j) \to \widehat{c}(e_j), \quad y_j \to \sqrt{t} y_j.$$

By (1.21), (1.31) and (1.32), one verifies easily that as $t \to 0$, after the above rescaling, $tD^{\pi^* F,2}_{\widetilde{M}/\pi^* E}$ has the limit

$$\begin{aligned} &-\sum_{r=0}^{\dim M}\Big(\partial_r + \frac{1}{8}\sum_{i,j,q=0}^{\dim M}\langle R^{T\widetilde{M}}(e_i,e_j)e_q, e_r\rangle y_q e^i \wedge e^j\Big)^2 \\ &+\frac{1}{8}\sum_{i,j=0}^{\dim M}\sum_{\alpha,\beta=0}^{k}\Big\langle R^{(\pi^*E)^\perp}(e_i,e_j)f_\beta, f_\alpha\Big\rangle e^i\wedge e^j \widehat{c}(f_\alpha)\widehat{c}(f_\beta) \\ &+\frac{1}{8}\sum_{i,j=0}^{\dim M}\sum_{s,t=1}^{l}\langle R^{\pi^*E}(e_i,e_j)h_t, h_s\rangle e^i\wedge e^j\widehat{c}(h_s)\widehat{c}(h_t) \\ &-\frac{1}{8}\sum_{i,j=0}^{\dim M}(\omega(\pi^*F, g^{\pi^*F}))^2(e_i,e_j)e^i\wedge e^j \end{aligned} \tag{1.45}$$

with the obvious notation for curvatures on $T\widetilde{M}$ and its sub-bundles.

Let $\widehat{c}(T\widetilde{M})$ be the Clifford algebra generated by the $\widehat{c}(e_j)$'s with the generating relation (1.1).

Let $\int^\wedge : \Omega^*(\widetilde{M})\widehat{\otimes}\widehat{c}(T\widetilde{M}) \to \Omega^*(\widetilde{M})$ be the Berezin integral defined by

$$\int^\wedge (\alpha\beta) = \alpha[\beta]^{\max}, \qquad \alpha \in \Omega^*(\widetilde{M}),\ \beta \in \widehat{c}(T\widetilde{M}), \tag{1.46}$$

where $[\beta]^{\max}$ is the coefficient of the term $\widehat{c}(e_0)\cdots\widehat{c}(e_{\dim M})$ in β.

Also we write the curvatures as the matrices of 2-forms as

$$R^{T\widetilde{M}} = \frac{1}{2}\sum_{i,j=0}^{\dim M} R^{T\widetilde{M}}(e_i,e_j)e^i\wedge e^j,$$

$$R^{\pi^*E} = \frac{1}{2}\sum_{i,j=0}^{\dim M} R^{\pi^*E}(e_i,e_j)e^i\wedge e^j,$$

$$R^{(\pi^*E)^\perp} = \frac{1}{2}\sum_{i,j=0}^{\dim M} R^{(\pi^*E)^\perp}(e_i,e_j)e^i\wedge e^j, \tag{1.47}$$

$$(\omega(\pi^*F, g^{\pi^*F}))^2 = \frac{1}{2}\sum_{i,j=0}^{\dim M}(\omega(\pi^*F, g^{\pi^*F}))^2(e_i,e_j)e^i\wedge e^j. \tag{1.48}$$

By (1.45)–(1.48), Proposition 1.1 and by proceeding the by now standard local index techniques (cf. [11]), one gets

$$\begin{aligned}&\lim_{t\to 0}\mathrm{tr}_s\left[\widehat{c}\left((\pi^*E)^\perp, g^{(\pi^*E)^\perp}\right)T_t(x,x)\right]dv_{\widetilde{M}}(x)\\ &=(-1)^{\frac{\dim M+1}{2}}2^{\dim M+1}\\ &\cdot\left(\frac{1}{4\pi}\right)^{\frac{\dim M+1}{2}}\Big\{\det{}^{1/2}\Big(\frac{R^{T\widetilde{M}}/2}{\sinh(R^{T\widetilde{M}}/2)}\Big)\mathrm{Tr}^F\Big[\exp\Big(\frac{1}{4}(\omega(\pi^*F,g^{\pi^*F}))^2\Big)\Big]\\ &\cdot\int^{\wedge}\Big\{\widehat{c}\left((\pi^*E)^\perp, g^{(\pi^*E)^\perp}\right)\exp\Big(\sum_{\alpha,\beta=0}^{k}\frac{1}{4}\left\langle R^{(\pi^*E)^\perp}(f_\alpha),f_\beta\right\rangle\widehat{c}(f_\alpha)\widehat{c}(f_\beta)\\ &+\frac{1}{4}\sum_{s,q=1}^{l}\langle R^{\pi^*E}(h_s),h_q\rangle\widehat{c}(h_s)\widehat{c}(h_q)\Big)\Big\}\Big\}^{\max}(x),\qquad x\in\widetilde{M}.\end{aligned}\tag{1.49}$$

Now one verifies easily the important fact that

$$\mathrm{Tr}^F\Big[\exp\Big(\frac{1}{4}(\omega(\pi^*F,g^{\pi^*F}))^2\Big)\Big]=\dim F\tag{1.50}$$

(cf. [10, (3.77)]). On the other hand, by a simple algebraic result (cf. [11, Proposition 3.13]), the Berezin integral in the right hand side of (1.49) is easily seen to be

$$(-1)^{l/2}\det{}^{1/2}\Big(\cosh\Big(\frac{R^{(\pi^*E)^\perp}}{2}\Big)\Big)\det{}^{1/2}\Big(\frac{\sinh(R^{\pi^*E}/2)}{R^{\pi^*E}/2}\Big)\mathrm{Pf}\Big(\frac{R^{\pi^*E}}{2}\Big).\tag{1.51}$$

(1.42) follows from (1.44), (1.49)–(1.51) and the fact that $k+1\equiv 0 \bmod 4\mathbf{Z}$.

The proof of Proposition 1.5 is completed.

From Proposition 1.5, the index theorem of Atiyah-Patodi-Singer [1, Theorem 3.10], as well as an easy parity consideration on the boundary similar to that in [1, p.63], one gets

$$\begin{aligned}&2\overline{\eta}\left(D^F_{M/E,\mathrm{sig}}(0)\right)-2\overline{\eta}\left(D^F_{M/E,\mathrm{sig}}(1)\right)\\ &+\mathrm{ind}\left(D^{\pi^*F}_{\widetilde{M}/\pi^*E,\mathrm{sig}},P\right)+2\dim\left(\ker D^F_{M/E,\mathrm{sig}}(1)\right)\\ &=2^{\frac{k+1}{2}}(\dim F)\int_{\widetilde{M}}\Big\{\det{}^{1/2}\Big(\frac{R^{T\widetilde{M}}/4\pi}{\sinh(R^{T\widetilde{M}}/4\pi)}\Big)\det{}^{1/2}\Big(\cosh\Big(\frac{R^{(\pi^*E)^\perp}}{4\pi}\Big)\Big)\\ &\cdot\det{}^{1/2}\Big(\frac{\sinh(R^{\pi^*E}/4\pi)}{R^{\pi^*E}/4\pi}\Big)\mathrm{Pf}\Big(\frac{R^{\pi^*E}}{2\pi}\Big)\Big\}.\end{aligned}\tag{1.52}$$

Now an easy application of [14, Theorem 1.1] and the parity consideration mentioned above shows that

$$\begin{aligned}&\mathrm{ind}\left(D^{\pi^*F}_{\widetilde{M}/\pi^*E,\mathrm{sig}},P\right)+2\dim\left(\ker D^F_{M/E,\mathrm{sig}}(1)\right)\\ &=2\mathrm{sf}\left\{D^F_{M/E,\mathrm{sig}}(0),D^F_{M/E,\mathrm{sig}}(1)\right\},\end{aligned}\tag{1.53}$$

where ‘sf’ is the notation for the spectral flow of Atiyah-Patodi-Singer [3].

From (1.52) and (1.53), one gets

$$\begin{aligned}&\overline{\eta}\left(D^F_{M/E,\mathrm{sig}}(0)\right)-\overline{\eta}\left(D^F_{M/E,\mathrm{sig}}(1)\right)\\ &\equiv 2^{\frac{k-1}{2}}(\dim F)\int_{\widetilde{M}}\Big\{\det{}^{1/2}\Big(\frac{R^{T\widetilde{M}}/4\pi}{\sinh(R^{T\widetilde{M}}/4\pi)}\Big)\det{}^{1/2}\Big(\cosh\Big(\frac{R^{(\pi^*E)^\perp}}{4\pi}\Big)\Big)\\ &\quad\cdot\det{}^{1/2}\Big(\frac{\sinh(R^{\pi^*E}/4\pi)}{R^{\pi^*E}/4\pi}\Big)\mathrm{Pf}\Big(\frac{R^{\pi^*E}}{2\pi}\Big)\Big\}\mod \mathbf{Z}.\end{aligned}\tag{1.54}$$

From (1.54) and its application to the trivial line bundle case, one gets

$$\begin{aligned}&\overline{\eta}\left(D^F_{M/E,\mathrm{sig}}(0)\right)-(\dim F)\overline{\eta}\left(D_{M/E,\mathrm{sig}}(0)\right)\\ &\equiv\overline{\eta}\left(D^F_{M/E,\mathrm{sig}}(1)\right)-(\dim F)\overline{\eta}\left(D_{M/E,\mathrm{sig}}(1)\right)\mod\mathbf{Z},\end{aligned}\tag{1.55}$$

which completes the proof of Theorem 1.2(b).

Remark 1.3. The local index result Proposition 1.5, in a complexified form, was first announced in [18] when F is the trivial line bundle.

With Theorem 1.2, we define a series of smooth invariants as follows.

Definition 1.4. *The smooth invariant $\phi(M/E,F)$ is defined by*
(a) *if* $\dim(TM/E)\equiv 1 \mod 4\mathbf{Z}$, *then*

$$\phi(M/E,F)=\dim\left(\ker D^F_{M/E,\mathrm{sig}}\right)\mod 2\mathbf{Z};\tag{1.56}$$

(b) *if* $\dim(TM/E)\equiv 3 \mod 4\mathbf{Z}$, *then*

$$\phi(M/E,F)=\overline{\eta}\left(D^F_{M/E,\mathrm{sig}}\right)-(\dim F)\overline{\eta}\left(D_{M/E,\mathrm{sig}}\right)\mod\mathbf{Z}.\tag{1.57}$$

Remark 1.4. As was mentioned in Introduction, when ∇^F preserves g^F, the invariant in (1.57) provides an example of the Atiyah-Patodi-Singer invariants defined in [3, Proposition 2.14]. While, when $\dim(TM/E)=1$ and F is the trivial line bundle, the mod 2 invariant in (1.56) was constructed in [19].

Appendix. Remarks on the η-Invariant (and/or the mod 2 Index) of Subsignature Operators

We first assume $\dim(TM/E)\equiv 3\mod 4\mathbf{Z}$. Then $D^F_{M/E,\mathrm{sig}}$ is formally self-adjoint. The following is an analogue of [8, Theorem 2.4].

Proposition A.1. *As $t\downarrow 0$,*

$$\mathrm{Tr}\left[D^F_{M/E,\mathrm{sig}}\exp\left(-tD^{F,2}_{M/E,\mathrm{sig}}\right)\right]=O(\sqrt{t}\,).\tag{A.1}$$

Proof. Let $D^F_{M/E}$ be defined as in (1.29). By (1.30), for any $t>0$,

$$\mathrm{Tr}\left[D^F_{M/E,\mathrm{sig}}\exp\left(-tD^{F,2}_{M/E,\mathrm{sig}}\right)\right]=\mathrm{Tr}^{\mathrm{even}}\left[\widehat{c}\left(E^\perp,g^{E^\perp}\right)D^F_{M/E}\exp\left(-tD^{F,2}_{M/E}\right)\right],\tag{A.2}$$

where by $\mathrm{Tr}^{\mathrm{even}}$ we mean the trace taken on $\Omega^{\mathrm{even}}(M,F)$.

For any $t>0$, let $P_t(x,y)$ be the C^∞ kernel associated with $D^F_{M/E}\exp\left(-tD^{F,2}_{M/E}\right)$ where we regard that the later acts on $\Omega^*(M,F)$. Then in order to prove (A.1), we need only to show that as $t\downarrow 0$,

$$\mathrm{Tr}^{\mathrm{even}}\left[\widehat{c}\left(E^\perp, g^{E^\perp}\right)P_t(x,x)\right]=O(\sqrt{t}\,),\qquad \text{uniformly on } M, \tag{A.3}$$

where the trace is taken on $(\wedge^{\mathrm{even}}(T^*M)\otimes F)_x$.

Let D^0 be defined as in (1.33). Then from (1.21) and (1.35), one sees easily that locally, one can view D^0 as a standard twisted Dirac operator.

From (1.29), (1.33), one has

$$D^F_{M/E}=D^0-\frac{1}{2}\sum_{j=1}^{l}\widehat{c}(h_j)\omega(F,g^F)(h_j). \tag{A.4}$$

Furthermore, by (1.1), each $\widehat{c}(h_j)$ anticommutes with each $c(e_i)$. Thus formally we are in the same situation as what considered in [6, Sections 2, 3]. In fact, by proceeding as in [6, Sections 2, 3] and [8, Theorem 2.4], one gets easily that as $t\downarrow 0$,

$$P_t(x,x)=\sum_{i=0}^{(\dim M-1)/2}a_i(x)t^{i-\dim M/2}+O(t^{1/2}),\qquad \text{uniformly on } M, \tag{A.5}$$

where each $a_i(x)$ can be written as a linear combination of terms of the form

$$c(e_{i_1}(x))\cdots c(e_{i_s}(x))\widehat{c}(e_{\widehat{i}_1}(x))\cdots\widehat{c}(e_{\widehat{i}_q}(x))f\qquad \text{with } f\in\mathrm{End}(F_x),\ \ s\le 2i. \tag{A.6}$$

On the other hand, from the Lichnerowicz type formula (1.32) and the above argument, one sees easily that the terms of the form $\widehat{c}(f_\alpha)$ appear even times in (A.6). Together with the fact that $\dim(TM/E)$ is odd, one sees easily that each $\widehat{c}(E^\perp,g^{E^\perp})a_i(x)$, $0\le i\le\frac{\dim M-1}{2}$, can be expressed as a linear combination of terms of form (A.6) with a further condition that $\widehat{i}_1,\cdots,\widehat{i}_q$ are not equal to each other and that $q\ge 1$. One then verifies by Proposition 1.2 that

$$\mathrm{Tr}^{\mathrm{even}}\left[\widehat{c}\left(E^\perp,g^{E^\perp}\right)a_i(x)\right]=0,\qquad 0\le i\le\frac{\dim M-1}{2}. \tag{A.7}$$

(A.1) follows from (A.2), (A.3), (A.5) and (A.7).

From Proposition A.1, one gets immediately the following formula for the η-invariant of $D^F_{M/E,\mathrm{sig}}$,

$$\eta\left(D^F_{M/E,\mathrm{sig}}\right)=\frac{1}{\sqrt{\pi}}\int_0^\infty\mathrm{Tr}\left[D^F_{M/E,\mathrm{sig}}\exp\left(-tD^{F,2}_{M/E,\mathrm{sig}}\right)\right]\frac{dt}{\sqrt{t}}. \tag{A.8}$$

Now observe that by the arguments in the proof of Proposition A.1, one sees easily that the following analogue of [6, Lemma 2.11] holds.

Proposition A.2. *For any $u\in\mathbf{R}$, as $t\downarrow 0$,*

$$\mathrm{Tr}^{\mathrm{even}}\Big[\widehat{c}\left(E^\perp,g^{E^\perp}\right)\Big(\sum_{j=1}^{l}\widehat{c}(h_j)\omega(F,g^F)(h_j)\Big)$$
$$\cdot\exp\Big(-t\Big(D^0-\frac{u}{2}\sum_{j=1}^{l}\widehat{c}(h_j)\omega(F,g^F)(h_j)\Big)^2\Big)\Big]=O(\sqrt{t}\,). \tag{A.9}$$

From (A.9) and the standard variation formula for η-invariants ([3, 8]), one gets

$$\overline{\eta}\left(D^F_{M/E,\mathrm{sig}}\right)=\overline{\eta}\left(\widehat{c}\left(E^\perp,g^{E^\perp}\right)D^0\right)\mod \mathbf{Z}, \tag{A.10}$$

where $\widehat{c}(E^\perp,g^{E^\perp})D^0$ is now acting on $\Omega^{\mathrm{even}}(M,F)$.

Similarly, when $\dim(M/E)\equiv 1 \mod 4\mathbf{Z}$, one verifies that $\widehat{c}(E^\perp,g^{E^\perp})D^0$, when acts on $\Omega^{\mathrm{even}}(M,F)$, is also skew-adjoint. Thus by the homotopy invariance of the mod 2 index (cf. [4]), one has

$$\dim\ker\left(D^F_{M/E,\mathrm{sig}}\right)\equiv\dim\ker\left(\widehat{c}\left(E^\perp,g^{E^\perp}\right)D^0\right)\mod 2\mathbf{Z}. \tag{A.11}$$

From (A.10), (A.11), one arrives at the interesting fact that the generalized Atiyah-Patodi-Singer invariants in Definition 1.4 can also be defined by using $\widehat{c}(E^\perp,g^{E^\perp})D^0$ which actually may also be viewed as a sub-signature operator (twisted by the Euclidean vector bundle $(F,g^F,\nabla^{F,e})$).

§ 2 . Adiabatic Limits of η-Invariants and a Riemann-Roch Theorem for Flat Vector Bundles

In this section, we apply the construction in Section 1 to the case where M is a fibered manifold with compact fibers. We use the method of adiabatic limits to compute the associated generalized Atiyah-Patodi-Singer invariants and prove Theorem 0.2 for these invariants.

This section is organized as follows. In (a), we restate Theorem 0.2 for convenience. In (b), we write the sub-signature operator on M in a form which is convenient for the evaluation of the adiabatic limits of the associated η-invariants. In (c), we recall the construction of a superconnection due to Bismut-Lott [9]. In (d), we prove a local index result which will identify the Bismut-Cheeger $\widehat{\eta}$-form [6, 7] in the computation of the adiabatic limit of η-invariants. In (e), we calculate the adiabatic limit of η-invariants. In (f), we prove Theorem 0.2.

(a) A Rieman-Roch Theorem for Flat Vector Bundles

In this section, we assume that M is a fibered manifold with compact base and fibers,

$$Z\to M\overset{\pi}{\to}B. \tag{2.1}$$

We assume that both TB and the vertical tangent bundle TZ are oriented, and that both $\dim M$ and $\dim B$ are odd integers. Then TM is also oriented. For any flat vector bundle F over M with the flat connection ∇^F, one can apply the construction in Section 1 to F and $E=TZ$ to obtain the smooth invariant $\phi(M/TZ,F)$. On the other hand, F induces canonically a $\mathbf{Z}$-graded flat vector bundle

$$H^*(Z;F|_Z)=\bigoplus_{i=0}^{\dim Z}H^i(Z;F|_Z), \tag{2.2}$$

from which one can construct a series of smooth invariants $\phi(B, H^i(Z; F|_Z))$, $0 \leq i \leq \dim Z$, as in Section 1 by setting $E = 0$.

The main result of this section is a Riemann-Roch type formula relating these invariants.

Theorem 2.1. *The following identity holds,*

$$\phi(M/TZ, F) = \sum_{i=0}^{\dim Z} (-1)^i \phi(B, H^i(Z, F|_Z)) - \mathrm{rk}(F) \sum_{i=0}^{\dim Z} (-1)^i \phi(B, H^i(Z, \mathbf{R}_Z)). \tag{2.3}$$

We will prove Theorem 2.1 by computing the adiabatic limits of the η-invariants involved in the definition of $\phi(M/TZ, F)$. For this purpose, we will first write the corresponding sub-signature operator in a suitable way in the next subsection.

(b) The Sub-signature Operator on a Fibered Manifold

Choose a splitting

$$TM = T^H M \oplus TZ. \tag{2.4}$$

We have

$$T^H M \simeq \pi^*(TB). \tag{2.5}$$

Let g^{TZ} (resp. g^{TB}, resp. g^F) be a metric on TZ (resp. TB, resp. F). Let

$$g^{TM} = \pi^* g^{TB} \oplus g^{TZ} \tag{2.6}$$

be the Euclidean metric on TM such that $T^H M \simeq \pi^*(TB)$ and TZ are orthogonal to each other with respect to the splitting (2.4).

By applying the constructions in Section 1 to the case of $E = TZ$, $E^\perp = T^H M$, we have the sub-signature operator $D^F_{M/TZ,\mathrm{sig}}$ with respect to the metrics g^{TM} and g^F. Let $D^F_{M/TZ}$ be the associated operator as defined in Definition 1.3. We now proceed as in [5], [13] to write $D^F_{M/TZ}$ in a form adapted to the splitting (2.4).

Let ∇^{TB} be the Levi-Civita connection of g^{TB}. Let $\nabla^{\wedge^*(T^*B)}$ be the canonical Euclidean connection on $\wedge^*(T^*B)$ induced by ∇^{TB}. Then ∇^{TB} (resp. $\nabla^{\wedge^*(T^*B)}$) induces an Euclidean connection on $T^H M \simeq \pi^*(TB)$ (resp. $\wedge^*(T^H M)$) denoted by $\pi^*\nabla^{T^*B}$ (resp. $\pi^*\nabla^{\wedge^*(T^*B)}$).

Recall that the connections ∇^{TZ}, $\nabla^{T^H M}$, as well as their liftings on

$$\wedge^*(T^*Z), \quad \wedge^*(T^{H*}M),$$

have been defined in (1.19). Also recall from (1.17) that $\nabla^{F,e}$ is an Euclidean connection on the flat vector bundle F over M, and from (1.21) that $\nabla^{e,E^\perp}$ is an Euclidean connection on $\wedge^*(T^*M) \otimes F$.

Following [5, (1.23)], let S^B be the tensor defined by

$$\nabla^{TM} = \pi^*\nabla^{TB} + \nabla^{TZ} + S^B. \tag{2.7}$$

Let T be the torsion of the connection $\pi^*\nabla^{TB}+\nabla^{TZ}$. Recall that the tensor S has been defined in (1.20). Let $\widetilde{S}$ be the tensor defined by

$$\widetilde{S}=S^B-S. \tag{2.8}$$

From (2.7), (2.8) and (1.20), one has

$$\widetilde{S}=\nabla^{T^HM}-\pi^*\nabla^{TB}. \tag{2.9}$$

Let $\widetilde{\nabla}$ be the Euclidean connection on $\wedge^*(T^*M)\otimes F$ obtained by the tensor product of $\pi^*\nabla^{TB}$, ∇^{TZ} and $\nabla^{F,e}$. We use the same notation as in Section 1 that $h_1,\cdots,h_l$ is an orthonormal basis of $E=TZ$, while $f_1,\cdots,f_k$ is an orthonormal basis of $E^\perp=T^HM$. Without loss of generality, we assume that $f_1,\cdots,f_k$ is a lift of an orthonormal basis, which we still denote by $f_1,\cdots,f_k$ when there is no confusion, of TB.

From (1.21), (1.29), (1.35) and (2.7)–(2.9), one deduces easily by proceeding as in [5], [6] and [13] that

$$\begin{aligned} D^F_{M/TZ}=&\sum_{\alpha=1}^k c(f_\alpha)\Big(\widetilde{\nabla}_{f_\alpha}-\frac{1}{2}\sum_{s=1}^l\langle S(h_s)h_s,f_\alpha\rangle\Big)\\ &+\sum_{s=1}^l c(h_s)\widetilde{\nabla}_{h_s}-\frac{1}{2}\sum_{s=1}^l\widehat{c}(h_s)\omega(F,g^F)(h_s)\\ &-\frac{1}{4}\sum_{\alpha<\beta}c\left(T(f_\alpha,f_\beta)\right)c(f_\alpha)c(f_\beta)\\ &-\frac{1}{4}\sum_{s=1}^l\sum_{\alpha\neq\beta}\left\langle\nabla^{TM}_{h_s}f_\alpha,f_\beta\right\rangle c(h_s)\widehat{c}(f_\alpha)\widehat{c}(f_\beta). \end{aligned}\tag{2.10}$$

By the standard formula for Levi-Civita connections, one finds

$$2\left\langle\nabla^{TM}_{h_s}f_\alpha,f_\beta\right\rangle=-\langle[f_\alpha,f_\beta],h_s\rangle=\langle T(f_\alpha,f_\beta),h_s\rangle\,. \tag{2.11}$$

Now for any $t>0$, let g^{TM}_t be the metric on TM given by

$$g^{TM}=\frac{1}{t}\pi^*g^{TB}\oplus g^{TZ}. \tag{2.12}$$

Let $D^F_{M/TZ,\mathrm{sig}}(t)$, $D^F_{M/TZ}(t)$ be constructed with respect to g^{TM}_t and g^F.

Let N_B be the number operator of $\Omega^*(B)$. Then it extends naturally to an operator on $\Omega^*(M,F)$. Set

$$\widehat{D}^F_{M/TZ}(t)=t^{N_B/2}D^F_{M/TZ}(t)t^{-N_B/2}. \tag{2.13}$$

From (2.10), (2.11), (2.13) and proceeding as in [5, (3.10), (3.11)], one deduces that

$$\begin{aligned} \widehat{D}^F_{M/TZ}(t)=&\sqrt{t}\sum_{\alpha=1}^k c(f_\alpha)\Big(\widetilde{\nabla}_{f_\alpha}-\frac{1}{2}\sum_{s=1}^l\langle S(h_s)h_s,f_\alpha\rangle\Big)+\sum_{s=1}^l c(h_s)\widetilde{\nabla}_{h_s}\\ &-\frac{1}{2}\sum_{s=1}^l\widehat{c}(h_s)\omega(F,g^F)(h_s)-\frac{t}{4}\sum_{\alpha<\beta}c(T(f_\alpha,f_\beta))c(f_\alpha)c(f_\beta)\\ &-\frac{t}{4}\sum_{\alpha<\beta}c(T(f_\alpha,f_\beta))\widehat{c}(f_\alpha)\widehat{c}(f_\beta). \end{aligned}\tag{2.14}$$

(c) The Bismut-Lott Superconnection

In this subsection, we recall from [9] the construction of a natural superconnection associated with flat vector bundles over fibered manifolds.

From (2.4), (2.5), we have that as bundles of **Z**-graded algebras over M,

$$\wedge^*(T^*M) \simeq \pi^*\left(\wedge^*(T^*B)\right)\widehat{\otimes}\wedge^*\left(T^*Z\right). \tag{2.15}$$

As in [9, Section 3a)], let W be the smooth infinite-dimensional **Z**-graded vector bundle over B whose fiber over $b \in B$ is $C^\infty(Z_b;(\wedge^*(T^*Z)\otimes F)|_{Z_b})$. That is,

$$C^\infty(B;W) \simeq C^\infty\left(M;\wedge^*(T^*Z)\otimes F\right). \tag{2.16}$$

Then W acquires a canonically induced Euclidean metric (cf. [5, (3.29)]).

Let $\Omega^{*V}(M;F)$ denote the subspace of $\Omega^*(M;F)$ which is annihilated by interior multiplication with horizontal vectors. Then there is an isomorphism

$$\Omega^{*V}(M;F) \simeq C^\infty(B;W), \tag{2.17}$$

where the isomorphism is given by sending an element of $\Omega^{*V}(M;F)$ to its fiberwise restrictions. From (2.15),

$$\Omega^*(M;F) \simeq \Omega^*(B)\widehat{\otimes}\Omega^{*V}(M;F). \tag{2.18}$$

Thus we have an isomorphism of **Z**-graged vector spaces

$$\Omega^*(M;F) \simeq \Omega^*(B;W). \tag{2.19}$$

If U is a smooth vector field on B, let $U^H \in C^\infty(M;T^HM)$ be its horizontal lift, so that $\pi_*U^H = U$. As in [9, Definition 3.2], let ∇^W be the connection on W defined by

$$\nabla^W_U s = L_{U^H}s, \qquad s \in C^\infty(B;W),\ \ U \in C^\infty(B;TB), \tag{2.20}$$

where L_{U^H} is the Lie differential operator acting on $C^\infty(M;\wedge^*(T^*Z)\otimes F)$.

Let d^Z denote the exterior differentiation along fibers of W. Let $(d^Z)^*$ (resp. $(\nabla^W)^*$) be the adjoint of d^Z (resp. ∇^W) with respect to the Euclidean metric on W.

Set as in [9, Definition 3.8] that

$$\begin{aligned} D^Z &= d^Z + (d^Z)^*, \\ \nabla^{W,u} &= \frac{1}{2}(\nabla^W + (\nabla^W)^*). \end{aligned} \tag{2.21}$$

Then $\ker(D^Z)$ forms a **Z**-graded vector bundle over B.

For any $t > 0$, we can define the (rescaled) Bismut-Lott superconnection A_t on the infinite dimensional vector bundle W as follows,

$$A_t = \nabla^{W,u} + \sqrt{t}D^Z - \frac{1}{4\sqrt{t}}\sum_{\alpha<\beta} c(T(f_\alpha, f_\beta))dy^\alpha dy^\beta, \tag{2.22}$$

where $dy^1,\cdots,dy^k$ is an orthonormal basis of T^*B dual to $f_1,\cdots,f_k$. In fact, up to an obvious rescaling, this A_t is exactly the superconnection C_t defined in [9, (3.50)] (cf. [9, Remark 3.10 and (3.59)]). For brievity of notation, we will also denote the term $\sum_{\alpha<\beta} c(T(f_\alpha, f_\beta))dy^\alpha dy^\beta$ by $c(T)$.

Remark 2.1. The critical observation is that this Bismut-Lott superconnection can be obtained from $\widehat{D}^F_{M/TZ}$ through Getzler rescaling (cf. [11]) in the same way as what the Bismut superconnection [5] be obtained from the Dirac operator (compare with [5], [11]). In fact, it is clear that the expression of $\widehat{D}^F_{M/TZ}(t)$ in (2.14) is compatible with the identification (2.19). In particular, one may view both $\widehat{D}^F_{M/TZ}(t)$ and A_t as operators acting on $\Omega^*(B;W)$. Now if one proceeds the Getzler rescaling

$$c(f_\alpha)\to\frac{dy^\alpha}{\sqrt{t}}-\sqrt{t}i_{f_\alpha}\qquad\text{in }\ \widehat{D}^F_{M/TZ}(t)$$

and takes $t\to 0$, then from (2.14) and [9, (3.36), (3.37)] one sees that the limiting operator is precisely A_1.

(d) A Local Index Computation

In this subsection, we prove a local index result which will be used in the next subsection to evaluate the adiabatic limit of η-invariants. We make the assumption in this subsection that $k\equiv 3 \bmod 4\mathbf{Z}$.

We first prove the following analogue of [6, (4.40)].

Proposition 2.1. *For any $u>0$, one has*

$$\begin{aligned}&\frac{1}{\sqrt{\pi}}\lim_{t\to 0}\mathrm{Tr}^{\mathrm{even}}\Big[\widehat{c}\left(T^HM,g^{T^HM}\right)\widehat{D}^F_{M/TZ}(t)\exp\Big(-u\left(\widehat{D}^F_{M/TZ}(t)\right)^2\Big)\Big]\\&=\frac{1}{2}\Big(\frac{1}{\pi}\Big)^{\frac{k+1}{2}}\int_B\det{}^{1/2}\Big(\frac{R^{TB}/2}{\tanh(R^{TB}/2)}\Big)\mathrm{Tr}_s\Big[\Big(D^Z+\frac{c(T)}{4u}\Big)\exp(-A_u^2)\Big],\end{aligned}\tag{2.23}$$

where by $\mathrm{Tr}^{\mathrm{even}}$ *we mean the trace taken on* $\Omega^{\mathrm{even}}(M,F)$*, while* Tr_s *is taken on* W*.*

Proof. We proceed as in [6, pp.59–63] to prove (2.23).

Let $e_1,\cdots,e_{\dim M}$ be an oriented (local) orthonormal basis of TM. As in [6, (4.48), (4.49)], if Q is a section of the bundle of linear maps from TM to $\wedge^*(T^*M)\otimes F$, we use the abbreviations

$$(\widetilde{\nabla}_{h_s}+Q(h_s))^2=\sum_s(\widetilde{\nabla}_{h_s}+Q(h_s))^2-\widetilde{\nabla}_{\sum_s\nabla^{TZ}_{h_s}h_s}-Q\Big(\sum_s\nabla^{TZ}_{h_s}h_s\Big),\tag{2.24}$$

$$(\widetilde{\nabla}_{f_\alpha}+Q(f_\alpha))^2=\sum_\alpha(\widetilde{\nabla}_{f_\alpha}+Q(f_\alpha))^2-\widetilde{\nabla}_{\sum_\alpha\pi^*\nabla^{TB}_{f_\alpha}f_\alpha}-Q\Big(\sum_\alpha\pi^*\nabla^{TB}_{f_\alpha}f_\alpha\Big),\tag{2.25}$$

$$\Big(\nabla^{e,E^\perp}_{e_i}+Q(e_i)\Big)^2=\sum_i\Big(\nabla^{e,E^\perp}_{e_i}+Q(e_i)\Big)^2-\nabla^{e,E^\perp}_{\sum_i\nabla^{TM}_{e_i}e_i}-Q\Big(\sum_i\nabla^{TM}_{e_i}e_i\Big).\tag{2.26}$$

Also, in the formulas which follow, we sum over all repeated indices.

When there will be no confusion, in some notation we will use a subscript t to indicate that the corresponding geometric object is with respect to g_t^{TM}. For example, K_t will denote the scalar curvature of g_t^{TM}.

From (2.9), one verifies easily that

$$\widetilde{S}_t=t\widetilde{S},\tag{2.27}$$

$$R^{T^HM}=\pi^*R^{TB}+(\pi^*\nabla^{TB}\widetilde{S})+\widetilde{S}\wedge\widetilde{S}.\tag{2.28}$$

Following [8] and [6], let z be an odd Grassmannian variable which anticommutes with $c(e_i)$'s and $\widehat{c}(e_i)$'s. We first prove the following extension of [6, (4.53)].

Proposition 2.2. *For $u>0$, $t>0$, the following identity holds,*

$$
\begin{aligned}
& u\widehat{D}^{F,2}_{M/TZ}(t) - zu^{1/2}\widehat{D}^{F}_{M/TZ}(t)\\
&= -u\Big(\widetilde{\nabla}_{h_s} + \frac{t^{1/2}}{2}\langle S(h_s)h_q, f_\alpha\rangle c(h_q)c(f_\alpha)\\
&\quad + \frac{t}{4}\langle S^B(h_s)f_\alpha, f_\beta\rangle c(f_\alpha)c(f_\beta) - \frac{t}{4}\langle \widetilde{S}(h_s)f_\alpha, f_\beta\rangle \widehat{c}(f_\alpha)\widehat{c}(f_\beta) + \frac{zc(h_s)}{2u^{1/2}}\Big)^2\\
&\quad - u\Big(t^{1/2}\widetilde{\nabla}_{f_\alpha} + \frac{t}{2}\langle S(f_\alpha)h_s, f_\beta\rangle c(h_s)c(f_\beta) + \frac{zc(f_\alpha)}{2u^{1/2}}\Big)^2\\
&\quad + \frac{ut^{3/2}}{2}\langle S(S(h_s)h_s)h_q, f_\alpha\rangle c(h_q)c(f_\alpha) - \frac{ut^2}{4}\langle \widetilde{S}(S(h_s)h_s)f_\alpha, f_\beta\rangle \widehat{c}(f_\alpha)\widehat{c}(f_\beta)\\
&\quad + ut\widetilde{\nabla}_{S(h_s)h_s} + \frac{zu^{1/2}t^{1/2}}{2}c(S(h_s)h_s) + \frac{zu^{1/2}}{2}\widehat{c}(h_s)\omega(F,g^F)(h_s) + \frac{uK_t}{4}\\
&\quad - \frac{ut}{8}c(f_\alpha)c(f_\beta)(\omega(F,g^F))^2(f_\alpha,f_\beta) - \frac{ut^{1/2}}{4}c(h_s)c(f_\alpha)(\omega(F,g^F))^2(h_s,f_\alpha)\\
&\quad + \frac{ut}{8}\langle R^{TB}(f_\gamma,f_\delta)f_\beta, f_\alpha\rangle c(f_\gamma)c(f_\delta)\widehat{c}(f_\alpha)\widehat{c}(f_\beta)\\
&\quad + \frac{ut^{3/2}}{8}\langle (\pi^*\nabla^{TB}\widetilde{S})(h_s,f_\gamma)f_\beta, f_\alpha\rangle c(h_s)c(f_\gamma)\widehat{c}(f_\alpha)\widehat{c}(f_\beta)\\
&\quad + \frac{ut^2}{8}\langle \widetilde{S}\wedge\widetilde{S}(h_s,h_q)f_\beta, f_\alpha\rangle c(h_s)c(h_q)\widehat{c}(f_\alpha)\widehat{c}(f_\beta)\\
&\quad + \frac{ut}{8}\langle R^{TZ}(f_\alpha,f_\beta)h_q, h_s\rangle c(f_\alpha)c(f_\beta)\widehat{c}(h_s)\widehat{c}(h_q)\\
&\quad + \frac{ut^{1/2}}{4}\langle R^{TZ}(h_p,f_\alpha)h_q, h_s\rangle c(h_p)c(f_\alpha)\widehat{c}(h_s)\widehat{c}(h_q)\\
&\quad + \frac{u}{8}\langle R^{TZ}(h_p,h_r)h_q, h_s\rangle c(h_p)c(h_r)\widehat{c}(h_s)\widehat{c}(h_q) + \frac{u}{4}\sum_s(\omega(F,g^F)(h_s))^2\\
&\quad - \frac{u}{8}c(h_s)c(h_q)(\omega(F,g^F))^2(h_s,h_q) + \frac{u}{8}\widehat{c}(h_q)\widehat{c}(h_s)(\omega(F,g^F))^2(h_q,h_s)\\
&\quad - \frac{ut^{1/2}}{4}c(f_\alpha)\widehat{c}(h_s)(\nabla^F_{f_\alpha}\omega(F,g^F)(h_s) + \nabla^F_{h_s}\omega(F,g^F)(f_\alpha))\\
&\quad - \frac{u}{4}c(h_q)\widehat{c}(h_s)(\nabla^F_{h_q}\omega(F,g^F)(h_s) + \nabla^F_{h_s}\omega(F,g^F)(h_q)). \qquad (2.29)
\end{aligned}
$$

Proof. As in [8, (2.5)], one verifies that

$$
-u\Big(\nabla^{e,E^\perp}_{e_i} + \frac{zc(e_i)}{2u^{1/2}}\Big)^2 = -u\Delta^{e,E^\perp} - zu^{1/2}c(e_i)\nabla^{e,E^\perp}_{e_i}. \qquad (2.30)
$$

From (1.29), (1.32), (2.28) and (2.30) one finds

$$
\begin{aligned}
& uD^{F,2}_{M/TZ} - zu^{1/2}D^F_{M/TZ}\\
&= -u\Big(\nabla^{e,E^\perp}_{e_i} + \frac{zc(e_i)}{2u^{1/2}}\Big)^2 + \frac{zu^{1/2}}{2}\widehat{c}(h_s)\omega(F,g^F)(h_s)\\
&\quad + \frac{uK}{4} - \frac{u}{8}c(e_i)c(e_j)(\omega(F,g^F))^2(e_i,e_j)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{u}{8}\langle R^{TB}(f_\gamma,f_\delta)f_\beta,f_\alpha\rangle c(f_\gamma)c(f_\delta)\widehat{c}(f_\alpha)\widehat{c}(f_\beta)\\
&+\frac{u}{8}\langle(\pi^*\nabla^{TB}\widetilde{S})(h_s,f_\gamma)f_\beta,f_\alpha\rangle c(h_s)c(f_\gamma)\widehat{c}(f_\alpha)\widehat{c}(f_\beta)\\
&+\frac{u}{8}\langle\widetilde{S}\wedge\widetilde{S}(h_s,h_q)f_\beta,f_\alpha\rangle c(h_s)c(h_q)\widehat{c}(f_\alpha)\widehat{c}(f_\beta)\\
&+\frac{u}{8}\langle R^{TZ}(e_i,e_j)h_q,h_s\rangle c(e_i)c(e_j)\widehat{c}(h_s)\widehat{c}(h_q)\\
&+\frac{u}{4}\sum_s(\omega(F,g^F)(h_s))^2+\frac{u}{8}\widehat{c}(h_q)\widehat{c}(h_s)(\omega(F,g^F))^2(h_q,h_s)\\
&-\frac{u}{4}c(e_i)\widehat{c}(h_s)(\nabla^F_{e_i}\omega(F,g^F)(h_s)+\nabla^F_{h_s}\omega(F,g^F)(e_i)). \qquad (2.31)
\end{aligned}
$$

From (1.21), (1.31), (1.35), (2.7)–(2.9), (2.13), (2.24)–(2.27), (2.31) and proceeding as in [5, (3.14)] and [6, (4.53)], one gets (2.29) easily.

We now turn to the proof of (2.23). As was pointed in [6, p.60], which uses [12, Section 3], we may replace the base space by $\mathbf{R}^k$ with a metric which is flat outside a compact set. We also assume that the bundle is isometrically a product on that region.

Following [6, (4.54)], if A, A' are of trace class in $\mathrm{End}(\Omega^*(M,F))$ with A' preserves $\Omega^{\mathrm{even}}(M,F)$, set

$$\mathrm{Tr}^{\mathrm{even},z}[A+zA']=z\mathrm{Tr}^{\mathrm{even}}[A']. \qquad (2.32)$$

One finds as in [8] and [6, (4.55)] that

$$
\begin{aligned}
&\exp\left(-u\widehat{D}^{F,2}_{M/TZ}(t)+zu^{1/2}\widehat{D}^F_{M/TZ}(t)\right)\\
&=\exp\left(-u\widehat{D}^{F,2}_{M/TZ}(t)\right)+zu^{1/2}\widehat{D}^F_{M/TZ}(t)\exp\left(-u\widehat{D}^{F,2}_{M/TZ}(t)\right), \qquad (2.33)\\
&\mathrm{Tr}^{\mathrm{even},z}\left[\widehat{c}\left(T^HM,g^{T^HM}\right)\exp\left(-u\widehat{D}^{F,2}_{M/TZ}(t)+zu^{1/2}\widehat{D}^F_{M/TZ}(t)\right)\right]\\
&=-zu^{1/2}\mathrm{Tr}^{\mathrm{even}}\left[\widehat{c}\left(T^HM,g^{T^HM}\right)\widehat{D}^F_{M/TZ}(t)\exp\left(-u\widehat{D}^{F,2}_{M/TZ}(t)\right)\right]. \qquad (2.34)
\end{aligned}
$$

Take $x_0\in M$. Let $P^{t,u}(x_0,\cdot)$ be the C^∞ kernel, with respect to g^{TM}, of

$$\exp\left(-u\widehat{D}^{F,2}_{M/TZ}(t)+zu^{1/2}\widehat{D}^F_{M/TZ}(t)\right).$$

Let $y_0=\pi(x_0)$ and take a system of geodesic coordinates $\{y_\alpha\}$ centered at y_0. We can assume that $\{y_\alpha\}$ is globally defined on $\mathbf{R}^k$. By using parallel transport along the horizontal lifts of geodesics in the base, we can trivialize the fibration $Z\to M\to\mathbf{R}^k$. Similarly, we use parallel transport along such geodesics to trivialize $\wedge^{\mathrm{even}}(T^*M)\otimes F$ (using the connection $\nabla^{e,E^\perp}$).

In what follows, as in [6, p.61], we put $\overline{y}=\sum_\alpha y_\alpha f_\alpha$. As in [8] and [6], we will conjugate the operator $u\widehat{D}^{F,2}_{M/TZ}(t)-zu^{1/2}\widehat{D}^F_{M/TZ}(t)$ by

$$e^{zc(\overline{y})/2(ut)^{1/2}}=1+\sum_\alpha\frac{zy_\alpha c(f_\alpha)}{2(ut)^{1/2}}. \qquad (2.35)$$

For $x\in M$, $\pi(x)=y$, set

$$\widehat{P}^{t,u}(x_0,x)=P^{t,u}(x_0,x)e^{-zc(\overline{y})/2(ut)^{1/2}}. \qquad (2.36)$$

Then, as in [6, (4.60)–(4.63)], one verifies that $\widehat{P}^{t,u}(x_0,x)$ is equal to the C^∞ kernel, with respect to g^{TM}, of

$$e^{zc(\overline{y})/2(ut)^{1/2}}\exp\left(-u\widehat{D}^{F,2}_{M/TZ}(t)+zu^{1/2}\widehat{D}^{F}_{M/TZ}(t)\right)e^{-zc(\overline{y})/2(ut)^{1/2}}$$

and that

$$\begin{aligned}&\operatorname{Tr}^{\mathrm{even},z}\left[\widehat{c}\left(T^HM,g^{T^HM}\right)\widehat{P}^{t,u}(x_0,x_0)\right]\\&=\operatorname{Tr}^{\mathrm{even},z}\left[\widehat{c}\left(T^HM,g^{T^HM}\right)P^{t,u}(x_0,x_0)\right].\end{aligned}\tag{2.37}$$

By using (2.29), one deduces easily that (compare with [6, p.62])

$$\begin{aligned}&e^{zc(\overline{y})/2(ut)^{1/2}}\left(u\widehat{D}^{F,2}_{M/TZ}(t)-zu^{1/2}\widehat{D}^{F}_{M/TZ}(t)\right)e^{-zc(\overline{y})/2(ut)^{1/2}}\\
&=-u\Big(\widetilde{\nabla}_{h_s}+\frac{t^{1/2}}{2}\langle S(h_s)h_q,f_\alpha\rangle c(h_q)c(f_\alpha)+\frac{1}{2u^{1/2}}\langle S(h_s)h_q,\overline{y}\rangle zc(h_q)\\
&\quad+\frac{t}{4}\langle S^B(h_s)f_\alpha,f_\beta\rangle c(f_\alpha)c(f_\beta)-\frac{t^{1/2}}{2u^{1/2}}\langle S^B(h_s)\overline{y},f_\beta\rangle zc(f_\beta)\\
&\quad-\frac{t}{4}\langle\widetilde{S}(h_s)f_\alpha,f_\beta\rangle\widehat{c}(f_\alpha)\widehat{c}(f_\beta)+\frac{zc(h_s)}{2u^{1/2}}\Big)^2\\
&\quad-u\Big(t^{1/2}\widetilde{\nabla}_{f_\alpha}+\frac{t}{2}\langle S(f_\alpha)h_s,f_\beta\rangle c(h_s)c(f_\beta)\\
&\quad+\frac{t^{1/2}}{2u^{1/2}}\langle S(f_\alpha)h_s,\overline{y}\rangle zc(h_s)+\frac{1}{u^{1/2}}\sum_\alpha(O(|y|^2)zc(f_\alpha))\Big)^2\\
&\quad+\frac{ut^{3/2}}{2}\langle S(S(h_s)h_s)h_q,f_\alpha\rangle c(h_q)c(f_\alpha)+\frac{u^{1/2}t}{2}\langle S(S(h_s)h_s)h_q,\overline{y}\rangle zc(h_q)\\
&\quad+ut\widetilde{\nabla}_{S(h_s)h_s}+(ut)^{1/2}\sum_\alpha(O(|y|)zc(f_\alpha))+\frac{zu^{1/2}}{2}\widehat{c}(h_s)\omega(F,g^F)(h_s)+\frac{uK_t}{4}\\
&\quad-\frac{ut}{8}c(f_\alpha)c(f_\beta)(\omega(F,g^F))^2(f_\alpha,f_\beta)-\frac{(ut)^{1/2}}{4}zc(f_\alpha)(\omega(F,g^F))^2(f_\alpha,\overline{y})\\
&\quad-\frac{ut^{1/2}}{4}c(h_s)c(f_\alpha)(\omega(F,g^F))^2(h_s,f_\alpha)-\frac{u^{1/2}}{4}zc(h_s)(\omega(F,g^F))^2(h_s,\overline{y})\\
&\quad-\frac{ut^2}{4}\langle\widetilde{S}(S(h_s)h_s)f_\alpha,f_\beta\rangle\widehat{c}(f_\alpha)\widehat{c}(f_\beta)-\frac{u}{8}c(h_s)c(h_q)(\omega(F,g^F))^2(h_s,h_q)\\
&\quad+\frac{ut}{8}\langle R^{TB}(f_\gamma,f_\delta)f_\beta,f_\alpha\rangle c(f_\gamma)c(f_\delta)\widehat{c}(f_\alpha)\widehat{c}(f_\beta)\\
&\quad+\frac{(ut)^{1/2}}{4}\langle R^{TB}(f_\gamma,\overline{y})f_\beta,f_\alpha\rangle zc(f_\gamma)\widehat{c}(f_\alpha)\widehat{c}(f_\beta)\\
&\quad+\frac{ut^{3/2}}{8}\langle(\pi^*\nabla^{TB}\widetilde{S})(h_s,f_\gamma)f_\beta,f_\alpha\rangle c(h_s)c(f_\gamma)\widehat{c}(f_\alpha)\widehat{c}(f_\beta)\\
&\quad+\frac{u^{1/2}t}{8}\langle(\pi^*\nabla^{TB}\widetilde{S})(h_s,\overline{y})f_\beta,f_\alpha\rangle zc(h_s)\widehat{c}(f_\alpha)\widehat{c}(f_\beta)\\
&\quad+\frac{ut^2}{8}\langle\widetilde{S}\wedge\widetilde{S}(h_s,h_q)f_\beta,f_\alpha\rangle c(h_s)c(h_q)\widehat{c}(f_\alpha)\widehat{c}(f_\beta)\\
&\quad+\frac{ut}{8}\langle R^{TZ}(f_\alpha,f_\beta)h_q,h_s\rangle c(f_\alpha)c(f_\beta)\widehat{c}(h_s)\widehat{c}(h_q)\end{aligned}$$

$$
\begin{aligned}
&+\frac{(ut)^{1/2}}{4}\langle R^{TZ}(f_\alpha,\overline{y})h_q,h_s\rangle zc(f_\alpha)\widehat{c}(h_s)\widehat{c}(h_q)\\
&+\frac{ut^{1/2}}{4}\langle R^{TZ}(h_p,f_\alpha)h_q,h_s\rangle c(h_p)c(f_\alpha)\widehat{c}(h_s)\widehat{c}(h_q)\\
&+\frac{u^{1/2}}{4}\langle R^{TZ}(h_p,\overline{y})h_q,h_s\rangle zc(h_p)\widehat{c}(h_s)\widehat{c}(h_q)\\
&+\frac{u}{8}\langle R^{TZ}(h_p,h_r)h_q,h_s\rangle c(h_p)c(h_r)\widehat{c}(h_s)\widehat{c}(h_q)\\
&+\frac{u}{4}\sum_s(\omega(F,g^F)(h_s))^2+\frac{u}{8}\widehat{c}(h_q)\widehat{c}(h_s)(\omega(F,g^F))^2(h_q,h_s)\\
&-\frac{ut^{1/2}}{4}c(f_\alpha)\widehat{c}(h_s)(\nabla^F_{f_\alpha}\omega(F,g^F)(h_s)+\nabla^F_{h_s}\omega(F,g^F)(f_\alpha))\\
&+\frac{u^{1/2}}{4}z\widehat{c}(h_s)(\nabla^F_{\overline{y}}\omega(F,g^F)(h_s)+\nabla^F_{h_s}\omega(F,g^F)(\overline{y}))\\
&-\frac{u}{4}c(h_q)\widehat{c}(h_s)(\nabla^F_{h_q}\omega(F,g^F)(h_s)+\nabla^F_{h_s}\omega(F,g^F)(h_q)).
\end{aligned}\tag{2.38}
$$

Now as in [6], we apply the Getzler's transformation $G_{(ut)^{1/2}}$ to the right hand side of (2.38) and let $t\to 0$.[2] We find that

$$
\begin{aligned}
&\lim_{t\to 0}G_{(ut)^{1/2}}\left[e^{zc(\overline{y})/2(ut)^{1/2}}\left(u\widehat{D}^{F,2}_{M/TZ}(t)-zu^{1/2}\widehat{D}^F_{M/TZ}(t)\right)e^{-zc(\overline{y})/2(ut)^{1/2}}\right]\\
=&-u\Big(\widetilde{\nabla}_{h_s}+\frac{1}{2u^{1/2}}\langle S^B(h_s)h_q,f_\alpha\rangle c(h_q)dy_\alpha\\
&+\frac{1}{4u}\langle S^B(h_s)f_\alpha,f_\beta\rangle dy_\alpha dy_\beta+\frac{zc(h_s)}{2u^{1/2}}\Big)^2\\
&-\Big(\partial_\alpha+\frac{1}{8}\langle R^{TB}(f_\alpha,f_\beta)f_\gamma,f_\delta\rangle y_\beta dy_\gamma dy_\delta\Big)^2\\
&+\frac{zu^{1/2}}{2}\widehat{c}(h_s)\omega(F,g^F)(h_s)+\frac{uK_Z}{4}-\frac{1}{8}dy_\alpha dy_\beta(\omega(F,g^F))^2(f_\alpha,f_\beta)\\
&-\frac{u^{1/2}}{4}c(h_s)dy_\alpha(\omega(F,g^F))^2(h_s,f_\alpha)-\frac{u}{8}c(h_s)c(h_q)(\omega(F,g^F))^2(h_s,h_q)\\
&+\frac{1}{8}\langle R^{TB}(f_\gamma,f_\delta)f_\beta,f_\alpha\rangle dy_\gamma dy_\delta\widehat{c}(f_\alpha)\widehat{c}(f_\beta)\\
&+\frac{1}{8}\langle R^{TZ}(f_\alpha,f_\beta)h_q,h_s\rangle dy_\alpha dy_\beta\widehat{c}(h_s)\widehat{c}(h_q)\\
&+\frac{u^{1/2}}{4}\langle R^{TZ}(h_p,f_\alpha)h_q,h_s\rangle c(h_p)dy_\alpha\widehat{c}(h_s)\widehat{c}(h_q)\\
&+\frac{u}{8}\langle R^{TZ}(h_p,h_r)h_q,h_s\rangle c(h_p)c(h_r)\widehat{c}(h_s)\widehat{c}(h_q)\\
&+\frac{u}{4}\sum_s(\omega(F,g^F)(h_s))^2+\frac{u}{8}\widehat{c}(h_q)\widehat{c}(h_s)(\omega(F,g^F))^2(h_q,h_s)\\
&-\frac{u^{1/2}}{4}dy_\alpha\widehat{c}(h_s)(\nabla^F_{f_\alpha}\omega(F,g^F)(h_s)+\nabla^F_{h_s}\omega(F,g^F)(f_\alpha))\\
&-\frac{u}{4}c(h_q)\widehat{c}(h_s)(\nabla^F_{h_q}\omega(F,g^F)(h_s)+\nabla^F_{h_s}\omega(F,g^F)(h_q)),
\end{aligned}\tag{2.39}
$$

[2] Recall that by Geztler's transformaton we mean the rescaling that $y_\alpha\to(ut)^{1/2}y_\alpha$, $\partial_\alpha\to\frac{1}{(ut)^{1/2}}\partial_\alpha$ and $c(f_\alpha)\to\frac{1}{(ut)^{1/2}}f^\alpha\wedge-(ut)^{1/2}i_{f_\alpha}$.

where K_Z is the scalar curvature of the fiber Z.

Set

$$\begin{aligned}\mathcal{H} = &-\Big(\partial_\alpha + \frac{1}{8}\langle R^{TB}(f_\alpha, f_\beta)f_\gamma, f_\delta\rangle y_\beta dy_\gamma dy_\delta\Big)^2 \\ &+ \frac{1}{8}\langle R^{TB}(f_\gamma, f_\delta)f_\beta, f_\alpha\rangle dy_\gamma dy_\delta \widehat{c}(f_\alpha)\widehat{c}(f_\beta).\end{aligned} \tag{2.40}$$

We claim similarly as in [6, (4.69)] that the right hand side of (2.39) is equal to

$$\mathcal{H} + A_u^2 - z\Big(u^{1/2}D^Z + \frac{c(T)}{4u^{1/2}}\Big). \tag{2.41}$$

In fact, when $z = 0$, (2.41) follows from (2.39), (2.40) and [9, (3.58)]. Thus, one needs only to check the term involving z which, in view of (2.11) and [9, (3.36)], is given by

$$\begin{aligned}&- u^{1/2}zc(h_s)\widetilde{\nabla}_{h_s} - \frac{1}{4}\langle S^B(h_s)f_\alpha, f_\beta\rangle dy_\alpha dy_\beta \frac{zc(h_s)}{u^{1/2}} + \frac{zu^{1/2}}{2}\widehat{c}(h_s)\omega(F, g^F)(h_s) \\ &= -z\Big(u^{1/2}D^Z + \frac{c(T)}{4u^{1/2}}\Big).\end{aligned} \tag{2.42}$$

Now by Proposition 1.2, (2.34), (2.37) and using the same arguments as in [11], [5] and [6], one deduces that as $t \to 0$,

$$\begin{aligned}&\mathrm{Tr}^{\mathrm{even}}\Big[\widehat{c}\Big(T^H M, g^{T^H M}\Big)\widehat{D}^F_{M/TZ}(t)\exp(-u(\widehat{D}^F_{M/TZ}(t))^2)\Big] \\ &\to \frac{2^{k-1}}{(4\pi)^{k/2}}\int_B \Big\{\det{}^{1/2}\Big(\frac{R^{TB}/2}{\sinh(R^{TB}/2)}\Big)\mathrm{Tr}_s\Big[\Big(D^Z + \frac{c(T)}{4u}\Big)\exp(-A_u^2)\Big] \\ &\quad \cdot \int^{\wedge} \widehat{c}\Big(T^H M, g^{T^H M}\Big)\exp\Big(\frac{1}{8}\langle R^{TB}(f_\gamma, f_\delta)f_\alpha, f_\beta\rangle dy_\gamma dy_\delta \widehat{c}(f_\alpha)\widehat{c}(f_\beta)\Big)\Big\},\end{aligned} \tag{2.43}$$

where $\int^{\wedge}$ is the obvious (odd dimensional) analogue of the Berezin integral (1.46) on B.

(2.23) then follows from (2.43) and the easy fact that the Berezin integral in (2.43) is equal to

$$\det{}^{1/2}\Big(\cosh\Big(\frac{R^{TB}}{2}\Big)\Big). \tag{2.44}$$

The next result further simplifies the right hand side of (2.23).

Proposition 2.3. *For any $u > 0$, the following identity over B holds,*

$$\mathrm{Tr}_s\Big[\Big(D^Z + \frac{c(T)}{4u}\Big)\exp(-A_u^2)\Big] = 0. \tag{2.45}$$

Proof. From (2.22), one finds

$$\frac{\partial A_u}{\partial u} = \frac{1}{2u^{1/2}}\Big(D^Z + \frac{c(T)}{4u}\Big). \tag{2.46}$$

We now proceed as in [11, Section 10.5] to consider the extended fibration $\widetilde{M} = M \times \mathbf{R}_+$ over $\widetilde{B} = B \times \mathbf{R}_+$, with the vertical metric at $(b, u) \in B \times \mathbf{R}_+$ given by $u^{-1}g^{TZ_b}$. The flat vector bundle F lifts canonically to a flat vector bundle over $\widetilde{M}$, and so is the Euclidean metric on it.

One verifies easily by proceeding as in [11, Lemma 10.33] that the Bismut-Lott superconnection on $\widetilde{B}$ is given by

$$\widetilde{A} = A_u + d_{\mathbf{R}_+} - \frac{\dim Z}{4u} du. \tag{2.47}$$

From (2.47), one finds in using the Duhamel formula that

$$\mathrm{Tr}_s[\exp(-\widetilde{A}^2)] = \mathrm{Tr}_s[\exp(-A_u^2)] - \mathrm{Tr}_s\Big[\frac{\partial A_u}{\partial u}\exp(-A_u^2)\Big]du. \tag{2.48}$$

(2.45) follows from (2.46), (2.48) and a refined local index result of Bismut and Lott [9, Theorem 3.15].

(e) The Adiabatic Limit of η-Invariants

In this subsection, we still make the assumption that $k \equiv 3 \bmod 4\mathbf{Z}$ and will calculate the limit of $\overline{\eta}\big(D^F_{M/TZ,\mathrm{sig}}(t)\big)$ as $t \to 0$.

From (1.14), (A.8), (2.13) and an obvious rescaling, one finds

$$\begin{aligned}\overline{\eta}\left(D^F_{M/TZ,\mathrm{sig}}(t)\right) = & \frac{\dim\left(\ker D^F_{M/TZ,\mathrm{sig}}(t)\right)}{2} \\ & + \frac{1}{\sqrt{\pi}}\int_0^\infty \mathrm{Tr}^{\mathrm{even}}\Big[\widehat{c}\left(T^HM, g^{T^HM}\right)\widehat{D}^F_{M/TZ}(t) \\ & \cdot\exp(-u(\widehat{D}^F_{M/TZ}(t))^2)\Big]\frac{du}{2\sqrt{u}}.\end{aligned} \tag{2.49}$$

Now since the fiberwise operators D^Z_b, $b \in B$, have constant rank kernels (cf. [9, Section 3f)]), in view of (2.14) and [9, (3.36)] one can proceed in exactly the same way as in Dai [13] to calculate the limit of $\overline{\eta}\big(D^F_{M/TZ,\mathrm{sig}}(t)\big)$ as $t \to 0$.

In fact, by [9, (3.66)], we have an isomorphism of smooth $\mathbf{Z}$-graded vector bundles over B:

$$H^*\left(Z; F|_Z\right) = \ker(D^Z). \tag{2.50}$$

As a sub-bundle of W, $\ker(D^Z)$ inherits a Hermitian metric from that of W. Also, recall that $H^*\left(Z; F|_Z\right)$ admits a canonically induced flat connection $\nabla^{H^*(Z;F|_Z)}$. Let $\nabla^{H^*(Z;F|_Z),e}$ be the corresponding Euclidean connection on $H^*\left(Z; F|_Z\right)$ in the sense of (1.16), (1.17). From [9, Proposition 3.14], one gets

$$\nabla^{H^*(Z;F|_Z),e} = P^{\ker\left(D^Z\right)}\nabla^{W,u}, \tag{2.51}$$

where $P^{\ker(D^Z)}$ is the orthogonal projection from W onto $\ker(D^Z)$. Clearly, $\nabla^{H^*(Z;F|_Z),e}$ still preserves the $\mathbf{Z}$-grading of $H^*\left(Z; F|_Z\right)$.

On the other hand, the local index results in the last subsection show that the corresponding $\widehat{\eta}$-form vanishes in this context.

From the above discussion and by mimicing the arguments in [13], one gets the following analogue of [13, Theorem 0.1'][3] in our context, where the definition of sub-signature operators in Section 1(b) has been incoorporated.

[3] Compare also with [17, p.298] for the precise counting of the mod $\mathbf{Z}$ term.

Proposition 2.4. *The following identity holds,*

$$\lim_{t\to 0}\overline{\eta}\left(D^F_{M/TZ,\mathrm{sig}}(t)\right)\equiv\sum_{i=0}^{\dim Z}(-1)^i\overline{\eta}\left(D^{H^i(Z,F|_Z)}_{B,\mathrm{sig}}\right)\mod \mathbf{Z}. \tag{2.52}$$

(f) A Proof of Theorem 2.1

We first assume again that $k\equiv 3 \mod 4\mathbf{Z}$. Then by Proposition 2.4 and its application to the trivial line bundle case, one deduces that

$$\begin{aligned}
&\lim_{t\to 0}\left\{\overline{\eta}\left(D^F_{M/TZ,\mathrm{sig}}(t)\right)-\mathrm{rk}(F)\overline{\eta}(D_{M/TZ,\mathrm{sig}}(t))\right\}\\
&\equiv\sum_{i=0}^{\dim Z}(-1)^i\overline{\eta}\left(D^{H^i(Z,F|_Z)}_{B,\mathrm{sig}}\right)-\mathrm{rk}(F)\sum_{i=0}^{\dim Z}(-1)^i\overline{\eta}\left(D^{H^i(Z,\mathbf{R}|_Z)}_{B,\mathrm{sig}}\right)\mod \mathbf{Z}\\
&=\sum_{i=0}^{\dim Z}(-1)^i\left(\overline{\eta}\left(D^{H^i(Z,F|_Z)}_{B,\mathrm{sig}}\right)-\mathrm{rk}(H^i(Z,F|_Z))\overline{\eta}\left(D_{B,\mathrm{sig}}\right)\right)\\
&\quad-\mathrm{rk}(F)\sum_{i=0}^{\dim Z}(-1)^i\left(\overline{\eta}\left(D^{H^i(Z,\mathbf{R}|_Z)}_{B,\mathrm{sig}}\right)-\mathrm{rk}(H^i(Z,\mathbf{R}|_Z))\overline{\eta}\left(D_{B,\mathrm{sig}}\right)\right)\\
&\equiv\sum_{i=0}^{\dim Z}(-1)^i\phi(B,H^i(Z,F|_Z))-\mathrm{rk}(F)\sum_{i=0}^{\dim Z}(-1)^i\phi(B,H^i(Z,\mathbf{R}_Z))\mod \mathbf{Z},
\end{aligned}\tag{2.53}$$

where we have used the obvious fact that

$$\sum_{i=0}^{\dim Z}(-1)^i\mathrm{rk}(H^i(Z,F|_Z))=\mathrm{rk}(F)\sum_{i=0}^{\dim Z}(-1)^i\mathrm{rk}(H^i(Z,\mathbf{R}|_Z))=\mathrm{rk}(F)\chi(Z). \tag{2.54}$$

(2.3) then follows from (1.57) and (2.53).

Now we assume that $k\equiv 1 \mod 4\mathbf{Z}$. Then both $D^F_{M/TZ,\mathrm{sig}}(t)$ and $D^{H^*(Z,F|_Z)}_{B,\mathrm{sig}}$ are skew-adjoint. By proceeding as in [13, Section 2], where a proof of [13, Theorem 1.5] is given, one finds that there exist constants $\varepsilon_0>0$ and $\lambda_0>0$, such that for any $0<t\leq\varepsilon_0$,

$$\sharp\left\{\lambda:\lambda\in\mathrm{Spec}\left(-D^{F,2}_{M/TZ,\mathrm{sig}}(t)\right),\quad 0\leq\lambda\leq\lambda_0 t\right\}=\dim\left(\ker D^{H^*(Z,F|_Z)}_{B,\mathrm{sig}}\right). \tag{2.55}$$

(2.3) then follows from (1.56), (2.55) and the trivial fact that the nonzero eigenvalues of $D^{F,2}_{M/TZ,\mathrm{sig}}(t)$ are of even multiplicities (as $D^F_{M/TZ,\mathrm{sig}}(t)$ is skew-adjoint).

The proof of Theorem 2.1 is completed.

§ 3. An Extension to the Case Where TM/E Is Spin

In this section, we discuss briefly the refinements for the case where TM/E in Section 1 is spin, which were mentioned in Introduction.

For brevity, we only discuss the case where F is a complex flat vector bundle and work in the complex coefficient category.

This section is organized as follows. In (a), we extend the result in Section 1 to the case where TM/E is spin. In (b), we extend the results in Section 2 to the case where B is spin.

(a) Sub-Dirac Operators and the Generalized Atiyah-Patodi-Singer Invariants for Flat Vector Bundles

Let M be an odd dimensional closed oriented manifold. Let E be an oriented subbundle of TM. Then TM/E admits an induced orientation. We make the assumption that $\dim TM/E$ is odd. We further assume in this section that TM/E is spin and carries a fixed spin structure.

Let g^{TM} be a metric on TM. Let $E^\perp \subset TM$ be the orthogonal completement to E with respect to g^{TM}. Then one has the orthogonal splitting (1.6). We identify TM/E with $E^\perp$. Let $S(E^\perp)$ denote the corresponding bundle of spinors.

Let (F, ∇^F) be a complex flat vector bundle over M. Let g^F be a Hermitian metric on F. Then as in (1.16) and (1.17), one gets a Hermitian connection $\nabla^{F,e}$ on F.

Let ∇^{TM} be the Levi-Civita connection of g^{TM}. Let ∇^E, $\nabla^{E^\perp}$ be the projection connection on E, $E^\perp$ defined as in (1.19). Let $\nabla^{S(E^\perp)}$ (resp. $\nabla^{\wedge^*(E^*)}$) be the canonical Hermitian connection on $S(E^\perp)$ (resp. $\wedge^*(E^*)$) induced by $\nabla^{E^\perp}$ (resp. ∇^E). Let ∇^u be the unitary connection on $S(E^\perp)\otimes\wedge^*(E^*)\otimes F$ obtained from the tensor product of $\nabla^{S(E^\perp)}$, $\nabla^{\wedge^*(E^*)}$ and $\nabla^{F,e}$.

For any $X \in E^\perp$, let $c(X)$ be the Clifford action of X on $S(E^\perp)$. It extends to an action on $S(E^\perp)\otimes\wedge^*(E^*)\otimes F$ as $c(X)\otimes(\mathrm{Id}_{\wedge^{\mathrm{even}}(E^*)} - \mathrm{Id}_{\wedge^{\mathrm{odd}}(E^*)})\otimes \mathrm{Id}_F$. For any $Y \in E$, let $c(Y) = Y^*\wedge - i_Y$ be the Clifford action of Y on $\wedge^*(E^*)$. It extends to an action on $S(E^\perp)\otimes\wedge^*(E^*)\otimes F$ by acting as identity on $S(E^\perp)\otimes F$.

Let $h_1, \cdots, h_{\dim E}$ (resp. $f_1, \cdots, f_{\dim E^\perp}$) be an oriented orthonormal basis of E (resp. $E^\perp$). Let S be the tensor defined as in (1.20).

Definition 3.1. *The sub-Dirac operator $D^F_{M/E,\mathrm{spin}}$ is the first order elliptic differential operator acting on $S(E^\perp)\otimes\wedge^*(E^*)\otimes F$ given by*

$$\begin{aligned} D^F_{M/E,\mathrm{spin}} = &\sum_{\alpha=1}^{\dim E^\perp} c(f_\alpha)\nabla^u_{f_\alpha} + \frac{1}{2}\sum_{s,q=1}^{\dim E}\sum_{\alpha=1}^{\dim E^\perp}\langle S(h_s)h_q, f_\alpha\rangle c(h_s)c(h_q)c(f_\alpha) \\ &+ \sum_{s=1}^{\dim E} c(h_s)\nabla^u_{h_s} + \frac{1}{2}\sum_{\alpha,\beta=1}^{\dim E^\perp}\sum_{s=1}^{\dim E}\langle S(f_\alpha)f_\beta, h_s\rangle c(f_\alpha)c(f_\beta)c(h_s). \end{aligned} \tag{3.1}$$

One verifies easily that $D^F_{M/E,\mathrm{spin}}$ is formally self-adjoint. Furthermore, it is formally of the same nature as D^0 in (1.33) (compare with (1.35)).

Remark 3.1. When both M and E are even dimensional and F is the trivial line bundle, such an operator has been constructed in [16]. Locally, it can certainly be looked as a twisted Dirac operator. The main point here is that it is globally well defined even if M is nonspin. The same remark holds for the sub-signature operators studied in Section 1, where neither $E^\perp$ nor M are supposed to be spin.

We can now state the following analogue of Theorem 0.1, which can be proved by mimicing the procedure in Section 1.

Theorem 3.1. *The quantity*

$$\widehat{\phi}(M/E,F)=\overline{\eta}\left(D^F_{M/E,\mathrm{spin}}\right)-\mathrm{rk}(F)\overline{\eta}\left(D^{\mathbf{R}_M}_{M/E,\mathrm{spin}}\right)\in\mathbf{R}/\mathbf{Z}$$

does not depend on the metrics g^{TM} and g^F.

Clearly, these quantities define a series of invariants generalizing those of Atiyah-Patodi-Singer [3, Proposition 2.14].

(b) A Riemann-Roch Theorem for Flat Vector Bundles

In this section, we assume that we have a fibration $Z\to M\stackrel{\pi}{\to}B$ such that $E=TZ$ verifies the assumptions in the above subsection. As in Section 2(a), one has a $\mathbf{Z}$-graded flat vector bundle

$$H^*(Z;F|_Z)=\bigoplus_{i=0}^{\dim Z}H^i(Z;F|_Z)$$

over B. We can now state the following analogue of Theorem 0.2.

Theorem 3.2. *The following identity holds,*

$$\begin{aligned}&\widehat{\phi}(M/TZ,F)\\&=\sum_{i=0}^{\dim Z}(-1)^i\widehat{\phi}(B,H^i(Z,F|_Z))-\mathrm{rk}(F)\sum_{i=0}^{\dim Z}(-1)^i\widehat{\phi}(B,H^i(Z,\mathbf{R}|_Z)).\end{aligned}\tag{3.2}$$

Proof. Let $\omega(F,g^F)$ be defined as in (1.16). Then one sees clearly that the operator

$$D^F_{M/TZ,\mathrm{spin}}-\frac{1}{2}\sum_{s=1}^{\dim Z}\widehat{c}(h_s)\omega(F,g^F)(h_s)\tag{3.3}$$

is of exactly the same nature as $D^F_{M/TZ}$ studied in Section 2.

Now it is clear that we can apply the arguments of Bismut-Cheeger [6, 7] and Dai [13] as in Section 2 to compute the adiabatic limit of the η invariants of the operators of form (3.3). In particular, the $\widehat{\eta}$-form is still seen to be zero. Thus, one has

$$\lim_{t\to 0}\overline{\eta}\Big(D^F_{M/TZ,\mathrm{spin}}(t)-\frac{1}{2}\sum_{s=1}^{\dim Z}\widehat{c}(h_s)\omega(F,g^F)(h_s)\Big)\equiv\sum_{i=0}^{\dim Z}(-1)^i\overline{\eta}\left(D^{H^i(Z,F|_Z)}_{B,\mathrm{spin}}\right),\tag{3.4}$$

where $D^F_{M/TZ,\mathrm{spin}}(t)$ is the sub-Dirac operator with respect to the metric g^{TM}_t in (2.12).

On the other hand, by proceeding as in (A.9) and (A.10), one gets easily that

$$\overline{\eta}\Big(D^F_{M/TZ,\mathrm{spin}}-\frac{1}{2}\sum_{s=1}^{\dim Z}\widehat{c}(h_s)\omega(F,g^F)(h_s)\Big)\equiv\overline{\eta}\left(D^F_{M/E,\mathrm{spin}}\right)\mod\mathbf{Z}.\tag{3.5}$$

(3.2) then follows from Theorem 3.1, (3.2), (3.4) and (3.5) as well as their applications to the trivial line bundle case.

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