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Geometric quantization for proper actions ☆

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Abstract

We first introduce an invariant index for G-equivariant elliptic differential operators on a locally compact manifold M admitting a proper cocompact action of a locally compact group G. It generalizes the Kawasaki index for orbifolds to the case of proper cocompact actions. Our invariant index is used to show that an analog of the Guillemin–Sternberg geometric quantization conjecture holds if M is symplectic with a Hamiltonian action of G that is proper and cocompact. This essentially solves a conjecture of Hochs and Landsman.

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1. Introduction

The main purpose of this paper is to generalize the Guillemin–Sternberg geometric quantization conjecture [5], which was proved in [12], to the case of non-compact spaces and group actions. Here we will consider the framework considered by Hochs and Landsman in [9].

☆ With an Appendix by Ulrich Bunke, NWF I - Mathematik, Universität Regensburg, 93040 Regensburg, Germany. *E-mail address:* ulrich.bunke@mathematik.uni-regensburg.de.

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To be more precise, let (M,ω) be a locally compact symplectic manifold. Assume that there exists a Hermitian line bundle (L,h^L) over X carrying a Hermitian connection ∇^L such that $\frac{\sqrt{-1}}{2\pi}(\nabla^L)^2=\omega$. Let J be an almost complex structure on TM such that $\omega(\cdot,J\cdot)$ defines a Riemannian metric g^{TM} on TM. Let $D^L:\Omega^{0,*}(M,L)\to\Omega^{0,*}(M,L)$ be the canonically associated Spin^c-Dirac operator (cf. [16, Section 1]). Let $D^L_\pm$ be the restrictions of D^L on $\Omega^{0,\frac{\text{even}}{\text{odd}}}(M,L)$ respectively.

Let G be a locally compact group with Lie algebra $\mathfrak{g}$. Suppose that G acts on M properly. The proper G-action ensures that the isotropy subgroups $G_x=\{g\in G \mid gx=x\}$ are compact subgroups of G for all $x\in M$, all the orbits $Gx=\{gx \mid g\in G\}\subseteq M$ are closed, and moreover the space of orbits M/G is Hausdorff (cf. [13]).

We assume that the action of G on M lifts on L. Moreover, we assume the G-action preserves the above metrics and connections on TM, L, and J. Then $D^L_\pm$ commute with the G-action.

The action of G on L induces naturally a moment map $\mu: M\to\mathfrak{g}^*$ such that for any $V\in\mathfrak{g}$, $s\in\Gamma(L)$, if V_M denotes the induced Killing vector field on M, then the following Kostant formula for Lie derivative holds,

$$L_{V_M}s=\nabla^L_{V_M}s-2\pi\sqrt{-1}\langle\mu,V\rangle s. \tag{1.1}$$

We make the assumption that $0\in\mathfrak{g}^*$ is a regular value of μ and that G acts freely on $\mu^{-1}(0)$. Then the Marsden–Weinstein symplectic reduction $(M_G=\mu^{-1}(0)/G,\omega_{M_G})$ is a symplectic manifold. Moreover, (L,∇^L) descends to (L_G,∇^{L_G}) over X_G so that the corresponding curvature condition $\frac{\sqrt{-1}}{2\pi}R^{L_G}=\omega_G$ holds. The G-invariant almost complex structure J also descends to an almost complex structure on TM_G, and h^L, g^{TM} descend to h^{L_G}, g^{TM_G} respectively. Let D^{L_G} denote the corresponding Spin^c-Dirac operator on M_G.

Following [9], we make the assumption that the quotient space M/G is compact, that is, the G-action on M is cocompact. Then $M_G=\mu^{-1}(0)/G$ is also compact.

In this paper, we will first define in Section 2 what we call the G-invariant index associated to D^L_+, denoted by $\operatorname{ind}_G(D^L_+)$, which generalizes the usual definition in the compact case to the non-compact case.

For any positive integer p, let L^p denote the p-th tensor power of L. Then it admits the canonically induced G-action as well as the G-invariant Hermitian metric and connection.

We can now state the main result of this paper, which might be thought of as a "quantization commutes with reduction" result, in the sense of a conjecture of Hochs and Landsman [9, Conjecture 1.1], as follows.

Theorem 1.1. *In the general case where G is merely assumed to be locally compact, there exists $p_0>0$ such that for any integer $p\geqslant p_0$,*

$$\operatorname{ind}_G\big(D_+^{L^p}\big)=\operatorname{ind}\big(D_+^{L_G^p}\big). \tag{1.2}$$

Moreover, if $\mathfrak{g}^*$ admits an Ad_G*-invariant metric, then one can take $p=1$ in* (1.2).

Remark 1.2. In the special case where G (and thus M) are compact, Theorem 1.1 is the Guillemin–Sternberg geometric quantization conjecture [5] first proved by Meinrenken in [12] (see the excellent survey of Vergne [18] for further related works). In the special case where G is non-compact and admits a normal discrete subgroup Γ such that Γ acts on M freely and that G/Γ is compact,[1] Theorem 1.1 is closely related to [9, Theorem 1.2]. While in the special case where G is semisimple and acts on $G \times_K N$, with K the maximal compact subgroup of G acting Hamiltonianly on a compact symplectic manifold N, Theorem 1.1 should be closely related to the quantization formula of Hochs obtained in [7]. In fact, in the formulas of Hochs and Hochs–Landsman, the left-hand side of the quantization formula is interpreted by using non-commutative K-theories. Thus, in combining with our result, in the case considered by them, our invariant index admits the non-commutative K-theoretic interpretation. In fact, in the Appendix to this paper, Bunke establishes such an interpretation. Combining with Kasparov's index theorem [10], one gets a topological counterpart to our analytic index.

Since here one is dealing with non-compact group actions on non-compact spaces, one cannot apply the Atiyah–Bott–Segal–Singer equivariant index theorem directly as in [12] to prove Theorem 1.1. Instead, we will generalize the analytic proof of the Guillemin–Sternberg conjecture due to Tian and Zhang [16] to the current situation.

On the other hand, one can show that the finiteness of $\dim(\operatorname{Ker} D^L_{\pm})^G$, the dimensions of the G-invariant subspaces of $\operatorname{Ker}(D^L_{\pm})$, holds for any equivariant Dirac type operator on spaces with proper cocompact actions. Moreover, we will show that the vanishing properties of the half kernel of Spin^c-Dirac operators due to Braverman [4], Borthwick and Uribe [2] and Ma and Marinescu [11] still hold for D^{L^p} when $p > 0$ is large. Combining with Theorem 1.1, one gets

Theorem 1.3. *There exists $p_1 > 0$ such that for any integer $p \geqslant p_1$, one has*

$$\dim\big(\operatorname{Ker} D_+^{L^p}\big)^G = \dim\big(\operatorname{Ker} D_+^{L_G^p}\big),$$
$$\dim\big(\operatorname{Ker} D_-^{L^p}\big)^G = \dim\big(\operatorname{Ker} D_-^{L_G^p}\big) = 0. \tag{1.3}$$

Remark 1.4. Just as in [16], one can twist L^p by an arbitrary G-equivariant vector bundle F over M carrying a G-invariant Hermitian metric and a G-invariant Hermitian connection (that is, replace L^p by $L^p \otimes F$). Then all the results of this paper still hold when $p > 0$ is large enough. In particular, if we chose $F = \Lambda^{*,0}(T^*M)$, then we get a non-compact quantization formula for the signature quantization considered in [17] and [6].

Remark 1.5. Our method also works for the case where the action of G on $\mu^{-1}(0)$ is not free. Then $M_G = \mu^{-1}(0)/G$ is an orbifold, and (1.2) and (1.3) still hold if we replace the right-hand sides by the corresponding orbifold indices. We leave this to the interested reader.

The rest of this paper is organized as follows. In Section 2, for any locally compact manifold M admitting a proper cocompact action of a locally compact group G, and any G-equivariant Dirac type operator D on M, we introduce what we call the G-invariant index $\operatorname{ind}_G(D)$. In Section 3, we generalize the analytic techniques developed in [16] to give a proof of Theorem 1.1. In Section 4, we prove the vanishing properties of the G-invariant part of the half kernel of the equivariant Dirac operator and as a consequence, get Theorem 1.3. Finally, in Section 5, we consider some examples and applications of our main results.

[1] In this case there is an Ad_G invariant metric on $\mathbf{g}^*$.

2. The invariant index for proper cocompact actions

Let M be a locally compact manifold. Let G be a locally compact group. Let dg be the left invariant Haar measure on G.

We make the assumption that G acts on M properly and cocompactly, where by proper action we mean that the following map

$$G \times M \to M \times M, \quad (g, x) \mapsto (x, gx) \tag{2.1}$$

is proper (that is, the inverse image of a compact subset is compact), while by cocompact we mean that the quotient M/G is a compact space.

One of the basic properties for such an action is that there exists a smooth, non-negative, compactly supported cut-off function c on M such that

$$\int_G c\big(g^{-1}x\big)^2\, dg = 1 \tag{2.2}$$

for any $x \in M$ (cf. [3, Section 7.2.4, Proposition 8]). This cut-off function allows one to get G-invariant objects from the originally not necessarily G-invariant ones.

As an example, for any Riemannian metric g^{TM} on TM one gets an "averaged" G-invariant metric

$$\int_G c\big(h^{-1}x\big)^2\big(h^* g^{TM}\big)(x)\, dh \tag{2.3}$$

on TM.

From now on, we assume that g^{TM} is G-invariant.

Let $\mathrm{cl}(TM)$ be the Clifford algebra bundle associated to (TM, g^{TM}). Then it admits a naturally induced G-action as well as a G-invariant Hermitian metric $g^{\mathrm{cl}(TM)}$.

Let E be a complex vector bundle over M such that $E = E_+ \oplus E_-$ is a $\mathbf{Z}_2$-graded $\mathrm{cl}(TM)$-module and that it admits a G-action, preserving the $\mathbf{Z}_2$-grading of E, lifted from the action of G on M. Let g^E be a $\mathbf{Z}_2$-graded G-invariant Hermitian metric on E, let ∇^E be a $\mathbf{Z}_2$-graded G-invariant Hermitian connection on E. The averaging procedure similar to that in (2.3) guarantees the existence of g^E and ∇^E.

Let $e_1, \dots, e_{\dim M}$ be an oriented orthonormal basis of TM. We define a Dirac type operator D^E to be the operator acting on $\Gamma(E)$,

$$D^E = \sum_{i=1}^{\dim M} c(e_i)\nabla^E_{e_i} + A : \Gamma(E) \to \Gamma(E), \tag{2.4}$$

where $A \in \Gamma(\mathrm{End}(E))$ exchanges $E_\pm$.

We make the assumption that D^E is G-equivariant.

Let $\Gamma(E)$ carry the natural inner product such that for any $s, s' \in \Gamma(E)$ with compact supports,

$$\langle s, s' \rangle = \int_M \langle s(x), s'(x) \rangle_E \, dx. \tag{2.5}$$

Let $\|\cdot\|_0$ denote the associated L^2-norm. Let $L^2(M, E)$ denote the completion of $\Gamma(E)$ with respect to the inner product $\|\cdot\|_0$.

Since M/G is compact, there exists a compact subset Y of M such that $G(Y) = M$ (cf. [15, Lemma 2.3]).

Let U, U' be two open subsets of M such that $Y \subset U$ and that the closures $\overline{U}$ and $\overline{U'}$ are both compact in M, and that $\overline{U} \subset U'$. The existence of U, U' is clear.

Then it is easy to construct a smooth function $f : M \to [0, 1]$ such that $f|_U = 1$ and $\mathrm{Supp}(f) \subset U'$.[2]

We now consider the space $\Gamma(E)^G$, the subspace of G-invariant sections of $\Gamma(E)$.

By using the property that $G(Y) = M$, it is easy to see that there exists a positive constant $C > 0$ such that for any $s \in \Gamma(E)^G$,

$$\|s\|_{U,0} \leqslant \|fs\|_0 \leqslant \|s\|_{U',0} \leqslant C\|s\|_{U,0}, \tag{2.6}$$

where for $V = U$ or U',

$$\|s\|^2_{V,0} = \int_V \langle s(x), s(x) \rangle_E \, dx. \tag{2.7}$$

Let $\mathbf{H}^0_f(M, E)^G$ be the completion of the space $\{fs: s \in \Gamma(E)^G\}$ under the norm $\|\cdot\|_0$ associated to the inner product (2.5). Let $\mathbf{H}^1_f(M, E)^G$ be the completion of $\{fs: s \in \Gamma(E)^G\}$ under a (fixed) G-invariant first Sobolev norm associated to the inner product (2.5).

Let P_f be the orthogonal projection from $L^2(M, E)$ to its subspace $\mathbf{H}^0_f(M, E)^G$.

It is clear that $P_f D^E$ maps an element of $\mathbf{H}^1_f(M, E)^G$ into $\mathbf{H}^0_f(M, E)^G$.

Proposition 2.1. *The induced operator*

$$P_f D^E : \mathbf{H}^1_f(M, E)^G \to \mathbf{H}^0_f(M, E)^G \tag{2.8}$$

is a Fredholm operator.

Proof. For any $s \in \Gamma(E)^G$, by (2.4), one has

$$D^E(fs) = fD^E s + c(df)s, \tag{2.9}$$

where we identify the one form df with its metric dual $(df)^*$.

Since D^E is G-equivariant, it is clear that $P_f(fD^E s) = fD^E s$, while in view of (2.6),

$$\left\| P_f\big(c(df)s\big) \right\|_0 \leqslant \left\| c(df)s \right\|_0 \leqslant C_1 \|s\|_{U,0} \tag{2.10}$$

for some constant $C_1 > 0$.

[2] With this function, one can construct the cut-off function c by $c(x) = \frac{f(x)}{(\int_G f(g^{-1}x)^2 \, dg)^{1/2}}$ for any $x \in M$.

Thus, one has, by also proceeding as in (2.6),

$$\left\| P_f D^E(fs)\right\|_0 \geqslant \left\| f D^E s\right\|_0 - C_1\|s\|_{U,0} \geqslant C_2\|s\|_{U,1} - C_3\|s\|_{U,0} \tag{2.11}$$

for some constants $C_2, C_3 > 0$.

On the other hand, by (2.9) and by proceeding as in (2.6), one verifies that

$$\|fs\|_1 \leqslant C_4\left(\|fs\|_0 + \left\|D^E(fs)\right\|_0\right) \leqslant C_5\|s\|_{U,0} + C_6\|s\|_{U,1} \tag{2.12}$$

for some constants $C_4, C_5, C_6 > 0$.

From (2.6), (2.11) and (2.12), one gets

$$\left\| P_f D^E(fs)\right\|_0 \geqslant C_7\|fs\|_1 - C_8\|fs\|_0 \tag{2.13}$$

for some constants $C_7, C_8 > 0$.

Since f is of compact support, from the Gärding type inequality (2.13) one sees that $P_f D^E$ is a Fredholm operator. □

Remark 2.2. Besides the Fredholm property in Proposition 2.1, the following self-adjoint property also holds: for any $s, s' \in \Gamma(E)^G$, one has

$$\left\langle P_f D^E(fs), fs'\right\rangle = \left\langle fs, P_f D^E\left(fs'\right)\right\rangle, \tag{2.14}$$

if D^E is formally self-adjoint.

Remark 2.3. If $(\tilde{U}, \tilde{U}', \tilde{f})$ is another triple of open subsets and the cut-off function as above, then by taking the deformation $f_t = (1-t)f + t\tilde{f}$, one gets easily a continuous family of Fredholm operators $P_{f_t} D^E$.

Now let $D^E_\pm : \Gamma(E_\pm) \to \Gamma(E_\mp)$ be the restrictions of D^E on $\Gamma(E_\pm)$ respectively.

Then by Proposition 2.1 and (2.14), the induced operator $P_f D^E_+ : \mathbf{H}^1_f(M, E_+)^G \to \mathbf{H}^0_f(M, E_-)^G$ is Fredholm. Moreover its index, $\operatorname{ind}(P_f D^E_+)$, does not depend on the choice of f, in view of Remark 2.3. Similarly, it is also easy to see that this index does not depend on the choices of G-invariant metrics and connections involved.

Definition 2.4. We call $\operatorname{ind}(P_f D^E_+)$ defined above the G-invariant index associated to D^E_+ and denote it by $\operatorname{ind}_G(D^E_+)$.

Remark 2.5. If M is compact and D^E is formally self-adjoint, then one can take $f \equiv 1$, so that one has

$$\operatorname{ind}_G\left(D^E_+\right) = \dim\left(\left(\ker D^E_+\right)^G\right) - \dim\left(\left(\ker D^E_-\right)^G\right). \tag{2.15}$$

Remark 2.6. For any $s \in \Gamma(E)^G$, it is clear that fs and s are determined by each other. That is, if $s, s' \in \Gamma(E)^G$ are such that $fs = fs'$, then $s = s'$ as $f \equiv 1$ on Y. Moreover, by (2.9), (2.10) and (2.13), one sees easily that the induced operator $\tilde{D}^E_f : \mathbf{H}^1_f(M, E)^G \to \mathbf{H}^0_f(M, E)^G$ such that

$$\tilde{D}_f^E(fs) := f D^E(s) \tag{2.16}$$

is a Fredholm operator. Thus one gets that

$$\dim\bigl(\ker\bigl(D^E|_{\Gamma(E)^G}\bigr)\bigr) = \dim\bigl(\bigl(\ker D^E\bigr)^G\bigr) < +\infty. \tag{2.17}$$

In fact, when G is unimodular, we can further identify $\operatorname{ind}_G(D_+^E)$ as follows.

Theorem 2.7. *If G is unimodular, then with the notation above, one has*

$$\operatorname{ind}\bigl(P_f D_+^E\bigr) = \dim\bigl(\bigl(\ker D_+^E\bigr)^G\bigr) - \dim\bigl(\bigl(\ker D_-^E\bigr)^G\bigr), \tag{2.18}$$

whenever D^E is formally self-adjoint.

Proof. We use the cut-off function f, and set for any $x \in M$,

$$c(x) = \frac{f(x)}{(\int_G f(g^{-1}x)^2\,dg)^{1/2}}. \tag{2.19}$$

Then

$$\int_G c\bigl(g^{-1}x\bigr)^2\,dg = 1 \tag{2.20}$$

for any $x \in M$. Let α denote the positive G-invariant function on M defined by

$$\alpha(x) = \biggl(\int_G f\bigl(g^{-1}x\bigr)^2\,dg\biggr)^{1/2}. \tag{2.21}$$

Let $\mathbf{H}_c^0(M,E)^G$ be the L^2 completion of the space $\{cs\colon s \in \Gamma(E)^G\}$. Let $\mathbf{H}_c^1(M,E)^G$ be the corresponding first Sobolev space associated to a (fixed) G-invariant first Sobolev norm.

Let $\beta_f : \mathbf{H}_f^0(M,E)^G \to \mathbf{H}_c^0(M,E)^G$ be the isomorphism such that for any $s \in \Gamma(E)^G$,

$$\beta_f : fs \mapsto f\frac{s}{\alpha} = cs. \tag{2.22}$$

Let P_c be the orthogonal projection from $L^2(M,E)$ onto $\mathbf{H}_c^0(M,E)^G$.

For any $s \in \Gamma(E)^G$, one verifies that

$$\begin{aligned}\beta_f\bigl(P_f D^E(fs)\bigr) &= cD^E s + \frac{1}{\alpha}P_f\bigl(c(df)s\bigr)\\ &= P_c D^E(cs) - P_c\bigl(c(dc)s\bigr) + \frac{1}{\alpha}P_f\bigl(c(df)s\bigr).\end{aligned} \tag{2.23}$$

Since $P_f D^E$ is Fredholm, from (2.23), one sees easily that $P_c D^E$ is a Fredholm operator and

$$\operatorname{ind}\bigl(P_f D_+^E\bigr) = \operatorname{ind}\bigl(P_c D_+^E\bigr). \tag{2.24}$$

Indeed, what Bunke does in his Appendix is to give a KK-theoretic interpretation of the right-hand side of (2.24).

Moreover, Bunke actually writes out explicitly the projection P_c when G is unimodular. According to Bunke, when G is unimodular, for any $\mu \in \Gamma(E)$ with compact support, one has (cf. Appendix D, where P_c here is exactly Q_ϵ there with $\epsilon = 1$)

$$(P_c\mu)(x) = c(x)\int_G c\bigl(g^{-1}x\bigr)\bigl(g^*\mu\bigr)(x)\,dg. \tag{2.25}$$

From (2.20) and (2.25), one computes that for any $s \in \Gamma(E)^G$, one has at $x \in M$ that

$$P_c\bigl(c(dc)s\bigr) = c\int_G c\bigl(g^{-1}x\bigr)\bigl(g^*\bigl(c(dc)s\bigr)\bigr)(x)\,dg = \frac{1}{2}c\cdot c\left(d\int_G c\bigl(g^{-1}x\bigr)^2\,dg\right)s(x) = 0. \tag{2.26}$$

From (2.26), one gets that for any $s \in \Gamma(E)^G$,

$$P_c D^E(cs) = cD^E s + P_c\bigl(c(dc)s\bigr) = cD^E s. \tag{2.27}$$

When D^E is formally self-adjoint, from (2.27) one gets immediately that

$$\dim\bigl(\operatorname{Ker} P_c D_\pm^E\bigr) = \dim\bigl(\bigl(\operatorname{Ker} D_\pm^E\bigr)^G\bigr). \tag{2.28}$$

Combining with (2.24), one gets

$$\operatorname{ind}_G\bigl(D^E\bigr) = \operatorname{ind}\bigl(P_c D^E\bigr) = \dim\bigl(\bigl(\ker D_+^E\bigr)^G\bigr) - \dim\bigl(\bigl(\ker D_-^E\bigr)^G\bigr), \tag{2.29}$$

which is exactly (2.18). □

Remark 2.8. Remark 2.5 and Theorem 2.7 fully justify the term "G-invariant index" in Definition 2.4. Moreover, by (2.28), one sees that $\ker(P_c D^E)$ consists of smooth elements.

Remark 2.9. When G is non-unimodular, Theorem 2.7 still holds if one inserts the modular factor δ (with $dg^{-1} = \delta(g)dg$) in the right-hand side of (2.18) as in Bunke's Appendix. We leave it to the interested reader.

We now consider the very special case where G acts on M freely, but we no longer assume that G is unimodular (thus we no longer have Theorem 2.7). Then M/G is a compact manifold, while E descends to a Hermitian vector bundle E_G over M/G carrying an induced Hermitian connection.

Let $D_+^{E_G} : \Gamma(E_{+,G}) \to \Gamma(E_{-,G})$ be associated Dirac operator.

Proposition 2.10. *The following identity holds,*

$$\operatorname{ind}_G\big(D_+^E\big) = \operatorname{ind}\big(D_+^{E_G}\big). \tag{2.30}$$

Proof. For any $s \in \Gamma(E_G)$, let $\tilde{s} \in \Gamma(E)^G$ be its canonical lift.

It is clear that the map $\Gamma(E_G) \to \{f\Gamma(E)^G\}$ such that $s \to f\tilde{s}$ extends canonically to a bounded linear isomorphism $\alpha_f : L^2(E_G) \to \mathbf{H}_f^0(M, E)^G$.

Lemma 2.11. *There exists a constant $C > 0$ such that for any $s \in \Gamma(E_G)$, one has*

$$\big\|(\alpha_f)^{-1} P_f D^E \alpha_f(s) - D^{E_G} s\big\| \leqslant C\|s\|. \tag{2.31}$$

Proof. From (2.9), one has

$$D^E \alpha_f s = D^E(f\tilde{s}) = f D^E \tilde{s} + c(df)\tilde{s}. \tag{2.32}$$

Now, on the principal fibre bundle $G \to M \to M/G$, the vertical directions are generated by elements in the Lie algebra $\mathbf{g}$. Thus, for any $X \in T^V M$ such that $|X| = 1$, since $\tilde{s}$ is G-invariant so that $L_X^E \tilde{s} = 0$, where L_X^E is the Lie derivative along X on E, one has

$$\big\|f\big(\nabla_X^E \tilde{s}\big)\big\| = \big\|f\big(\nabla_X^E \tilde{s} - L_X^E \tilde{s}\big)\big\| \leqslant C_1\|f\tilde{s}\| \tag{2.33}$$

for a (fixed) positive constant $C_1 > 0$.

From (2.33), one verifies easily that

$$\big\|(\alpha_f)^{-1}\big(f D^E \tilde{s}\big) - D^{E_G} s\big\| \leqslant C_2\|s\| \tag{2.34}$$

for a (fixed) positive constant $C_2 > 0$.

From (2.34), one sees immediately that

$$\big\|(\alpha_f)^{-1}\big(P_f\big(f D^E \tilde{s}\big)\big) - D^{E_G} s\big\| \leqslant C_2\|s\|. \tag{2.35}$$

On the other hand, one finds easily that there are positive constants $C_3 > 0$, $C_4 > 0$, $C_5 > 0$ such that

$$\big\|(\alpha_f)^{-1} P_f\big(c(df)\tilde{s}\big)\big\| \leqslant C_3\big\|c(df)\tilde{s}\big\| \leqslant C_4\|f\tilde{s}\| \leqslant C_5\|s\|. \tag{2.36}$$

From (2.32), (2.35) and (2.36), one gets (2.31). □

We now return to the proof of (2.30).

By restricting (2.31) to $\Gamma(E_{+,G})$, one deduces that

$$\operatorname{ind}\big(D_+^{E_G}\big) = \operatorname{ind}\big((\alpha_f)^{-1} P_f D_+^E \alpha_f\big) = \operatorname{ind}\big(P_f D_+^E\big), \tag{2.37}$$

which is exactly (2.30). □

Remark 2.12. It is easy to see that for Proposition 2.10 to hold, one need only to assume that (the G-equivariant operator) D_+^E is G-transversally elliptic.

3. The geometric quantization formula for proper actions

We now turn back to the situation as in Section 1. In this case, $E = \Lambda^{0,*}(T^*M) \otimes L^p$ and $\Omega^{0,*}(M, L^p) = \Gamma(\Lambda^{0,*}(T^*M) \otimes L^p)$.

Let D^{L^p} denote the corresponding Spin^c-Dirac operator (cf. [16, Section 1]). Clearly, D^{L^p} is G-equivariant.

Let $\Omega^{0,*}(M, L^p)^G$ be the subspace of G-invariant sections of $\Omega^{0,*}(M, L^p)$. It is clear that any section in $\Omega^{0,*}(M, L^p)^G$ is determined by its restriction to Y.

Note that since in the general case where G is assumed to be only locally compact, there might not be any Ad_G-invariant metric on $\mathfrak{g}$.

Choose any metric on $\mathfrak{g}^*$. Let $h_1, \ldots, h_{\dim G}$ be an orthonormal basis of $\mathfrak{g}^*$. Denote by V_i the Killing vector field on X generated by the dual of h_i ($1 \leqslant i \leqslant \dim G$). The point here is that the function

$$\mathcal{H} = \|\mu\|^2 = \sum_{i=1}^{\dim G} \mu_i^2 \tag{3.1}$$

might not be G-invariant, thus the associated Hamiltonian vector field $X^{\mathcal{H}}$ might not be G-invariant. We first construct an invariant one out of it.

Recall that the cut-off function c has been defined in (2.2).

Let

$$X_G^{\mathcal{H}} = \int_G c\big(g^{-1}x\big)^2 X_g^{\mathcal{H}}\, dg \tag{3.2}$$

denote the averaged *G-invariant* vector field on M, where $X_g^{\mathcal{H}}$ denotes the pullback of $X^{\mathcal{H}}$ by $g \in G$.

For any $T \geqslant 0$, set

$$D_T^{L^p} = D^{L^p} + \frac{\sqrt{-1}T}{2} c\big(X_G^{\mathcal{H}}\big). \tag{3.3}$$

Then it is G-equivariant. Moreover, it is a formally self-adjoint Dirac type operator in the sense of (2.4) and thus the results in Section 2 apply here to $D_T^{L^p}$.

From (3.2) and the fact that

$$X^{\mathcal{H}} = 2 \sum_{i=1}^{\dim G} \mu_i V_i \tag{3.4}$$

(cf. [16, (1.19)]), it is clear that at any $x \in M$, $X_G^{\mathcal{H}}$ lies in $T_x(Gx)$. Moreover, by (1.1) and (3.4), one verifies that for any $s \in \Omega^{0,*}(M, L^p)^G$,

$$\nabla_{X_G^{\mathcal{H}}}^{\Lambda^{0,*}(T^*M)\otimes L^p} s = \big(A \otimes \mathrm{Id}_{L^p} + 4p\pi\sqrt{-1}\mathcal{H}_G(x)\big)s, \tag{3.5}$$

where

$$\mathcal{H}_G = \int_G c\big(g^{-1}x\big)^2 \mathcal{H}\big(g^{-1}x\big)\, dg \tag{3.6}$$

and

$$A = \nabla^{\Lambda^{0,*}(T^*M)}_{X^{\mathcal{H}}_G} - \int_G c\big(g^{-1}x\big)^2 \left(2\sum_{i=1}^{\dim G} \mu_i\big(g^{-1}x\big) L^{\Lambda^{0,*}(T^*M)}_{g^* V_i}\right) dg \tag{3.7}$$

is of order zero and does not involve p.

Let U' be constructed as in Section 2. Let W be any open neighborhood of $\mu^{-1}(0)$ in M. We first show that the following analogue of [16, Theorem 2.1] holds.

Proposition 3.1. *There exists $p_0 \geqslant 1$ such that for any integer $p \geqslant p_0$, there exist $C > 0$, $b > 0$ verifying the following property: for any $T \geqslant 1$ and $s \in \Omega^{0,*}(M, L^p)^G$ with* $\mathrm{Supp}(s) \cap \overline{U'} \subset \overline{U'} \setminus W$, *one has*

$$\big\| P_f D^{L^p}_T (fs)\big\|_0^2 \geqslant C\big(\|fs\|_1^2 + (T-b)\|fs\|_0^2\big). \tag{3.8}$$

Moreover, if $\mathbf{g}^*$ *admits an* Ad_G*-invariant metric, then one can take $p_0 = 1$.*

Proof. One computes first that

$$\big\| P_f D^{L^p}_T (fs)\big\|_0 = \big\| P_f\big(f D^{L^p}_T s + c(df)s\big)\big\|_0 \geqslant \big\| f D^{L^p}_T s\big\|_0 - C_1\|fs\|_0 \tag{3.9}$$

for some constant $C_1 > 0$.

On the other hand, one has

$$\big\| f D^{L^p}_T s\big\|_0 = \big\| D^{L^p}_T (fs) - c(df)s\big\|_0 \geqslant \big\| D^{L^p}_T (fs)\big\|_0 - C_2\|fs\|_0 \tag{3.10}$$

for some constant $C_2 > 0$.

From (3.9) and (3.10), one sees that there exists $C_3 > 0$ such that

$$\big\| P_f D^{L^p}_T (fs)\big\|_0^2 \geqslant \frac{1}{2}\big\| D^{L^p}_T (fs)\big\|_0^2 - C_3\|fs\|_0^2. \tag{3.11}$$

Since fs has compact support, one has

$$\big\| D^{L^p}_T (fs)\big\|_0^2 = \big\langle D^{L^p}_T (fs), D^{L^p}_T (fs)\big\rangle = \big\langle D^{L^p,2}_T (fs), fs\big\rangle. \tag{3.12}$$

Now since s is G-invariant, from [16, (1.26)], (3.3), (3.5) and (3.7), one computes that

$$\begin{aligned} D^{L^p,2}_T(fs) &= D^{L^p,2}(fs) + \frac{\sqrt{-1}T}{2}\sum_{j=1}^{\dim M} c(e_j)c\big(\nabla^{TM}_{e_j} X^{\mathcal{H}}_G\big)(fs) \\ &\quad - \sqrt{-1}T(A \otimes \mathrm{Id}_{L^p})(fs) + 4p\pi T\mathcal{H}_G fs \\ &\quad - \sqrt{-1}T X^{\mathcal{H}}_G(f)s + \frac{T^2}{4}\big|X^{\mathcal{H}}_G\big|^2(fs). \end{aligned} \tag{3.13}$$

Lemma 3.2. *One has* $\mathcal{H}_G^{-1}(0)=\mu^{-1}(0)$.

Proof. By the definition (3.6) of $\mathcal{H}_G$, it is clear that for any $x\in\mu^{-1}(0)$, $\mathcal{H}_G(x)=0$.

Conversely, for any $x\notin\mu^{-1}(0)$, by (2.2), there exists $g\in G$ such that $c(g^{-1}x)\neq 0$. Thus, $c(g^{-1}x)\mu(g^{-1}x)=c(g^{-1}x)g^*\mu(x)\neq 0$, from which and from (3.6) one gets that $\mathcal{H}_G(x)\neq 0$. □

By Lemma 3.2, there exists a constant $\alpha>0$ such that

$$\mathcal{H}_G(x)\geqslant\alpha \tag{3.14}$$

for any $x\in\overline{U'}\setminus W$.

Clearly,

$$\mathrm{Re}\big(\big\langle\sqrt{-1}X_G^{\mathcal{H}}(f)s,\,fs\big\rangle\big)=0. \tag{3.15}$$

On the other hand, since f has compact support in U' and $\mathrm{Supp}(s)\cap\overline{U'}\subset\overline{U'}\setminus W$, it is easy to see that there exists a constant $C_4>0$ such that

$$\begin{aligned}&\mathrm{Re}\left(\left\langle\frac{\sqrt{-1}}{2}\sum_{j=1}^{\dim M}c(e_j)c\big(\nabla_{e_j}^{TM}X_G^{\mathcal{H}}\big)(fs)-\sqrt{-1}(A\otimes\mathrm{Id}_{L^p})(fs),\,fs\right\rangle\right)\\&\quad+4p\pi\langle\mathcal{H}_G fs,\,fs\rangle\geqslant(4p\pi\alpha-C_4)\|fs\|_0^2.\end{aligned} \tag{3.16}$$

From (3.12)–(3.16), one sees that there exist constants $C_5, C_6>0$ such that

$$\big\|D_T^{L^p}(fs)\big\|_0^2=\big\langle D_T^{L^p,2}(fs),\,fs\big\rangle\geqslant C_5\|fs\|_1^2-C_6\|fs\|_0^2+(4p\pi\alpha-C_4)T\|fs\|_0^2. \tag{3.17}$$

Formula (3.8) follows from (3.11) and (3.17) by taking $p_0=\frac{C_4}{\pi\alpha}$ in (3.17).

For the remaining situation where $\mathbf{g}^*$ admits an Ad_G-invariant metric, in this case, both $\mathcal{H}$ and $X^{\mathcal{H}}$ are G-invariant, so we are in an exactly similar situation as in [16, Section 2].

Set

$$F_T^L=D_T^{L,2}+2\sqrt{-1}T\sum_{i=1}^{\dim G}\mu_i L_{V_i} \tag{3.18}$$

as in [16, (1.30)].

One verifies directly in this case, in view of (3.4), that

$$\big\langle D_T^{L,2}(fs),\,fs\big\rangle=\big\langle F_T^L(fs),\,fs\big\rangle-\sqrt{-1}T\big\langle X^{\mathcal{H}}(f)s,\,fs\big\rangle. \tag{3.19}$$

Now since f has compact support in U' and $\mathrm{Supp}(s)\cap\overline{U'}\subset\overline{U'}\setminus W$, by proceeding in exactly the same way as in [16, Section 2], one sees that there exist constants $C_7, C_8>0$ such that for any $T\geqslant 1$,

$$\mathrm{Re}\big(\big\langle F_T^L(fs),\,fs\big\rangle\big)\geqslant C_7\big(\|fs\|_1^2+(T-C_8)\|fs\|_0^2\big). \tag{3.20}$$

From (3.11), (3.12), (3.15), (3.19) and (3.20), one sees that Proposition 3.1 holds for $p_0 = 1$ in the case where $\mathbf{g}^*$ admits an Ad_G-invariant metric.

The proof of Proposition 3.1 is completed. □

Remark 3.3. Note that in the general case where $\mathbf{g}^*$ does not admit an Ad_G-invariant metric, $X_G^{\mathcal{H}}$ might not be the Hamiltonian vector field associated to $\mathcal{H}_G$. This makes it difficult here to get the pointwise estimates like in [16, Proposition 2.2] at zeroes of $X_G^{\mathcal{H}}$, and partially explains why we need to pass to the uniform estimate for $p > 0$ large.

Remark 3.4. Proposition 3.1 allows us to localize the proof of Theorem 1.1 to an arbitrarily small open neighborhood of $\mu^{-1}(0) \cap \overline{U'}$ in $\overline{U'}$, just as in [16] which relies on techniques developed in [1].

Indeed, for any $r > 0$, let W_r denote the G-invariant open neighborhood of $\mu^{-1}(0)$ in M such that $W_r = \{x \in M\colon \mathcal{H}_G(x) < r\}$.

Since $0 \in \mathbf{g}^*$ is a regular value of μ and G acts on $\mu^{-1}(0)$ freely, one sees easily that when $r > 0$ is small enough, G also acts on W_r freely.

Lemma 3.5. *One has that* $0 \in \mathbf{R}$ *is a non-degenerate critical value of* $\mathcal{H}_G : M \to \mathbf{R}$.

Proof. Let N be the normal bundle to $\mu^{-1}(0)$ in M. Let g^N be the G-invariant metric on N induced by the G-invariant orthogonal decomposition

$$TM|_{\mu^{-1}(0)} = T\mu^{-1}(0) \oplus N, \qquad g^{TM|_{\mu^{-1}(0)}} = g^{T\mu^{-1}(0)} \oplus g^N. \tag{3.21}$$

Let $P^{T\mu^{-1}(0)}$, P^N denote the orthogonal projections from $TM|_{\mu^{-1}(0)}$ to $T\mu^{-1}(0)$ and N respectively with respect to (3.21). Let ∇^N be the connection on N defined by $\nabla^N = P^N(\nabla^{TM}|_{\mu^{-1}(0)})$ where ∇^{TM} is the (G-invariant) Levi-Civita connection associated to g^{TM}.

For any $x \in \mu^{-1}(0)$, $Z \in N_x$, we identify Z with $\exp^{T_xM}(Z) \in M$. Since M/G is compact, one verifies easily that when $\varepsilon > 0$ is small enough, the above map induces an identification from $N_\varepsilon = \{Z \in N\colon |Z| < \varepsilon\}$ to its image in M.

For any $y \in \mu^{-1}(0)$ with $c(y) \neq 0$, let U_y be a small enough open neighborhood of y in $\mu^{-1}(0)$ such that $c(y') \geqslant \frac{1}{2}c(y) > 0$ for any $y' \in U_y$, and that there exists $\varepsilon_y > 0$ and $C_y > 0$ such that $\mathcal{H}(y', Z) \geqslant C_y|Z|^2$ for any $Z \in N_{y'}$ with $|y'| \leqslant \varepsilon_y$. Moreover, there is an open neighborhood G_y of e in G such that $c(g^{-1}y') \geqslant \frac{1}{4}c(y)$ and $\mathcal{H}(g^{-1}y', Z) \geqslant \frac{1}{2}C_y|Z|^2$ for any $g \in G_y$ and $y' \in U_y$. The existence of U_y is clear.

For any $x \in \mu^{-1}(0)$, let $h \in G$ be such that $c(h^{-1}x) \neq 0$. Let $U_{h^{-1}x}$ be the open neighborhood of $h^{-1}x$ constructed above. Then $hU_{h^{-1}x}$ is an open neighborhood of x such that for any $x' \in hU_{h^{-1}x}$ and $Z' \in N_{x'}$ with $|Z'| \leqslant \varepsilon_{h^{-1}x}$, one has

$$\mathcal{H}_G(x', Z') = \int_G c(g^{-1}hh^{-1}x')^2 \mathcal{H}(g^{-1}x', g^*Z')\,dg \tag{3.22}$$

$$\geqslant \frac{1}{32} c(h^{-1}x)^2 \operatorname{vol}(G_{h^{-1}x}) C_{h^{-1}x} |Z'|^2. \tag{3.23}$$

By using again the fact that $\overline{U'}$ is compact, one can cover $U' \cap \mu^{-1}(0)$ by finite open subsets of M verifying (3.22), from which one sees that there exist $C > 0$ and $\varepsilon' > 0$ such that for any $x \in U' \cap \mu^{-1}(0)$ and $Z \in N_x$ with $|Z| \leqslant \varepsilon'$, one has

$$\mathcal{H}_G(x, Z) \geqslant C|Z|^2. \tag{3.24}$$

From (3.24) and the G-invariance of $\mathcal{H}_G$, Lemma 3.5 follows. □

From Lemma 3.5, one deduces the following key property.

Lemma 3.6. *There exist $\varepsilon_0 > 0$ and $C > 0$ such that for any $x \in \mu^{-1}(0)$ and $Z \in N_x$ with $|Z| \leqslant \varepsilon_0$, one has*

$$\left|X_G^{\mathcal{H}}(x, Z)\right| \geqslant C|Z|. \tag{3.25}$$

Proof. Recall that for any $z \in M$,

$$X_G^{\mathcal{H}}(z) = \int_G c\left(g^{-1}z\right)^2 X_g^{\mathcal{H}}\, dg. \tag{3.26}$$

From (3.26) and [16, (1.14)], one verifies that

$$(d\mathcal{H}_G)^*(z) = \left(d\int_G c\left(g^{-1}z\right)^2 \mathcal{H}\left(g^{-1}z\right) dg\right)^* \tag{3.27}$$

$$= JX_G^{\mathcal{H}}(z) + 2\int_G \mathcal{H}\left(g^{-1}z\right)c\left(g^{-1}z\right)\left(\left(g^*(dc)\right)(z)\right)^* dg. \tag{3.28}$$

Thus, one has

$$X_G^{\mathcal{H}}(z) = -J(d\mathcal{H}_G)^*(z) + 2J\int_G \mathcal{H}\left(g^{-1}z\right)c\left(g^{-1}z\right)\left(\left(g^*(dc)\right)(z)\right)^* dg. \tag{3.29}$$

From (3.24), one sees that there exists $C' > 0$ such that when $z = (x, Z)$ is close enough to $\mu^{-1}(0)$, one has

$$\left|(d\mathcal{H}_G)^*(z)\right| \geqslant C'|Z|. \tag{3.30}$$

On the other hand, since $\mathcal{H}(\mu^{-1}(0)) = 0$ and $(d\mathcal{H})|_{\mu^{-1}(0)} = 0$, one verifies easily that

$$\frac{\partial}{\partial Z}\left(\int_G \mathcal{H}\left(g^{-1}z\right)c\left(g^{-1}z\right)\left(\left(g^*(dc)\right)(z)\right)^* dg\right)\bigg|_{Z=0} = 0. \tag{3.31}$$

From (3.29)–(3.31), the G-invariance property, as well as the assumption that M/G is compact, one gets (3.25). □

Formula (3.25) is a direct analogue of [16, (3.17)] and [1, Proposition 8.14]. By this and by Proposition 3.1, one sees that one can proceed in exactly the same way as in [1, Sections 8 and 9] and [16, Section 3] to prove that for $p > 0$ verifying Proposition 3.1, when $T > 0$ is large enough, one has

$$\operatorname{ind}\big(P_f D^{L^p}_{+,T}\big) = \operatorname{ind}\big(D^{L^p_G}_{+}\big). \tag{3.32}$$

Indeed, all one need is to modify suitably according to the appearance of the cut-off function f. And it is easy to see that this only causes a modification of adding a compact operator to the Fredholm operators involved and thus does not alter the indices in due course.

From (3.32), the obvious invariance of the independence of $\operatorname{ind}(P_f D^{L^p}_{+,T})$ with respect to T (which follows from the Fredholm properties) and Definition 2.4, one completes the proof of Theorem 1.1.

4. Vanishing properties of cokernels for large p

We take $E = \Lambda^{0,*}((T^*M) \otimes L^p)$ as in Section 2 and Section 3.

Following Remark 2.6, let

$$\tilde{D}^{L^p}_{+} : \mathbf{H}^1_f\big(M, \Lambda^{0,\mathrm{even}}\big(\big(T^*M\big) \otimes L^p\big)\big)^G \to \mathbf{H}^0_f\big(\Lambda^{0,\mathrm{odd}}\big(\big(T^*M\big) \otimes L^p\big)\big)^G$$

be the operator defined by

$$\tilde{D}^{L^p}_{+}(fs) = f D^{L^p}_{+} s. \tag{4.1}$$

From (4.1) and (2.9), one verifies that for any $s \in \Gamma(\Lambda^{0,\mathrm{even}}((T^*M) \otimes L^p))^G$,

$$\tilde{D}^{L^p}_{+}(fs) = P_f D^{L^p}_{+}(fs) - P_f\big(c(df)s\big). \tag{4.2}$$

By (4.2) and Proposition 2.1, one finds

$$\operatorname{ind}\big(\tilde{D}^{L^p}_{+}\big) = \operatorname{ind}\big(P_f D^{L^p}_{+}\big). \tag{4.3}$$

From (2.14) and (4.2), one finds that for any $s' \in \Gamma(\Lambda^{0,\mathrm{odd}}((T^*M) \otimes L^p))^G$,

$$\begin{aligned}\big(\tilde{D}^{L^p}_{+}\big)^*\big(fs'\big) &= P_f D^{L^p}_{-}\big(fs'\big) + P_f\big(c(df)s'\big) = f D^{L^p}_{-} s' + 2P_f\big(c(df)s'\big)\\ &= D^{L^p}_{-}\big(fs'\big) - c(df)s' + 2P_f\big(c(df)s'\big).\end{aligned} \tag{4.4}$$

From (2.6) and (4.4), one finds that there exists $C_1 > 0$ such that

$$\big\|\big(\tilde{D}^{L^p}_{+}\big)^*\big(fs'\big)\big\|_0 \geqslant \big\|D^{L^p}_{-}\big(fs'\big)\big\|_0 - C_1\big\|fs'\big\|_0. \tag{4.5}$$

On the other hand, since U' has compact closure and f has compact support in U', by proceeding in exactly the same way as in [11, Section 2], one sees that there exist $C_2, C_3 > 0$ such that for any $s' \in \Gamma(\Lambda^{0,\mathrm{odd}}((T^*M) \otimes L^p))^G$,

$$\big\|D^{L^p}_{-}\big(fs'\big)\big\|_0^2 \geqslant (C_2 p - C_3)\big\|fs'\big\|_0^2. \tag{4.6}$$

From (4.5) and (4.6), one sees that when $p \geqslant \frac{2C_3}{C_2}$, one has

$$\left\|\left(\tilde{D}_+^{L^p}\right)^*\left(fs'\right)\right\|_0 \geqslant \left(\sqrt{\frac{C_2 p}{2}} - C_1\right)\left\|fs'\right\|_0. \tag{4.7}$$

From (4.7), one sees that when $p \geqslant \max\{\frac{8C_1^2}{C_2}, \frac{2C_3}{C_2}\}$, one has

$$\left\|\left(\tilde{D}_+^{L^p}\right)^*\left(fs'\right)\right\|_0 \geqslant \frac{1}{2}\sqrt{\frac{C_2 p}{2}}\left\|fs'\right\|_0, \tag{4.8}$$

from which one gets

$$\ker\left(\tilde{D}_+^{L^p}\right)^* = 0. \tag{4.9}$$

From (4.1) and (4.9), one deduces the following result.

Theorem 4.1. *There exists $p_0 \geqslant 0$ such that for any integer $p \geqslant p_0$,*

$$\operatorname{ind}\left(\tilde{D}_+^{L^p}\right) = \dim\left(\operatorname{Ker} D_+^{L^p}\right)^G. \tag{4.10}$$

Proof of Theorem 1.3. By the vanishing theorem due to Borthwick and Uribe [2], Braverman [4] and Ma and Marinescu [11], one sees that when $p > 0$ is large enough,

$$\operatorname{ind}\left(D_+^{L_G^p}\right) = \dim\left(\operatorname{Ker} D_+^{L_G^p}\right). \tag{4.11}$$

From Theorem 1.1, Definition 2.4, (4.3), (4.10) and (4.11), one sees that the first equality in (1.3) holds when $p > 0$ is large enough.

By using (4.6), one can proceed as in the proof of (4.8) to see that when $p > 0$ large enough, the second equality in (1.3) also holds.

The proof of Theorem 1.3 is completed. □

Remark 4.2. Formula (4.9) and the second equality in (1.3) might be regarded as extensions of the vanishing theorem of the half kernel of Spin^c-Dirac operators due to Borthwick and Uribe [2], Braverman [4] and Ma and Marinescu [11] to the non-compact case.

5. Examples and applications

The main source of examples can be found in the papers of Hochs and Landsman [9] and Hochs [7], [8] and [19]. In some of their examples, zero is not in the image of the moment map. In this case, by Theorem 1.3, for p sufficiently large, we deduce that the G-invariant kernels of $D_+^{L^p}$ and of $D_-^{L^p}$ both vanish. That is, for p sufficiently large, D^{L^p} is invertible on the G-invariant sections in this case.

Another collection of examples consists of a finitely generated discrete group G acting properly on a locally compact symplectic manifold (M,ω) such that M/G is compact. In this case, the moment map is trivial, so that the symplectic reduction of M is just M/G, which is generally only an orbifold. In the special case when G acts freely and properly discontinuously on M, so that M/G is a manifold, then Theorem 1.1 is well known and for instance it can be deduced from a result of Pierrot [14].

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Appendix: A KK-theoretic interpretation of the G-invariant index of Mathai–Zhang

by
Ulrich Bunke

Appendix A. The class $[D]$

We consider a locally compact group G which acts properly on a manifold M with compact quotient. On M we consider a G-invariant Riemannian metric and a G-invariant generalized Dirac operator D which acts on sections of a G-equivariant bundle $F \to M$ equipped with a G-equivariant Dirac bundle structure. The G-C^*-algebra $C_0(M)$ acts on the G-Hilbert space $\mathcal{E} := L^2(M, F)$. The Riemannian manifold M is complete, and therefore by [3] the operator D is essentially self-adjoint on this Hilbert space with domain $C_c(M, F)$. We consider the G-invariant operator $\mathcal{F} := D(D^2 + 1)^{-1/2}$ defined by function calculus applied to the unique self-adjoint extension of D. The pair $(L^2(M, F), \mathcal{F})$ is a G-equivariant Kasparov module (see [5], [1]) over $(C_0(M), \mathbb{C})$ and represents a class $[D] \in KK^G(C_0(M), \mathbb{C})$.

The group

$$K^G(M) := KK^G\big(C_0(M), \mathbb{C}\big)$$

is called the G equivariant K-homology group of M.

Appendix B. Descent and assembly

Let $C^*(G)$ denote the maximal group C^*-algebra. In this appendix we provide an explicit description of the Baum–Connes assembly map

$$\mu : K^G(M) \to K\big(C^*(G)\big).$$

Let us first fix some conventions. If A is a C^*-algebra with an action $\rho : G \to \mathrm{Aut}(A)$, then we define the convolution product on $C_0(G, A)$ by

$$\phi * \psi(h) = \int_G \phi(g)\rho(g)\big[\psi\big(h^{-1}g\big)\big]\, dg,$$

where dg denotes the left-invariant Haar measure on G. The modular character $\delta : G \to \mathbb{R}^*$ is defined by

$$d\big(g^{-1}\big) = \delta(g)dg.$$

The adjoint $*_* : A \to A$ is the anti-involution given by

$$\phi^{*_*}(g) := \rho(g)\big[\phi\big(g^{-1}\big)^*\big],$$

where $a \mapsto a^*$ is the anti-involution of A. The C^*-algebra $C^*(G, A)$ is the maximal cross product of G with A and defined as the closure of the convolution algebra $C_c(G, A)$ with respect to the norm

$$\|\phi\| = \sup_{\kappa} \big\| \kappa(\phi) \big\|,$$

where the supremum is taken over all $*$-representations κ of $C_c(G, A)$. The maximal group C^*-algebra $C^*(G) = C^*(G, \mathbb{C})$ is obtained in the special case where $A := \mathbb{C}$ has the trivial action of G. The other important example for the present note is the C^*-algebra $C_0(M)$ with the action $(g, f) \mapsto g^* f :\overset{!}{=} f \circ g^{-1}$, $g \in G$, $f \in C_0(M)$.

There is the descent homomorphism

$$j^G : K^G(M) \cong KK^G\big(C_0(M), \mathbb{C}\big) \to KK\big(C^*\big(G, C_0(M)\big), C^*(G)\big)$$

introduced in [5, 3.11] (we will give the explicit description in the proof of Lemma E.1). Following [4, Chapter 10] we choose a non-negative cut-off function $c \in C_c^\infty(M)$ such that $\int_G g^* c^2 \, dg \equiv 1$. Then we define the projection $P \in C^*(G, C_0(M))$ by

$$P(g) = c g^* c \delta(g)^{1/2} \in C_c(M). \tag{B.1}$$

Since G acts properly on M we observe that $P \in C_c(G, C_0(M))$. The relations $P^2 = P = P^*$ are straightforward to check,

$$\begin{aligned} P^2(g, m) &= \int_G c\big(h^{-1}m\big) c(m) \delta(h)^{1/2} c\big(g^{-1}hh^{-1}m\big) c\big(h^{-1}m\big) \delta\big(h^{-1}g\big)^{1/2} \, dh \\ &= c\big(g^{-1}m\big) c(m) \delta(g)^{1/2} = P(g, m), \\ P^*(g, m) &= c\big(g^{-1}m\big) c\big(gg^{-1}m\big) \delta\big(g^{-1}\big)^{1/2} \delta(g) = P(g, m). \end{aligned}$$

Let $[P] \in K_0(C^*(G, C_0(M))) \cong KK(\mathbb{C}, C^*(G, C_0(M)))$ be the class induced by P, which is independent of the choice of c, since any two such functions c_0, c_1 can be joined by a path $c_t := \sqrt{t c_0^2 + (1-t) c_1^2}$ which induces a corresponding path of projections.

Definition B.1. The assembly map $\mu : K^G(M) \to K(C^*(G)) \cong KK(\mathbb{C}, C^*(G))$ is defined as the composition

$$KK^G\big(C_0(M), \mathbb{C}\big) \xrightarrow{j^G} KK\big(C^*\big(G, C_0(M)\big), C^*(G)\big) \overset{[P]\otimes_{C^*(G, C_0(M))} \cdots}{\longrightarrow} KK\big(\mathbb{C}, C^*(G)\big).$$

Appendix C. The index

The non-reduced group C^*-algebra of G has the universal property that any unitary representation of G extends to a representation of $C^*(G)$. In particular, the trivial representation of G on $\mathbb{C}$ has an extension $1 : C^*(G) \to \mathbb{C}$. On the level of K-theory it induces a homomorphism $I : K_0(C^*(G)) \to K_0(\mathbb{C}) \cong \mathbb{Z}$.

If we identify

$$K\big(C^*(G)\big) \cong KK\big(\mathbb{C}, C^*(G)\big),$$

then the homomorphism I can be written as a Kasparov product $\cdots \otimes_{C^*(G)} [1]$, where $[1] \in KK(C^*(G), \mathbb{C})$ is represented by the Kasparov module $(\mathbb{C}, 0)$.

Definition C.1. We define index : $K^G(M) \to \mathbb{Z}$ to be the composition

$$K^G(M) \xrightarrow{\mu} K\big(C^*(G)\big) \xrightarrow{I} \mathbb{Z}.$$

Appendix D. A model

Let $\epsilon : G \to \mathbb{C}^*$ be a character. By $L^2_{loc}(M, F)^G_\epsilon$ we denote the space of locally square integrable sections of F which transform under G with character ϵ, i.e. which satisfy

$$g^*\phi = \epsilon(g)\phi$$

for all $g \in G$. Multiplication by c defines a map $c : L^2_{loc}(M, F)^G_\epsilon \to L^2(M, F)$. This map is actually injective and has a closed range $H_\epsilon \subseteq L^2(M, F)$. In order to see this we define the continuous maps $E_\epsilon : L^2(M, F) \to L^2_{loc}(M, F)^G_\epsilon$ and $Q_\epsilon : L^2(M, F) \to H_\epsilon$ by

$$E_\epsilon(\phi) := \int_G \epsilon^{-1}(g) g^*(c\phi)\, dg, \qquad Q_\epsilon(\phi) := cE_\epsilon(\phi).$$

Indeed, we have

$$\begin{aligned} h^* E_\epsilon(\phi)(l) &= \int_G \epsilon^{-1}(g)(c\phi)\big(g^{-1}h^{-1}l\big)\, dg \\ &= \int_G \epsilon^{-1}\big(h^{-1}z\big)(c\phi)\big(z^{-1}l\big)\, dz \\ &= \epsilon(h) E_\epsilon(\phi)(l). \end{aligned}$$

For $\phi \in L^2_{loc}(M, F)^G_\epsilon$ we have $E_\epsilon(c\phi) = \phi$, since

$$\begin{aligned} E_\epsilon(c\phi)(h) &= \int_G \epsilon^{-1}(g) g^* c^2(h) g^*\phi\, dg \\ &= \int_G g^* c^2(h)\phi(h)\, dg \\ &= \phi(h). \end{aligned}$$

This implies injectivity of c, and furthermore $Q_\epsilon(c\phi) = c\phi$. Therefore Q_ϵ is a projection onto H_ϵ which is in fact orthogonal for $\epsilon = \delta^{-1/2}$, the square root of the modular character,

$$\begin{aligned}\langle Q_{\delta^{-1/2}}\phi,\psi\rangle &= \Big\langle c\int_G g^*(c\phi)\delta^{1/2}(g)\,dg,\psi\Big\rangle = \int_G \big\langle g^*(c\phi),c\psi\big\rangle\delta^{1/2}(g)\,dg \\ &= \int_G \big\langle c\phi,\big(g^{-1}\big)^*(c\psi)\big\rangle\delta^{1/2}(g)\,dg \overset{!}{=} \int_G \big\langle c\phi,g^*(c\psi)\big\rangle\delta^{1/2}(g)\,dg \\ &= \Big\langle \phi,c\int_G g^*(c\psi)\delta^{1/2}(g)\,dg\Big\rangle = \langle\phi,Q_{\delta^{-1/2}}\psi\rangle,\end{aligned}$$

where we use for the marked equality that the measure $\delta^{1/2}(g)\,dg$ is invariant with respect to inversion $g\mapsto g^{-1}$.

The map c further induces an injective map of Sobolev spaces

$$c : H^1_{loc}(M,F)^G_\epsilon \to H^1(M,F),$$

and we let $H^1_\epsilon \subseteq H^1(M,F)$ denote the closed image under c.

We define the operator $\tilde{D}_\epsilon : H^1_\epsilon \to H_\epsilon$ by

$$\tilde{D}_\epsilon(cf) = Q_\epsilon D(cf). \tag{D.1}$$

Lemma D.1. *The operator $\tilde{D}_\epsilon : H^1_\epsilon \to H_\epsilon$ is Fredholm.*

Proof. To see this we first choose a G-invariant parametrix R for D which is a G-invariant pseudo-differential operator of order -1 with finite propagation. This operator induces a G-equivariant map $R : L^2_{loc}(M,F)\to H^1_{loc}(M,F)$, and therefore

$$\tilde{R}_\epsilon := cRE_{\epsilon|H_\epsilon} : H_\epsilon \to H^1_\epsilon.$$

We have

$$\tilde{D}_\epsilon\tilde{R}_\epsilon = Q_\epsilon DcRE_\epsilon = Q_\epsilon cE_\epsilon + Q_\epsilon[D,c]RE_\epsilon + Q_\epsilon c(DR-1)E_\epsilon.$$

Using that $Q_\epsilon cE_{\epsilon|H_\epsilon} = 1_{H_\epsilon}$, and that $[D,c]=c(dc)$ and c are compactly supported operators of order zero, and R and $DR-1$ are of order -1 we see that $\tilde{D}_\epsilon\tilde{R}_\epsilon - 1_{H_\epsilon}$ is compact. In a similar manner we show that $\tilde{R}_\epsilon\tilde{D}_\epsilon - 1_{H^1_\epsilon}$ is compact. □

Note that the index of the operator $\tilde{D}_1$ associated to the trivial character is studied in the main text by Mathai–Zhang. On the other hand, we will see in Proposition D.3 that the index of the operator $\tilde{D}_{\delta^{-1/2}}$ is equal to index$([D])$. The following result connects both cases.

Lemma D.2. *We have* $\operatorname{index}(\tilde{D}_1) = \operatorname{index}(\tilde{D}_{\delta^{-1/2}})$.

Proof. The main idea is that $\delta^{1/2}$ can be connected with the trivial character by the continuous path of characters $\epsilon_t := \delta^{-t/2}$, $t\in[0,1]$. Let $I:=[0,1]$ and consider the $C(I)$-Hilbert-modules $C(I,L^2(M,F))$ and $C(I,H^1(M,F))$. The family of operators Q_{ϵ_t} defines projections Q on $C(I,L^2(M,F))$ and $C(I,H^1(M,F))$ with images H and H^1. Furthermore, the family $\tilde{D}_{\epsilon_t}$ induces an operator $\tilde{D}: H^1 \to H^0$ whose parametrix $\tilde{R}$ is given by the family $\tilde{R}_{\epsilon_t}$. These data give a Kasparov module

$$\left(H^1 \oplus H, \begin{pmatrix} 0 & \tilde{R} \\ \tilde{D} & 0 \end{pmatrix}\right)$$

over $C(I)$ which is a homotopy (see [1]) between the Kasparov modules

$$\left(H_1^1 \oplus H_1, \begin{pmatrix} 0 & \tilde{R}_1 \\ \tilde{D}_1 & 0 \end{pmatrix}\right), \qquad \left(H_{\delta-1/2}^1 \oplus H_{\delta-1/2}, \begin{pmatrix} 0 & \tilde{R}_{\delta-1/2} \\ \tilde{D}_{\delta-1/2} & 0 \end{pmatrix}\right)$$

representing $\mathrm{index}(\tilde{D}_1)$ and $\mathrm{index}(D_{\delta-1/2})$. □

We can now formulate the main assertion of this note:

Proposition D.3. *We have* $\mathrm{index}([D]) = \mathrm{index}(\tilde{D}_{\delta-1/2})$.

Appendix E. Proof of Proposition D.3

We first apply j^G to the Kasparov module $(L^2(M,F), \mathcal{F})$ representing $[D]$. Note that by the universal property of the maximal crossed product the compatible G and $C_0(M)$-actions on $L^2(M,F)$ extend to an action of $C^*(G, C_0(M))$.

Lemma E.1. $j^G([D]) \otimes_{C^*(G)} [1] \in KK(C^*(G, C_0(M)), \mathbb{C})$ *is represented by the Kasparov module* $(L^2(M,F), \mathcal{F})$.

Proof. According to [5, 3.11], $j^G([D])$ is represented by $(C^*(G, L^2(M,F)), \tilde{\mathcal{F}})$, where $C^*(G, L^2(M,F))$ is a right $C^*(G)$-module with a left action by $C^*(G, C_0(M))$. It is a closure of the space of compactly supported continuous functions $f : G \to L^2(M,F)$. The operator $\tilde{\mathcal{F}}$ is given by $(\tilde{\mathcal{F}} f)(g) = (\mathcal{F} f)(g)$. The $C^*(G)$-valued scalar product is given by

$$\langle f_1, f_2\rangle(g) = \int_D \big\langle f_1(h), f_2(hg)\big\rangle \, dg.$$

Furthermore, the left action of $C^*(G, C_0(M))$ is given by

$$(\phi f)(g) = \int_G \phi(h)\big(h^* f\big)(g)\, dh.$$

Note that $C^*(G, L^2(M,F)) \otimes_{C^*(G)} \mathbb{C} \cong L^2(M,F)$ via

$$f \otimes v \mapsto \int_G f(g)\, dg v.$$

Therefore $j^G([D]) \otimes_{C^*(G)} [1]$ is represented by the Kasparov module $(L^2(M,F), \mathcal{F})$, where the left action of $C^*(G, C_0(M))$ on $L^2(M,F)$ is given by

$$(\phi f) = \int_G \phi(h) h^* f \, dh. \qquad \Box \tag{E.1}$$

We now compute $[P] \otimes_{C^*(G,C_0(M))} (j^G([D]) \otimes_{C^*(G)} [1])$. We represent $[P]$ by the Kasparov module $(PC^*(G, C_0(M)), 0)$. We must understand $PC^*(G, C_0(M)) \otimes_{C_0(M)} L^2(M,F)$. The operator $Q_{\delta-1/2} := cE_{\delta-1/2}$ is the orthogonal projection $L^2(M,F) \to H_{\delta-1/2}$. If we combine the formula (E.1) for the action of $C^*(G, C_0(M))$ on $L^2(M,F)$ with the definition (B.1) of P we see that the projection P acts as $Q_{\delta-1/2}$, and

$$\begin{aligned} PC^*\big(G, C_0(M)\big) \otimes_{C^*(G,C_0(M))} L^2(M,F) &= PL^2(M,F) \\ &= Q_{\delta-1/2} L^2(M,F) = H_{\delta-1/2}. \end{aligned}$$

Let $L : H_{\delta-1/2} \to L^2(M,F)$ denote the inclusion. Since $Q_{\delta-1/2}$ is orthogonal, the operator (D.1) can be written in the form

$$\tilde{D} = L^* D L$$

which makes clear that it is self-adjoint. We form $\tilde{\mathcal{F}} := \tilde{D}_{\delta-1/2}(1 + \tilde{D}^*_{\delta-1/2}\tilde{D}_{\delta-1/2})^{-1/2}$. The Kasparov module $(H, \tilde{\mathcal{F}})$ over $(\mathbb{C}, \mathbb{C})$ represents $\mathrm{index}(\tilde{D}_{\delta-1/2}) \in \mathbb{Z}$ under $KK^0(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$. The assertion of Proposition D.3 immediately follows from

Lemma E.2. $[P] \otimes_{C^*(G,C_0(M))} (j^G([D]) \otimes_{C^*(G)} [1])$ *is represented by the Kasparov module* $(H, \tilde{\mathcal{F}})$.

Proof. In order to show the claim we employ the characterization of the Kasparov product in terms of connections (see [5, 2.10]). In our situation we have only to show that $\tilde{\mathcal{F}}$ is an $\mathcal{F}$-connection.

For Hilbert-C^*-modules X, Y over some C^*-algebra A let $L(X,Y)$ and $K(X,Y)$ denote the spaces of bounded and compact adjoinable A-linear operators (see [1] for definitions). For $\xi \in PC^*(G, C_0(M))$ we define $\theta_\xi \in L(L^2(M,F), H)$ by $\theta_\xi(f) = \xi f$. Since $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are self-adjoint, we only must show that

$$\theta_\xi \circ \mathcal{F} - \tilde{\mathcal{F}} \circ \theta_\xi \in K\big(L^2(M,F), H\big).$$

We have $\xi\mathcal{F} - \tilde{\mathcal{F}}\xi = [\xi, \mathcal{F}] + (\mathcal{F} - \tilde{\mathcal{F}})P\xi$. Since $[\xi, \mathcal{F}]$ is compact it suffices to show that $(\mathcal{F} - \tilde{\mathcal{F}})P$ is compact. We consider $\overline{D} := (1-P)D(1-P) + L\tilde{D}L^*$. Then by a simple calculation we have $\overline{D} = D + R$, where R is a bounded operator. Let $\overline{\mathcal{F}} := \overline{\mathcal{D}}(1 + \overline{\mathcal{D}}^2)^{-1/2}$. Then $(\mathcal{F} - L\tilde{\mathcal{F}}L^*)P = (\mathcal{F} - \overline{\mathcal{F}})P$. Let $\tilde{c} \in C_c^\infty(M)$ be such that $c\tilde{c} = c$. Then we have $(\hat{\mathcal{F}} - \overline{\mathcal{F}})P = (\mathcal{F} - \overline{\mathcal{F}})\tilde{c}P$. Therefore it suffices to show that $(\mathcal{F} - \overline{\mathcal{F}})\tilde{c}$ is compact. This can be done using the integral representations for $\mathcal{F}$ and $\overline{\mathcal{F}}$ as in [2]. $\Box$

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