A note on the Lichnerowicz vanishing theorem for proper actions

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Abstract. We prove a Lichnerowicz type vanishing theorem for non-compact spin manifolds admitting proper cocompact actions. This extends a previous result of Ziran Liu who proves it for the case where the acting group is unimodular.

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1. Introduction

A classical theorem of Lichnerowicz [3] states that if an even dimensional closed smooth spin manifold admits a Riemannian metric of positive scalar curvature, then the index of the associated Dirac operator vanishes. In this note we prove an extension of this vanishing theorem to the case where a (possibly non-compact) spin manifold $M$ admitting a proper cocompact action by a locally compact group $G$.

To be more precise, recall that for such an action, a so called $G$-invariant index has been defined by Mathai–Zhang in [5]. Thus it is natural to ask whether this index vanishes if $M$ carries a $G$-invariant Riemannian metric of positive scalar curvature. Such a result has indeed been proved by Liu in [4] for the case of unimodular $G$. In this note we extend Liu’s result to the case of general $G$.

We will recall the definition of the Mathai–Zhang index [5] and state the main result as Theorem 2.2 in Section 2; and then prove Theorem 2.2 in Section 3.

2. A vanishing theorem for the Mathai–Zhang index

Let $M$ be an even dimensional spin manifold. Let $G$ be a locally compact group which acts on $M$ properly and cocompactly, where by proper we mean that the map

$$G \times M \rightarrow M \times M, \quad (g, x) \mapsto (x, gx),$$


is proper (the pre-image of a compact subset is compact), while by cocompact we mean that the quotient \( M/G \) is compact. We also assume that \( G \) preserves the spin structure on \( M \).

Given a \( G \)-invariant Riemannian metric \( g^{TM} \) (cf. [5, (2.3)]), we can construct canonically a \( G \)-equivariant Dirac operator \( D : \Gamma(S(TM)) \to \Gamma(S(TM)) \) (cf. [2] and [5]), acting on the Hermitian spinor bundle \( S(TM) = S_+(TM) \oplus S_-(TM) \). Let \( D_\pm : \Gamma(S_\pm(TM)) \to \Gamma(S_\mp(TM)) \) be the obvious restrictions.

Let \( || \cdot ||_0 \) be the standard \( L^2 \)-norm on \( \Gamma(S(TM)) \), let \( || \cdot ||_1 \) be a (fixed) \( G \)-invariant Sobolev 1-norm. Let \( \mathcal{H}^0(M, S(TM)) \) be the completion of \( \Gamma(S(TM)) \) under \( || \cdot ||_0 \).

Recall that by the compactness of \( M/G \), there exists a compact subset \( Y \) of \( M \) such that \( G.Y = M \) (cf. [6, Lemma 2.3]). Let \( U, U' \) be two open subsets of \( M \) such that \( Y \subseteq U \) and that the closures \( \overline{U} \) and \( \overline{U'} \) are both compact in \( M \), and that \( \overline{U} \subseteq U' \). Following [5], let \( f \in C^\infty(M) \) be a nonnegative function such that \( f|_U = 1 \) and \( \text{Supp}(f) \subseteq U' \).

Let \( \mathcal{H}^0_\gamma(M, S(TM))^G \) and \( \mathcal{H}^1_\gamma(M, S(TM))^G \) be the completions of

\[ \{ fs : s \in \Gamma(S(TM))^G \} \]

under \( || \cdot ||_0 \) and \( || \cdot ||_1 \) respectively. Let \( P_f \) denote the orthogonal projection from \( \mathcal{H}^0_\gamma(M, S(TM))^G \) to \( \mathcal{H}^1_\gamma(M, S(TM))^G \). Clearly, \( P_f D \) maps \( \mathcal{H}^1_\gamma(M, S(TM))^G \) into \( \mathcal{H}^0_\gamma(M, S(TM))^G \).

We recall a basic result from [5, Proposition 2.1].

**Proposition 2.1.** The operator \( P_f D : \mathcal{H}^1_\gamma(M, S(TM))^G \to \mathcal{H}^0_\gamma(M, S(TM))^G \) is a Fredholm operator.

It has been shown in [5] that \( \text{ind}(P_f D_+) \) is independent of the choice of the cut-off function \( f \), as well as the \( G \)-invariant metric involved. Following [5, Definition 2.4], we denote \( \text{ind}(P_f D_+) \) by \( \text{ind}_G(D_+) \).

The main result of this note can be stated as follows.

**Theorem 2.2.** If there is a \( G \)-invariant metric \( g^{TM} \) on \( TM \) such that its scalar curvature \( k^{TM} \) is positive over \( M \), then \( \text{ind}_G(D_+) = 0 \).

**Remark 2.3.** If \( G \) is unimodular, then Theorem 2.2 has been proved in [4]. Our proof of Theorem 2.2 combines the method in [4] with a simple observation that in order to prove the vanishing of the index, one need not restrict to self-adjoint operators.

3. Proof of Theorem 2.2

Following [5, (2.16)], let \( \widetilde{D}_{f,\pm} : \mathcal{H}^1_\gamma(M, S_\pm(TM))^G \to \mathcal{H}^0_\gamma(M, S_\mp(TM))^G \) be defined by that for any \( s \in \Gamma(S_\pm(TM))^G \),

\[ \widetilde{D}_{f,\pm}(fs) = f D_{\pm} s. \]
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Since one verifies easily that (cf. [5, (4.2)])

$$\widetilde{D}_{f,\pm}(fs) - P_f D_{\pm}(fs) = -P_f (c(df)s),$$

one sees that $\widetilde{D}_{f,\pm}$ is a compact perturbation of $P_f D_{\pm}$. Thus, one has

$$\text{ind} (\widetilde{D}_{f,\pm}) = \text{ind} (P_f D_+).$$

(3.3)

Now by (3.1), if $fs \in \ker (\widetilde{D}_{f,\pm})$, then $s \in \ker (D_+)$. Thus, by the standard Lichnerowicz formula [3], one has (cf. [1, p. 112] and [4, (3.6)])

$$\frac{1}{2} \Delta (|s|^2) = \left| \nabla^{S_+_{TM}}|s|^2 \right|^2 + \frac{k_{TM}}{4}|s|^2 \geq \frac{k_{TM}}{4}|s|^2,$$

(3.4)

where $\Delta$ is the negative Laplace operator on $M$ and $\nabla^{S_+_{TM}}$ is the canonical Hermitian connection on $S_+_{TM}$ induced by $g_{TM}$.

As has been observed in [4], since the $G$-action on $M$ is cocompact and $|s|$ is clearly $G$-invariant, there exists $x \in M$ such that

$$|s(x)| = \max \{|s(y)| : y \in M\}.$$

(3.5)

By the standard maximum principle, one has at $x$ that

$$\Delta (|s|^2) \leq 0.$$

(3.6)

Combining (3.6) with (3.4), one sees that if $k_{TM} > 0$ over $M$, one has

$$s(x) = 0,$$

(3.7)

which implies that $s \equiv 0$ on $M$. Thus, one has $\ker (\widetilde{D}_{f,\pm}) = \{0\}$, and, consequently,

$$\text{ind} (\widetilde{D}_{f,\pm}) \leq 0.$$

(3.8)

On the other hand, for any $s, s' \in \Gamma(S(TM))$, one verifies that

$$\{fDs, fs'\} = \{s, D\{(f^2s')\} = \{fs, D\{(fs') + c(df)s'\}.$$

(3.9)

Let $\widetilde{D}_{f,\pm} : H^1_f(M, S_{\pm(TM)})^G \rightarrow H^0_f(M, S_{\mp(TM)})^G$ be defined by that for any $s \in \Gamma(S_{\pm(TM)})^G$,

$$\widetilde{D}_{f,\pm}(fs) = P_f (D_{\pm}(fs) + c(df)s).$$

(3.10)

Clearly, $\widetilde{D}_{f,\pm}$ is a compact perturbation of $P_f D_+$. Thus one has

$$\text{ind}(\widetilde{D}_{f,\pm}) = \text{ind}(P_f D_+).$$

(3.11)
Now by (3.9), one sees that the formal adjoint of $\tilde{D}_{f,+}$ is $\tilde{D}_{f,-}$, while by proceeding as in (3.4)–(3.7), one finds that $\ker(\tilde{D}_{f,-}) = \{0\}$. Thus, one has

$$\text{ind} \left( \tilde{D}_{f,+} \right) \geq 0. \quad \text{(3.12)}$$

From (3.3), (3.8), (3.11) and (3.12), one gets $\text{ind} \left( P_f D_+ \right) = 0$, which completes the proof of Theorem 2.2.

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**References**


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