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A note on the Kervaire semi-characteristic

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We present a potential generalization of the Kervarie semi-characteristic (with real coefficient) to the case of non-orientable manifolds.

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1. The Kervaire Semi-Characteristic on Oriented Manifolds

Let M be a closed oriented smooth manifold of dimension $4q+1$. Following [1], we define the Kervaire semi-characteristic (with real coefficient) $k(M) \in \mathbf{Z}_2$ of M by

$$k(M) = \sum_{i=0}^{q} \dim H^{2i}(M, \mathbf{R}) \mod 2\mathbf{Z}. \tag{1.1}$$

Recall that by a classical theorem of Hopf, there always exists a nowhere vanishing vector field on M, as $\chi(M) = 0$, where $\chi(M)$ denotes the Euler characteristic of M.

Atiyah proved in [1] the following vanishing theorem for $k(M)$.

Theorem 1.1 (Atiyah 1970). *If there exists a pair of two linearly independent vector fields over M, then $k(M) = 0$.*

Atiyah made use of the mod 2 index of Atiyah-Singer [3] to prove Theorem 1.1.

Atiyah and Dupont then studied in [2] the case where the two vector fields can have a finite number of singularities on which the two vector fields in question become linearly dependent. We recall that this happens when the $4q$th Stiefel–Whitney class $w_{4q}(TM)$ of M vanishes.

In [9], we proved a generic counting formula for $k(M)$ on any oriented $4q+1$ manifolds, which we briefly recall as follows.

Let $V \in \Gamma(TM)$ be a nowhere vanishing vector field over M. Let $[V]$ denote the line bundle generated by V. We consider the quotient bundle $TM/[V]$, which is a vector bundle of rank $4q$ over M.

Take any transversal section X of $TM/[V]$, then the zero set of X, denoted by $\mathrm{zero}(X)$, consists of a disjoint union of circles in M.

Let $TM/[V]$ be quipped with a Euclidean metric. Take one circle F in $\mathrm{zero}(X)$. At any point $y \in F$, the transversal section X induces an automorphism of $T_yM/[V_y]$, which is the restriction of $TM/[V]$ at y. Then by a linear result due to Novikov (cf. [10, Lemma 4.8]), one can determine a one-dimensional subspace in $\Lambda^*((T_yM/[V_y])^*)$. Moreover, these linear subspaces form a real line bundle $o_F(X)$ over F, not depending on the choice of the given metric on $TM/[V]$.

We define a mod 2 index, denoted by $\mathrm{ind}_2(X, F)$, on F by

$$\mathrm{ind}_2(X, F) = 1 \quad \text{if } o_F(X) \text{ is orientable over } F \tag{1.2}$$

and

$$\mathrm{ind}_2(X, F) = 0 \quad \text{if } o_F(X) \text{ is non-orientable over } F. \tag{1.3}$$

We can now state the main result of [9] as follows.

Theorem 1.2 (Zhang 2000). *The following identity in* $\mathbf{Z}_2$ *holds*

$$k(M) = \sum_{F \in \mathrm{zero}(X)} \mathrm{ind}_2(X, F). \tag{1.4}$$

In particular, when $\mathrm{zero}(X) = \emptyset$, *one recovers Theorem* 1.1.

The proof in [9] of Theorem 1.2, which is purely analytic, makes use of a Witten type deformation (which will be recalled in the next section), besides the mod 2 index of Atiyah-Singer [3].

Remark 1.3. There is an alternate proof of Theorem 1.2 due to Tang [5], which makes use of Theorem 1.1, but not the Witten type deformation in [9].

2. An Extension to the Case of Non-Orientable Manifolds

Let now M be a $4q+1$-dimensional closed non-orientable smooth manifold. Since $\chi(M) = 0$, by the Hopf theorem there exist nowhere vanishing vector fields on M. Indeed, the set of homotopy classes of nowhere vanishing vector fields is called the set of Euler structures on M (cf. [6]).

Let V be a nowhere vanishing vector field on M. We give M a Riemannian metric g^{TM} such that $|V| = 1$. Let $V^* \in \Gamma(T^*M)$ denote the dual of V. Let $\widehat{c}(V) = V^* \wedge + i_V$ denote the Clifford element acting on $\Omega^*(M) = \Gamma(\Lambda^*(T^*M))$ (cf. [10]).

Let d denote the exterior differential operator and d^* be the formal adjoint of d determined by g^{TM} (cf. [10]).

Following [8, Definition 2.1], let D_V be the operator acting on $\Omega^{\mathrm{even}}(M)$ defined by

$$D_V = \widehat{c}(V)(d + d^*) - (d + d^*)\widehat{c}(V) : \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{even}}(M). \tag{2.1}$$

As is observed in [8], D_V is a real formally skew-adjoint elliptic differential operator, thus determines a mod 2 index in the sense of Atiyah-Singer [3] which we denote by $\alpha(V) \in \mathbf{Z}_2$ (cf. [8, (2.6)]), emphasizing that it does not depend on the choice of the metric g^{TM}.

By [8, Theorem 2.5], when M is oriented, one has $\alpha(V) = k(M)$. However, if M is non-orientable, then the proof of [8, Theorem 2.5] does not go through. Thus, in the case where M is non-orientable, what we get is a map from the set of Euler structures to $\mathbf{Z}_2$:

$$\alpha : \{\text{Euler structures}\} \to \mathbf{Z}_2, \tag{2.2}$$

which maps a class $\{V\}$ with representiative V to $\alpha(V)$.

Thus, it remains an interesting question that whether the map in (2.2) is non-constant.

On the other hand, the method in [9] still applies to give a similar generic counting formula for $\alpha(V)$.

Indeed, take any transversal section X of $TM/[V]$, one can proceed as in Sec. 1 to defined $\mathrm{ind}_2(X, F)$ for any $F \in \mathrm{zero}(X)$.

The following result maybe viewed as an analogue in the non-orientable case of Theorem 1.2.

Theorem 2.1. *The following identity in* $\mathbf{Z}_2$ *holds*

$$\alpha(V) = \sum_{F \in \mathrm{zero}(X)} \mathrm{ind}_2(X, F). \tag{2.3}$$

In particular, if X *has no zero, then* $\alpha(V) = 0$, *which extends Theorem* 1.1.

Proof. Let $TM = [V] \oplus [V]^\perp$ be the orthogonal decomposition of TM under g^{TM}. Without loss of generality we assume that $X \in \Gamma([V]^\perp)$.

Following [9, Definition 2.1], which is inspired by [7], for any $T \in \mathbf{R}$, we define the Witten type deformation of D_V by

$$D_{V,T} = D_V + T\widehat{c}(V)\widehat{c}(X) : \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{even}}(M). \tag{2.4}$$

Then $D_{V,T}$ is a real skew-adjoint operator whose mod 2 index equals to $\alpha(V)$.

We can then proceed as in [9], by taking $T > 0$ sufficiently large and applying the analytic localization techniques developed in [4] to complete the proof of (2.3). Details are easy to fill and are left to the interested reader. $\square$

Combining (2.3) with the arguments in [10, Sec. 7.5], we get the following formula computing $k(M)$ for certain oriented $4q + 1$-dimensional manifolds, which is of independent interest.

Corollary 2.2. *Let* $\widehat{M}$ *be a closed oriented manifold of dimension* $4q + 1$ *such that* $\widehat{M}$ *is the orientation double cover of a non-orientable manifold* M. *Then the* *following identity holds*

$$k(\widehat{M}) = \langle w_1(TM)w_{4q}(TM), [M]\rangle. \tag{2.5}$$

Remark 2.3. It is natural to ask whether Tang's method in [5] applies to give a proof of (2.3) without using the Witten type deformation (2.4). The more interesting question, we recall, is that whether $\alpha(V)$, which now has a more topological form given by the right-hand side of (2.3), depends on the homotopy class of V.

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