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Symplectic Reduction and Family Quantization

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§1 Introduction and the statement of the main results

The purpose of this paper is to generalize the Guillemin-Sternberg geometric quantization conjecture [GS], which has been proved in various generalities in [DGMW], [G], [GS], [JK], [M1], [M2], [V] to the case of families.

To be more precise, let $Z \to M \xrightarrow{\pi} B$ be a smooth fibration with connected closed fibers and base, and let TZ be the associated vertical tangent bundle. For any $b \in B$, we denote by $i_b : Z_b = \pi^{-1}(b) \hookrightarrow M$ the canonical embedding. We make the basic assumption that there exists a smooth 2-form $\omega \in \Gamma(\wedge^2(T^*Z))$ such that for any $b \in B$, $\omega_b = i_b^*\omega$ is a symplectic form over Z_b. Then what we have is a smooth fibration of symplectic manifolds.

Let J be an almost complex structure on TZ such that

$$g(u, v) = \omega(u, Jv), \quad u, v \in \Gamma(TZ) \tag{1.1}$$

defines a smooth Euclidean metric on TZ. We assume the existence of J. Let $TZ \otimes \mathbf{C} = T^{(1,0)}Z \oplus T^{(0,1)}Z$ be the canonical splitting of the complexification of TZ with respect to J.

Let E be a Hermitian vector bundle over M carrying a Hermitian connection ∇^E. Then for any $b \in B$, $\nabla^{E_b} = i_b^*\nabla^E$ is a Hermitian connection on $E_b = i_b^*E$.

For any $b \in B$, one can construct from the geometric objects ω_b, $J_b = J|_{TZ_b}$, and (E_b, ∇^{E_b}) the Spin^c-Dirac operator

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$$\begin{aligned} D_+^{E_b}: \Omega^{0,\mathrm{even}}(Z_b, E_b) &= \Gamma\left(\wedge^{\mathrm{even}}\left(T^{(0,1)*}Z_b \otimes E_b\right)\right) \\ &\longrightarrow \Omega^{0,\mathrm{odd}}(Z_b, E_b) = \Gamma\left(\wedge^{\mathrm{odd}}\left(T^{(0,1)*}Z_b \otimes E_b\right)\right) \end{aligned} \tag{1.2}$$

in a standard way (see [TZ1, Sect. 1(a)]). Then $D^E_{M/B,+} = \{D_+^{E_b}\}_{b\in B}$ forms a smooth family of Spin^c-Dirac operators parametrized by B.

Now let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. We make the assumption that the total manifold M admits a smooth G-action such that G preserves every Z_b, $b \in B$, and that there is a smooth map $\mu : M \to \mathfrak{g}^*$ such that the G action on each Z_b, $b \in B$, is Hamiltonian with the moment map given by $\mu_b = \mu|_{Z_b} \to \mathfrak{g}^*$. Furthermore, we assume that $0 \in \mathfrak{g}^*$ is a regular value of μ. For simplicity, we also assume that the G-action on $\mu^{-1}(0)$ is free. Then on each fiber Z_b, one can construct in the usual way the Marsden-Weinstein reduction $(Z_{G,b} = \mu_b^{-1}(0)/G, \omega_{G,b})$ (see [GS] and [TZ1, Sect. 3(a)]). These reduction spaces together form a smooth fibration of symplectic manifolds

$$(Z_G, \omega_G) \longrightarrow M_G = \frac{\mu^{-1}(0)}{G} \xrightarrow{\pi_G} B. \tag{1.3}$$

Also, we assume that the G action preserves the almost complex structure J and the Hermitian vector bundle (E, ∇^E), which in turn induce canonically the almost complex structure J_G on TZ_G and the Hermitian vector bundle (E_G, ∇^{E_G}) over M_G, respectively. In particular, one can construct as in (1.1) a smooth family of Spin^c-Dirac operators $D^{E_G}_{M_G/B,+} = \{D_+^{E_{G,b}}\}_{b\in B}$ with each $D_+^{E_{G,b}}$ acting on

$$D_+^{E_{G,b}} : \Omega^{0,\mathrm{even}}(Z_{G,b}, E_{G,b}) \longrightarrow \Omega^{0,\mathrm{odd}}(Z_{G,b}, E_{G,b}). \tag{1.4}$$

Now let $\mathfrak{g}$ (and thus $\mathfrak{g}^*$ also) be equipped with an $\mathrm{Ad}\,G$-invariant metric. Let $\mathcal{H} = |\mu|^2$ be the norm square of μ. Let $X^{\mathcal{H}} \in \Gamma(TZ)$ be such that for any $b \in B$, $X_b^{\mathcal{H}} = X^{\mathcal{H}}|_{Z_b}$ is the Hamiltonian vector field associated to $\mathcal{H}_b = \mathcal{H}|_{Z_b}$.

Set $B(X^{\mathcal{H}}) = \{x \in M : X^{\mathcal{H}}(x) = 0\}$.

Let h_i, $1 \le i \le \dim G$, be an orthonormal base of $\mathfrak{g}^*$. Let V_i, $1 \le i \le \dim G$, be the dual base of h_i, $1 \le i \le \dim G$. Then we can write μ as

$$\mu = \sum_{i=1}^{\dim G} \mu_i h_i, \tag{1.5}$$

with each μ_i a real function on M.

For any $V \in \mathfrak{g}$, set[1]

$$r_V^E = L_V^E - \nabla_V^E, \tag{1.6}$$

[1]We use the same notation V to denote the vector field it generates on M.

where L_V^E denotes the infinitesimal action of V on E.

Assumption 1.1 (see [TZ1, (4.4)]). For any $x \in B(X^{\mathcal{H}})$, one has

$$\sqrt{-1} \sum_{i=1}^{\dim G} \mu_i(x) r_{V_i}^E(x) \geq 0. \tag{1.7}$$

Now for any $b \in B$, denote by $\Omega^{0,*}(Z_b, E_b)^G$ the G-invariant subspace of $\Omega^{0,*}(Z_b, E_b)$. Then, since G preserves everything, the restriction of the Spin^c-Dirac operator in (1.1) to $\Omega^{0,*}(Z_b, E_b)^G$,

$$D_{G,+}^{E_b} : \Omega^{0,\mathrm{even}}(Z_b, E_b)^G \longrightarrow \Omega^{0,\mathrm{odd}}(Z_b, E_b)^G, \tag{1.8}$$

is well defined. Furthermore, one verifies easily that the smooth family of operators $D_{M/B,G,+}^E = \{D_{G,+}^{E_b}\}_{b \in B}$ admits a well-defined index bundle

$$\mathrm{ind}\, D_{M/B,G,+}^E \in K(B) \tag{1.9}$$

in the sense of Atiyah and Singer (see [AS]).

Similarly, the smooth family of operators $D_{M_G/B,+}^{E_G} = \{D_+^{E_{G,b}}\}_{b \in B}$ has a well-defined index bundle

$$\mathrm{ind}\, D_{M_G/B,+}^{E_G} \in K(B). \tag{1.10}$$

We can now state the main result of this paper as follows.

Theorem 1.2. If $\mu^{-1}(0) \neq \emptyset$, and Assumption 1.1 holds, then one has the following identity in $K(B)$:

$$\mathrm{ind}\, D_{M/B,G,+}^E = \mathrm{ind}\, D_{M_G/B,+}^{E_G}. \tag{1.11}$$

□

When $B = \{\mathrm{pt.}\}$, Theorem 1.2 was proved in [TZ1, Theorem 4.2] as an extension of the Guillemin-Sternberg geometric quantization conjecture (see [GS]).[2] Thus, Theorem 1.2 may be thought of as a generalization to families of the Guillemin-Sternberg conjecture.

We combine the analytic approach to the Guillemin-Sternberg conjecture developed in [TZ1] with the homotopy invariance property of the index bundle (see [AS]) to

[2]The original Guillemin-Sternberg conjecture was stated for the case where E is the prequantum line bundle over the underlying symplectic manifold. It has been proved in various generalities in [DGMW], [G], [GS], [JK], [M1], [M2], [V], with the most general result first obtained by Meinrenken (see [M2]).

prove Theorem 1.2. The proof turns out to be surprisingly simpler than what has been expected.

An important class of examples of fibrations of symplectic manifolds is the class of symplectic fibrations studied in [GLS]. Furthermore, when $Z \to M \xrightarrow{\pi} B$ is also a holomorphic fibration, one can refine (1.11) to an identity between sheafs of direct images, extending the holomorphic quantization formulas in [Te] and [Z] to the family case.

The rest of this paper is organized as follows. In Section 2, we present our proof of Theorem 1.2 and describe some of its immediate consequences. In Section 3, we discuss the above-mentioned refinement in the holomorphic case. There is also an Appendix in which we prove a family extension of the rigidity theorem of the canonical Spin^c-Dirac operators on symplectic manifolds, which plays a role in Section 2(c) for one of the applications of Theorem 1.2.

§2 Proof of Theorem 1.2

This section is organized as follows. In (a), we recall the analytic arguments in [TZ1], which now apply to the fiberwise Spin^c-Dirac operators in our situation. In (b), we prove Theorem 1.2. Section (c) contains some immediate consequences of Theorem 1.2.

We make the same assumption and use the same notation as in Section 1.

(a) Deformations of the fiberwise Dirac operators and the associated estimates

Let $b \in B$ be fixed temporarily.

Following [TZ1, Definition 1.2], for any $T \in \mathbf{R}$, let $D^{E_b}_{G,+,T}$ be the deformation of $D^{E_b}_{G,+}$ given by

$$D^{E_b}_{G,+,T} = D^{E_b}_{G,+} + \frac{\sqrt{-1}T}{2}\, c\left(X^{\mathcal{H}}_b\right) : \Omega^{0,\mathrm{even}}(Z_b, E_b)^G \longrightarrow \Omega^{0,\mathrm{odd}}(Z_b, E_b)^G, \tag{2.1}$$

where $c(X^{\mathcal{H}}_b)$ is the notation for the Clifford action of $X^{\mathcal{H}}_b$ (see [TZ1, Sect. 1(a)]).

We now describe the analytic arguments in [TZ1], which rely heavily on [BL, Sects. 8, 9], in a little more detailed version than what is seen in [TZ1].

As in [BL], we first introduce some notation of Sobolev spaces. For $q \geq 0$, let $\mathbf{E}^q_{\pm,b}$ (resp., $\mathbf{F}^q_{\pm,b}$) be the set of sections of $\wedge^{\mathrm{even}\,/\,\mathrm{odd}}(T^{(0,1)*} Z_b) \otimes E_b$ over Z_b (resp., $\wedge^{\mathrm{even}\,/\,\mathrm{odd}}(T^{(0,1)*} Z_{G,b}) \otimes E_{G,b}$ over $Z_{G,b}$) that lie in the q-th Sobolev space. We always use $\|\cdot\|_0$ as the notation for the standard L^2-norms. We denote the G-invariant part of $\mathbf{E}^q_{\pm,b}$ by $\mathbf{E}^{q,G}_{\pm,b}$.

Now since $\mu_b^{-1}(0) \neq \emptyset$ and Assumption 1.1 holds, one sees easily that one can proceed as in the proof in [TZ1] of [TZ1, Theorem 4.2] to get the following results.

There exists a sufficiently small G-invariant open neighborhood $U_b \subset Z_b$ of $\mu_b^{-1}(0)$ and a linear map $J_{T,b} : \mathbf{F}^q_{\pm,b} \to \mathbf{E}^{q,G}_{\pm,b}$ for $T > 0$, which is the analogue of the map defined in [BL, Definition 9.4][3], such that for any $u \in \Omega^{0,*}(Z_{G,b}, E_{G,b})$, one has $J_{T,b}u \in \Omega^{0,*}(Z_b, E_b)^G$ with $\mathrm{Supp}\, J_{T,b}u \subset U_b$. Let $\mathbf{E}^{q,G}_{T,\pm,b}$ be the image of $\mathbf{F}^q_{\pm,b}$ in $\mathbf{E}^{q,G}_{\pm,b}$ by $J_{T,b}$. Then $J_{T,b} : \mathbf{F}^0_{\pm,b} \to \mathbf{E}^{0,G}_{T,\pm,b}$ is an isometry.

Let $\mathbf{E}^{0,G}_{T,\pm,b,\perp}$ be the orthogonal space to $\mathbf{E}^{0,G}_{T,\pm,b}$ in $\mathbf{E}^{0,G}_{\pm,b}$. Let $p_{T,\pm,b}$, $p_{T,\pm,b,\perp}$ be the orthogonal projection operators from $\mathbf{E}^{0,G}_{\pm,b}$ to $\mathbf{E}^{0,G}_{T,\pm,b}$, $\mathbf{E}^{0,G}_{T,\pm,b,\perp}$, respectively. Then we have the following decomposition of $D^{E_b}_{G,+,T}$:

$$D^{E_b}_{G,+,T} = \sum_{i=1}^{4} D_{T,b,i} \tag{2.2}$$

with

$$\begin{aligned} &D_{T,b,1} = p_{T,-,b} D^{E_b}_{G,+,T} p_{T,+,b}, \qquad D_{T,b,2} = p_{T,-,b} D^{E_b}_{G,+,T} p_{T,+,b,\perp}, \\ &D_{T,b,3} = p_{T,-,b,\perp} D^{E_b}_{G,+,T} p_{T,+,b}, \qquad D_{T,b,4} = p_{T,-,b,\perp} D^{E_b}_{G,+,T} p_{T,+,b,\perp}. \end{aligned} \tag{2.3}$$

By proceeding as in [TZ1, Sects. 2, 3, and 4(a)] and [BL, Sects. 8, 9], we obtain the following proposition, which consists of the fiberwise analogues of [BL, Theorems 9.8, 9.10 and 9.14] in our situation.

Proposition 2.1. (a) As $T \to +\infty$,

$$J^{-1}_{T,b} D_{T,b,1} J_{T,b} = D^{E_{G,b}}_{Q,+} + O\left(\frac{1}{\sqrt{T}}\right) : \Omega^{0,\mathrm{even}}(Z_{G,b}, E_{G,b}) \longrightarrow \Omega^{0,\mathrm{odd}}(Z_{G,b}, E_{G,b}), \tag{2.4}$$

where $D^{E_{G,b}}_{Q,+}$ is the Dirac type operator on $Z_{G,b}$ defined in [TZ1, Definition 3.12].

(b) There exist $C_{b,1} > 0$, $C_{b,2} > 0$, and $T_{b,0} > 0$ such that for any $T \geq T_{b,0}$, and any $s \in \mathbf{E}^{1,G}_{T,+,b,\perp} = \mathbf{E}^{0,G}_{T,+,b,\perp} \cap \mathbf{E}^{1,G}_{+,b}$, $s' \in \mathbf{E}^{1,G}_{T,+,b}$, then

$$\|D_{T,b,2}s\|_0 \leq C_{b,1}\left(\frac{\|s\|_1}{\sqrt{T}} + \|s\|_0\right),$$

$$\|D_{T,b,3}s'\|_0 \leq C_{b,1}\left(\frac{\|s'\|_1}{\sqrt{T}} + \|s'\|_0\right), \tag{2.5}$$

[3]See [Z, Sect. 1(b)] for a more explicit description of the construction of $J_{T,b}$.

and

$$\|D_{T,b,4}s\|_0 \geq C_{b,2}\left(\|s\|_1 + \sqrt{T}\|s\|_0\right). \tag{2.6}$$

□

The easy but important observation is that the above fiberwise construction can be made to depend on $b \in B$ continuously. We have then a continuous family of Fredholm operators

$$D^{E_b}_{G,+,T} : \mathbf{E}^{1,G}_{+,b} \longrightarrow \mathbf{E}^{0,G}_{-,b}, \quad b \in B, \tag{2.7}$$

admitting the decompositions (2.2), (2.3) with each $\{D_{T,b,i}\}_{b\in B}$ a continuous family of bounded linear operators mapping from $\mathbf{E}^{1,G}_{+,b}$'s to $\mathbf{E}^{0,G}_{-,b}$'s.

Furthermore, the positive constants $T_{b,0}$, $C_{b,1}$, $C_{b,2}$ in Proposition 2.1 can be chosen to be independent of $b \in B$. In what follows, we simply denote them by T_0, C_1, C_2, respectively.

(b) Proof of Theorem 1.2

We work in the category of continuous families of Fredholm operators rather than that of differential operators.

For any $u \in \mathbf{R}$, $b \in B$, set

$$D^{E_b}_{G,+,T}(u) = D_{T,b,1} + D_{T,b,4} + u\,(D_{T,b,2} + D_{T,b,3}) : \mathbf{E}^{1,G}_{+,b} \longrightarrow \mathbf{E}^{0,G}_{-,b}. \tag{2.8}$$

The following easy lemma plays a key role in our proof of Theorem 1.2.

Lemma 2.2. There exists $T_1 > 0$ such that for any $u \in [0,1]$ and $T \geq T_1$, $\{D^{E_b}_{G,+,T}(u)\}_{b\in B}$ is a continuous family of Fredholm operators over B. □

Proof. We need only to show the Fredholm property on each fiber.

From the second part of Proposition 2.1 and from (2.2), (2.8), one sees that there exists $C_3 > 0$ such that for any $T \geq T_0$, $u \in [0,1]$, and $s \in \mathbf{E}^{1,G}_{+,b}$, one has

$$\left\|D^{E_b}_{G,+,T}s - D^{E_b}_{G,+,T}(u)s\right\|_0 \leq C_3\left(\frac{\|s\|_1}{\sqrt{T}} + \|s\|_0\right). \tag{2.9}$$

On the other hand, by the Bochner type formula for $D^{E_b,2}_{G,+,T}$ proved in [TZ1, Corollary 1.7], one deduces easily that there exist $C_4, C_5 > 0$ such that for any $T \geq T_0$,

$$\left\|D^{E_b}_{G,+,T}s\right\|_0 \geq C_4\|s\|_1 - C_5\sqrt{T}\|s\|_0. \tag{2.10}$$

From (2.9), (2.10), one gets

$$\left\| D^{E_b}_{G,+,T} s - D^{E_b}_{G,+,T}(u) s \right\|_0 \leq \frac{C_3}{C_4 \sqrt{T}} \left\| D^{E_b}_{G,+,T} s \right\|_0 + (C_3 + C_5) \|s\|_0. \tag{2.11}$$

From (2.11) and the Fredholm property of $D^{E_b}_{G,+,T}$, one obtains the Fredholm property of $D^{E_b}_{G,+,T}(u)$ for sufficiently large T. ■

Recall that the index bundle construction [AS] applies well to continuous families of Fredholm operators and that the homotopy invariance property for index bundles still holds in this situation.

Thus by Lemma 2.2, we have the following identity of index bundles:

$$\begin{aligned} \operatorname{ind}\left\{ D^{E_b}_{G,+,T} \right\}_{b \in B} &= \operatorname{ind}\left\{ D^{E_b}_{G,+,T} \right\}_{b \in B}(0) = \operatorname{ind}\{D_{T,b,1} + D_{T,b,4}\}_{b \in B} \\ &= \operatorname{ind}\{D_{T,b,1}\}_{b \in B} + \operatorname{ind}\{D_{T,b,4}\}_{b \in B} \quad \text{in } K(B), \end{aligned} \tag{2.12}$$

where in the last line, each $D_{T,b,1}$ (resp., $D_{T,b,4}$), $b \in B$, is now regarded as a Fredholm operator mapping from $\mathbf{E}^{1,G}_{T,+,b}$ (resp., $\mathbf{E}^{1,G}_{T,+,b,\perp}$) to $\mathbf{E}^{0,G}_{T,-,b}$ (resp., $\mathbf{E}^{0,G}_{T,-,b,\perp}$).

On the other hand, by an obvious analogue of the first part of [BL, Prop. 9.16], which follows from the third part of Proposition 2.1 and its adjoint analogue, one has

$$\operatorname{ind}\{D_{T,b,4}\}_{b \in B} = 0 \quad \text{in } K(B). \tag{2.13}$$

From (2.12), (2.13), one gets

$$\operatorname{ind}\left\{ D^{E_b}_{G,+,T} \right\}_{b \in B} = \operatorname{ind}\{D_{T,b,1}\}_{b \in B} \quad \text{in } K(B). \tag{2.14}$$

Now by the first part of Proposition 2.1, one deduces easily that when T is large enough, one has

$$\operatorname{ind}\left\{ J^{-1}_{T,b} D_{T,b,1} J_{T,b} \right\}_{b \in B} = \operatorname{ind}\left\{ D^{E_{G,b}}_{Q,+} \right\}_{b \in B} \quad \text{in } K(B). \tag{2.15}$$

From (2.14), (2.15), and again the homotopy invariance property of the index bundle, one gets for T sufficiently large that

$$\begin{aligned} \operatorname{ind}\left\{ D^{E_{G,b}}_{+} \right\}_{b \in B} &= \operatorname{ind}\left\{ D^{E_{G,b}}_{Q,+} \right\}_{b \in B} = \operatorname{ind}\left\{ D^{E_b}_{G,+,T} \right\}_{b \in B} \\ &= \operatorname{ind}\left\{ D^{E_b}_{G,+} \right\}_{b \in B} \quad \text{in } K(B), \end{aligned} \tag{2.16}$$

which is exactly (1.11).

The proof of Theorem 1.2 is completed. ■

Remark 2.3. The deformation trick in (2.8) allows us to avoid the small eigenvalue problem (see [TZ1, Theorem 3.13]), which is not easy to handle in the family case when the kernel of the involved Spin^c-Dirac type operators jumps. However, this may destroy the $\mathbf{Z}$-grading nature in holomorphic contexts. Thus, as we see in the next section, the full strength of the analytic arguments in [TZ1] as well as its holomorphic refinement in [Z] is in force in holomorphic situations.

(c) Some immediate consequences

The first example is certainly when E is a fiberwise prequantum line bundle and verifies fiberwise the Kostant formula (see [Ko]; see also [TZ1, (1.13)]). In this case, one gets a direct generalization of the Guillemin-Sternberg conjecture (see [GS]) to the family case.

For the second example, we take $E = \mathbf{C}$, the trivial complex line bundle with the trivial G-action on it. Assumption 1.1 clearly holds for it. On the other hand, by a family rigidity theorem to be proved in the Appendix, we have

$$\operatorname{ind} D^{\mathbf{C}}_{M/B,+} = \operatorname{ind} D^{\mathbf{C}}_{M/B,G,+} \quad \text{in } K(B). \tag{2.17}$$

From (1.11) and (2.17), we get the following consequence, which extends the corresponding result of Tian and Zhang (see [TZ1, Theorem 0.3]) and Meinrenken and Sjamaar (see [MS]) to the family case.

Corollary 2.4. If $\mu^{-1}(0) \neq \emptyset$, then the following identity holds:

$$\operatorname{ind} D^{\mathbf{C}}_{M/B,+} = \operatorname{ind} D^{\mathbf{C}_G}_{M_G/B,+} \quad \text{in } K(B). \tag{2.18}$$

□

When $\mu^{-1}(0) = \emptyset$, one has the following vanishing result, which holds, for example, for the fiberwise prequantum line bundles.

Theorem 2.5. If $\mu^{-1}(0) = \emptyset$ and the inequality (1.7) is strict, then one has

$$\operatorname{ind} D^{E}_{M/B,G,+} = 0 \quad \text{in } K(B). \tag{2.19}$$

□

Proof. In this case, one can apply the arguments in [TZ1, Sect. 2] to show that when T is large enough, every $D^{E_b}_{G,+,T}$, $b \in B$, is invertible. Thus one has

$$\operatorname{ind} \left\{ D^{E_b}_{G,+,T} \right\}_{b \in B} = 0 \quad \text{in } K(B). \tag{2.20}$$

Formula (2.19) then follows from (2.1), (2.20), and the homotopy invariance of the index bundle. ■

Remark 2.6. By using the methods in this paper, one can also prove a family extension of the weighted quantization formula obtained in [TZ2].

§3 Holomorphic family quantizations

In this section, we further assume that $Z \to M \xrightarrow{\pi} B$ is a holomorphic fibration with J the complex structure on TZ. We also assume that G acts holomorphically on the total space M, as well as on the Hermitian (now assuming) holomorphic vector bundle E, and ∇^E is the holomorphic Hermitian connection on E.

For any $b \in B$, let $H^{0,*}(Z_b, E_b)$ (resp., $H^{0,*}(Z_{G,b}, E_{G,b})$) be the Dolbeault cohomology with coefficient E_b (resp., $E_{G,b}$) (see [TZ1, Sect. 4(d)]). Let $H^{0,*}(Z_b, E_b)^G$ be the G-invariant part of $H^{0,*}(Z_b, E_b)$. We make the asumption that $\dim H^{0,*}(Z_b, E_b)^G$ (and thus $\dim H^{0,*}(Z_{G,b}, E_{G,b})$ also, by the results in [Z]), is locally constant. Then $H^{0,*}(Z_b, E_b)^G$ (resp., $H^{0,*}(Z_{G,b}, E_{G,b})$), $b \in B$, form a holomorphic vector bundle $H^{0,*}(Z, E)^G$ (resp., $H^{0,*}(Z_G, E_G)$) over B.

The following result refines Theorem 1.2 in this situation.

Theorem 3.1. If $\mu^{-1}(0) \neq \emptyset$ and Assumption 1.1 holds, then one has the following identification of holomorphic vector bundles over B:

$$H^{0,*}(Z, E)^G \simeq H^{0,*}(Z_G, E_G). \tag{3.1}$$

□

Proof. Let $b \in B$ be fixed temporarily. Following [TZ1, (1.21)], for any $T \in \mathbf{R}$, set

$$\bar{\partial}_T^{E_b} = e^{-T|\mu_b|^2/2} \bar{\partial}^{E_b} e^{T|\mu_b|^2/2} : \Omega^{0,*}(Z_b, E_b) \longrightarrow \Omega^{0,*}(Z_b, E_b). \tag{3.2}$$

Similarly, let

$$\bar{\partial}_Q^{E_{G,b}} : \Omega^{0,*}(Z_{G,b}, E_{G,b}) \longrightarrow \Omega^{0,*}(Z_{G,b}, E_{G,b}) \tag{3.3}$$

be defined as in [TZ1, (3.54)] (see also [Z, (1.8)]).

The main result of [Z] is an explicit construction of the quasi-isomorphisms

$$r_b : \left(\Omega^{0,*}(Z_b, E_b)^G, \bar{\partial}_T^{E_b}\right) \longrightarrow \left(\Omega^{0,*}(Z_{G,b}, E_{G,b}), \bar{\partial}_Q^{E_{G,b}}\right), \quad b \in B, \tag{3.4}$$

when T is sufficiently large.

Now, for the present holomorphic fibration situation, the fiberwise quasi-isomorphism r_b is holomorphic with respect to $b \in B$. Thus, under the assumptions on the constant dimension of the fiberwise Dolbeault cohomologies, $\{r_b\}_{b\in B}$ provides an identification of the corresponding holomorphic vector bundles over B,

$$H_T^{0,*}(Z,E)^G \simeq H_Q^{0,*}(Z_G,E_G). \tag{3.5}$$

On the other hand, one has the obvious identifications of holomorphic vector bundles

$$H_T^{0,*}(Z,E)^G \simeq H^{0,*}(Z,E)^G \tag{3.6}$$

and

$$H_Q^{0,*}(Z_G,E_G) \simeq H^{0,*}(Z_G,E_G). \tag{3.7}$$

Formula (3.1) follows from (3.5), (3.6), and (3.7). ■

As a holomorphic refinement of Corollary 2.4, we take again $E = \mathbf{C}$, the trivial complex line bundle over M with the trivial G-action. In this case, one has the following holomorphic refinement of the rigidity property (2.17),

$$H^{0,*}(Z,\mathbf{C}) = H^{0,*}(Z,\mathbf{C})^G, \tag{3.8}$$

which can be verified directly.

From Theorem 3.1 and (3.8), one gets the following corollary.

Corollary 3.2. If $\mu^{-1}(0) \neq \emptyset$ and $\dim H^{0,*}(Z_b,\mathbf{C}_b)$, $b \in B$, are constant, then one has the following identification of holomorphic vector bundles over B:

$$H^{0,*}(Z,\mathbf{C}) \simeq H^{0,*}(Z_G,\mathbf{C}_G). \tag{3.9}$$

□

Remark 3.3. When the assumption that $\dim H^{0,*}(Z_b,\mathbf{C}_b)$, $b \in B$, are constant does not hold, then (3.1) and (3.9) should hold at the level of analytic sheafs of the corresponding direct images, which may also be proved by using the quasi-isomorphisms in (3.4). We leave this to the interested reader.

Remark 3.4. The results in this section are closely related to the relative quantization formula proved in the projective case by Teleman (see [Te, (5.2)]).

Appendix A family rigidity theorem for Spin^c-Dirac operators

The purpose of this Appendix is to present a proof of the rigidity result (2.17). We make the same assumptions and use the same notation as in Section 1.

It is easy to see that in order to prove (2.17), we need only to assume that $G = S^1$, the circle group.

Let (E, ∇^E) be the Hermitian vector bundle in Section 1. We no longer assume that it verifies Assumption 1.1, unless this is emphasized otherwise.

Let V be the unit base of the Lie algebra of S^1. We use the same notation, V, to denote the vector field it generates on M. Then for any $b \in B$, $V_b = V|_b \in \Gamma(TZ_b)$ is a Killing vector field on Z_b.

Let F be one of the connected components of the fixed point set of the S^1 action on M. Then the projection $\pi : M \to B$ induces a fibration $\pi_F : F \to B$, with each fiber $F_b = \pi_F^{-1}(b)$ being one of the connected components of the fixed point set of the S^1 action on Z_b. Let N be the normal bundle to F in M. We identify it with the orthogonal complement to $TZ \cap TF$ in TZ. Then each $N_b = N|_{Z_b}$ is the normal bundle to F_b in Z_b and carries an almost complex structure J_{N_b}, an induced Hermitian metric, as well as an induced Hermitian connection.

Let $b \in B$ be fixed temporarily.

The fiberwise Lie derivative $\sqrt{-1}L_{V,b} = \sqrt{-1}L_V|_{Z_b}$ acts on N_b as a covariantly constant invertible selfadjoint operator commuting with $J_{N,b}$. Let $N_{b,+}$, $N_{b,-}$ be the positive and negative eigenbundles of $\sqrt{-1}L_{V,b}|_{N_b}$, respectively. Then J_{N_b} preserves $N_{b,\pm}$, and one has the canonical splittings

$$N_{b,\pm} \otimes \mathbf{C} = N_{b,\pm}^{(1,0)} \oplus N_{b,\pm}^{(0,1)}. \tag{A.1}$$

Let $\mathrm{Sym}(N_{b,+}^{(1,0)})$ (resp., $\mathrm{Sym}(N_{b,-}^{(0,1)})$) be the total symmetric power of $N_{b,+}^{(1,0)}$ (resp., $N_{b,-}^{(0,1)}$). Then $\mathrm{Sym}(N_{b,-}^{(0,1)}) \otimes \mathrm{Sym}(N_{b,+}^{(1,0)}) \otimes \det(N_{b,+}^{(1,0)}) \otimes E_b|_{F_b}$ is an infinite dimensional vector bundle over F_b, on which $\sqrt{-1}L_{V,b}$ acts as a covariantly constant selfadjoint operator. Furthermore, for any $\lambda \in \mathbf{Z}$, its λ-eigen-subbundle, denoted by $(\mathrm{Sym}(N_{b,-}^{(0,1)}) \otimes \mathrm{Sym}(N_{b,+}^{(1,0)}) \otimes \det(N_{b,+}^{(1,0)}) \otimes E_b|_{F_b})^\lambda$, is finite dimensional.

Let $D_{F_b,\lambda,+}^{E_b}(V)$ be the twisted Spin^c-Dirac operator defined by

$$\begin{aligned} D_{F_b,\lambda,+}^{E_b}(V) : \Omega^{0,\mathrm{even}}\Big(F_b, \Big(\mathrm{Sym}\Big(N_{b,-}^{(0,1)}\Big) \otimes \mathrm{Sym}\Big(N_{b,+}^{(1,0)}\Big) \otimes \det\Big(N_{b,+}^{(1,0)}\Big) \otimes E_b|_{F_b}\Big)^\lambda\Big) \\ \longrightarrow \Omega^{0,\mathrm{odd}}\Big(F_b, \Big(\mathrm{Sym}\Big(N_{b,-}^{(0,1)}\Big) \otimes \mathrm{Sym}\Big(N_{b,+}^{(1,0)}\Big) \otimes \det\Big(N_{b,+}^{(1,0)}\Big) \otimes E_b|_{F_b}\Big)^\lambda\Big). \end{aligned} \tag{A.2}$$

Then $\{D^{E_b}_{F_b,\lambda,+}(V)\}_{b\in B}$ form a smooth family of elliptic differential operators over B, which admits an index bundle

$$\operatorname{ind}\left\{D^{E_b}_{F_b,\lambda,+}(V)\right\}_{b\in B}\in K(B). \tag{A.3}$$

On the other hand, one can also construct a smooth family of the restrictions of the Spin^c-Dirac operators

$$D^{E_b}_{\lambda,+}:\Omega^{0,\mathrm{even}}(Z_b,E_b)^\lambda\longrightarrow\Omega^{0,\mathrm{odd}}(Z_b,E_b)^\lambda, \tag{A.4}$$

where $\Omega^{0,*}(Z_b,E_b)^\lambda$ is the eigen-subspace of $\Omega^{0,*}(Z_b,E_b)$ on which $\sqrt{-1}L_{V,b}$ acts as multiplication by λ, which admits an index bundle

$$\operatorname{ind}\left\{D^{E_b}_{\lambda,+}\right\}_{b\in B}\in K(B). \tag{A.5}$$

The first main result of this Appendix can be stated as follows.

Theorem A.1. The following identity holds in $K(B)$ for any $\lambda\in\mathbf{Z}$:

$$\operatorname{ind}\left\{D^{E_b}_{\lambda,+}\right\}_{b\in B}=\sum_F\operatorname{ind}\left\{D^{E_b}_{F_b,\lambda,+}(V)\right\}_{b\in B}. \tag{A.6}$$

□

Proof. Following Witten (see [W]; see also [T, Sect. 2] and [WZ, (3.5)]), for any $T\in\mathbf{R}$, we consider the following family of operators parametrized by B:

$$\widetilde{D}^{E_b}_{\lambda,T,+}=D^{E_b}_{\lambda,+}+\sqrt{-1}Tc(V):\Omega^{0,\mathrm{even}}(Z_b,E_b)^\lambda\longrightarrow\Omega^{0,\mathrm{odd}}(Z_b,E_b)^\lambda,\quad b\in B. \tag{A.7}$$

Recall that when $B=\{\mathrm{pt.}\}$, a proof of (A.6) has been given in the holomorphic context in [WZ], in which the arguments work with obvious modifications in the symplectic case, by taking $T\to+\infty$ in (A.7). Furthermore, the method in [WZ] is again based on [BL] and in particular on estimates similar to those in Proposition 2.1 of this paper (see [WZ, Prop. 3.5]).

Formula (A.6) then follows easily by combining the arguments in [WZ], which apply here fiberwise, with the method in Section 2(b). ■

Now we take $E=\mathbf{C}$. One verifies easily that for any $b\in B$, the restriction of $\sqrt{-1}L_{V,b}$ to $\mathrm{Sym}(N^{(1,0)}_{b,+})\otimes\det(N^{(1,0)}_{b,+})$ is positive, while its restriction to $\mathrm{Sym}(N^{(0,1)}_{b,-})$ is nonnegative. Thus, if $\lambda<0$, then one has

$$\left(\mathrm{Sym}\left(N^{(0,1)}_{b,-}\right)\otimes\mathrm{Sym}\left(N^{(1,0)}_{b,+}\right)\otimes\det\left(N^{(1,0)}_{b,+}\right)\right)^\lambda=0. \tag{A.8}$$

From (A.6) and (A.8), one finds that if $\lambda < 0$, then

$$\operatorname{ind}\left\{D^{E_b}_{\lambda,+}\right\}_{b\in B} = 0 \quad \text{in } K(B). \tag{A.9}$$

Similarly, by changing V to $-V$, one can show that (A.9) also holds for $\lambda > 0$. Since (A.9) holds for every nonzero λ, one deduces (2.17) easily. ∎

Remark A.2. In fact, the above arguments work for families of almost complex manifolds as well. One then gets (2.17) as a family extension of the rigidity theorem for the canonical Spin^c-Dirac operators on almost complex manifolds (see [H]).

Remark A.3. Now if we assume that E in Theorem A.1 also verifies Assumption 1.1 in Section 1, then by combining Theorems 1.2 and A.1, one gets that if $\mu^{-1}(0) \neq \emptyset$, then

$$\operatorname{ind} D^{E_G}_{M_G/B,+} = \sum_F \operatorname{ind}\left\{D^{E_b}_{F_b,\lambda=0,+}(V)\right\}_{b\in B} \quad \text{in } K(B), \tag{A.10}$$

which is of interest itself.

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DISCRETE OSCILLATION THEORY

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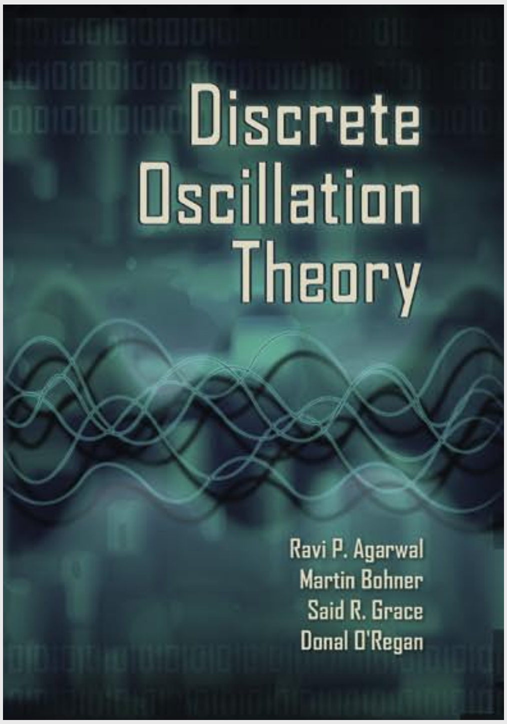

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