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Hopf cyclic cohomology and Hodge theory for proper actions

Xiang Tang, Yi-Jun Yao, and Weiping Zhang

Abstract. We introduce a Hopf algebroid associated to a proper Lie group action on a smooth manifold. We prove that the cyclic cohomology of this Hopf algebroid is equal to the de Rham cohomology of invariant differential forms. When the action is cocompact, we develop a generalized Hodge theory for the de Rham cohomology of invariant differential forms. We prove that every cyclic cohomology class of the Hopf algebroid is represented by a generalized harmonic form. This implies that the space of cyclic cohomology of the Hopf algebroid is finite dimensional. As an application of the techniques developed in this paper, we discuss properties of the Euler characteristic for a proper cocompact action.

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1. Introduction

Let G be a Lie group, and M be a smooth manifold. We assume that G acts on M properly. As the G-action is proper, the quotient M/G is a Hausdorff stratified space. Some of the examples of such spaces are considered already in [14].

In this paper, inspired by Connes and Moscovici's Hopf cyclic theory [2], [3], we introduce a Hopf algebroid to study the "local symmetries" of this stratified space.

Hopf algebroid was introduced by Lu [11] in generalizing the notion of Hopf algebra. Connes and Moscovici [4] applied this concept to generalize that of symmetry of "noncommutative spaces". They developed a beautiful theory of cyclic cohomology for a Hopf algebroid, and used it to study the transverse index theory.

Our Hopf algebroid associated to the G-action on M is a generalization of the Hopf algebroid introduced in the first author's joint work with Kaminker [7]. It is shown in [7] that if Γ is a discrete group acting on a smooth manifold M, the graded commutative algebra of differential forms on the action groupoid $M \rtimes \Gamma$ is a topological Hopf algebroid with the coalgebra and antipode structures defined by taking the dual of the groupoid structure. In the case of a Lie group G action, instead of considering the algebra of differential forms on the groupoid $M \rtimes G$, we consider the algebra $\mathcal{H}(G, M)$ of differential forms valued functions on G. The similar construction as in [7] defines a Hopf algebroid structure on this algebra.

We are able to compute the cyclic cohomology of this Hopf algebroid, which is equal to the differentiable cohomology of the groupoid $M \rtimes G \rightrightarrows M$ with coefficient in differential forms on M considered by Crainic [5]. As the G-action is proper, Crainic's result implies that the cyclic cohomology of the Hopf algebroid $\mathcal{H}(G, M)$ is equal to the de Rham cohomology of G-invariant differential forms on M.

Our main result of this paper is to prove a "Hodge theorem" for G-invariant differential forms on M when the G-action is cocompact. Our approach to this generalized Hodge theory is inspired from the third author's joint work with Mathai [12]. Our strategy is to study a generalized de Rham Laplace–Beltrami operator on the space of G-invariant differential forms on M. With some elliptic estimates, we are able to prove that this operator has essentially the same properties as the standard Laplace-Beltrami operator on a compact manifold. This allows to prove that every cyclic cohomology class of the Hopf algebroid is uniquely represented by a harmonic form of our generalized Laplace-Beltrami operator, which implies that the cyclic cohomology of our Hopf algebroid is finite dimensional.

Theorem 1.1. *Let G be a Lie group acting properly and cocompactly on a smooth manifold M. The cyclic cohomology groups of $\mathcal{H}(G, M)$ are of finite dimension.*

The above result allows us to introduce the Euler characteristic for a proper cocompact action of a Lie group G as the alternating sum of the dimensions of the de Rham cohomology groups of G-invariant differential forms. We are able to generalize the following two classical results about Euler characteristic to the case of a proper cocompact G-action.

(1) The Poincaré duality theorem holds for twisted de Rham cohomology groups of G-invariant differential forms. In particular, when the dimension of M is odd, the Euler characteristic of a proper cocompact G-action on M is 0;

(2) When there is a nowhere vanishing G-invariant vector field on M, the Euler characteristic of a proper cocompact G-action is also 0.

The paper is organized as follows. In Section 2, we introduce the Hopf algebroid $\mathcal{H}(G, M)$ and compute its Hopf cyclic cohomology. In Section 3, we study a generalized Laplace–Beltrami operator, and prove that every Hopf cyclic cohomology class of $\mathcal{H}(G, M)$ can be uniquely represented by a generalized harmonic form. In Section 4, we introduce and study the Euler characteristic for a proper cocompact G-action.

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2. Cyclic cohomology of Hopf algebroids

2.1. Hopf algebroids. In [11], Lu introduced the notion of a Hopf algebroid as a generalization of a Hopf algebra. Connes and Moscovici [4] introduced cyclic cohomology for Hopf algebroids. Since then, many authors have studied cyclic theory for Hopf algebroids, e.g. [8]–[10]. Oriented by our application, we take the simplest approach for the definition of cyclic cohomology of a Hopf algebroid, cf. [8], which is also close to Connes and Moscovici's original approach. We refer the interested readers to [9] and [10] for the beautiful systematic study of the general theory.

Let A and B be unital topological algebras. A (topological) bialgebroid structure on A, over B, consists of the following data.

i) A continuous algebra homomorphism $\alpha\colon B \to A$ called the *source map* and a continuous algebra anti-homomorphism $\beta\colon B \to A$ called the *target map*, satisfying $\alpha(a)\beta(b) = \beta(b)\alpha(a)$ for all $a, b \in B$.

In this paper, by tensor product $\otimes$ we always mean topological tensor product. Let $A \otimes_B A$ be the quotient of $A \otimes A$ by the right $A \otimes A$ ideal generated by $\beta(a) \otimes 1 - 1 \otimes \alpha(a)$ for all $a \in B$.

ii) A continuous B-B bimodule map $\Delta\colon A \to A \otimes_B A$, called the *coproduct*, satisfying
   (a) $\Delta(1) = 1 \otimes 1$;
   (b) $(\Delta \otimes_B \mathrm{Id})\Delta = (\mathrm{Id} \otimes_B \Delta)\Delta\colon A \to A \otimes_B A \otimes_B A$,
   (c) $\Delta(a)(\beta(b) \otimes 1 - 1 \otimes \alpha(b)) = 0$ for $a \in A$, $b \in B$,
   (d) $\Delta(a_1 a_2) = \Delta(a_1)\Delta(a_2)$ for $a_1, a_2 \in A$.

iii) A continuous B-B bimodule map $\epsilon\colon A \to B$, called the *counit*, satisfying
   (a) $\epsilon(1) = 1$;
   (b) $\ker \epsilon$ is a left A ideal;
   (c) $(\epsilon \otimes_B \mathrm{Id})\Delta = (\mathrm{Id} \otimes_B \epsilon)\Delta = \mathrm{Id}\colon A \to A$;
   (d) $\epsilon(\alpha(b)\beta(b')a) = b\epsilon(a)b'$ and $\epsilon(aa') = \epsilon(a\alpha(\epsilon(a'))) = \epsilon(a\beta(\epsilon(a')))$ for any $a, a' \in A$, $b, b' \in B$.

A topological *para-Hopf algebroid* is a topological bialgebroid A, over B, which admits a continuous algebra anti-isomorphism $S\colon A \to A$ such that

$$S^2 = \mathrm{Id}, \quad S\beta = \alpha, \quad m_A(S \otimes_B \mathrm{Id})\Delta = \beta\epsilon S\colon A \to A$$

and

$$S(a^{(1)})^{(1)} a^{(2)} \otimes_B S(a^{(1)})^{(2)} = 1 \otimes_B S(a).$$

In the above formula we have used Sweedler's notation for the coproduct $\Delta(a) = a^{(1)} \otimes_B a^{(2)}$.

We note that in the above definition one may allow A and B to be differential graded algebras and require all of the above maps to be compatible with the differentials and to be of degree 0. Thus one would have a differential graded (para) Hopf algebroid (cf. [6]).

We remark that as is pointed out in [9], Sec.2.6.13, with our definition any para-Hopf algebroid is a Hopf algebroid as was used in [9] and [10]. Therefore, for simplicity, in the following, we will abbreviate "para-Hopf algebroid" to "Hopf algebroid".

2.2. Hopf algebroid $\mathcal{H}(G, M)$. Let G be a Lie group acting on a smooth manifold M.

Define B to be the algebra of differential forms on M, and A to be the algebra of B-valued functions on G. Both A and B are differential graded algebras with the de Rham differential. We fix the notation that for a group element g in G and a smooth function a on M, $g^*(a)(x) := a(gx)$.

We define the source and target maps $\alpha, \beta \colon B \to A$ as follows:

$$\alpha(b)(g) = b \quad \text{and} \quad \beta(b)(g) = g^*(b).$$

It is easy to check that α (resp. β) is an algebra (resp. anti-) homomorphism.

When we consider the projective tensor product, the space $A \otimes_B A$ is isomorphic to the space of B-valued functions on $G \times G$, i.e.,

$$(\phi \otimes_B \psi)(g_1, g_2) = \phi(g_1) g_1^*(\psi(g_2))$$

for $\phi, \psi \in A$. We define the bimodule map $\Delta \colon A \to A \otimes_B A$ by

$$\Delta(\phi)(g_1, g_2) = \phi(g_1 g_2)$$

and define the counit map $\epsilon \colon A \to B$ by $\epsilon(\phi) = \phi(1)$ for $\phi \in A$.

It is straightforward to check that $(A, B, \alpha, \beta, \Delta, \epsilon)$ is a differential graded topological bialgebroid.

To make $(A, B, \alpha, \beta, \Delta, \epsilon)$ into a Hopf algebroid, we define the antipode on A by

$$S(\phi)(g) = g^*(\phi(g^{-1})).$$

It is easy to check that S satisfies properties for an antipode of a para-Hopf algebroid:

- $S(\beta(b))(g) = g^*(\beta(b)(g^{-1})) = g^*(g^{-1})^*(b) = b$.
- $(m_A(S \otimes \mathrm{Id})\Delta)(\phi)(g) = g^*\phi(1)$, $(\beta\epsilon S)(\phi)(g) = g^*(S(\phi)(1)) = g^*\phi(1)$.
- One computes that $(S \otimes \mathrm{Id})\Delta(a)(g_1, g_2) = {g_1}^*(a((g_1)^{-1} g_2))$. Therefore, one has

$$\begin{aligned}
(\Delta S \otimes \mathrm{Id})\Delta(a)(g_1, g_2, g_3) &= (g_1 g_2)^*(a(g_2^{-1} g_1^{-1} g_3)),\\
S(a^{(1)})^{(1)} a^{(2)} \otimes_B S(a^{(1)})^{(2)}(g_1, g_2) &= (g_1 g_2)^*(a(g_2^{-1} g_1^{-1} g_1))\\
&= g_1^*(g_2^*(a(g_2^{-1}))) = 1 \otimes_B S(a)(g_1, g_2).
\end{aligned}$$

We denote this Hopf algebroid by $\mathcal{H}(G, M)$.

2.3. Cyclic cohomology. In this part, we briefly recall the definition of the cyclic cohomology of a Hopf algebroid.

We denote by Λ the cyclic category and recall the cyclic module $A^{\natural}$ for $(A, B, \alpha, \beta, \Delta, \epsilon, S)$ introduced by Connes–Moscovici [3].

Define

$$C^0 = B, \quad C^n = \underbrace{A \otimes_B A \otimes_B \cdots \otimes_B A}_{n}, \; n \geq 1.$$

Faces and degeneracy operators are defined as follows:

$$\delta_0(a^1 \otimes_B \cdots \otimes_B a^{n-1}) = 1 \otimes_B a^1 \otimes_B \cdots \otimes_B a^{n-1};$$
$$\delta_i(a^1 \otimes_B \cdots \otimes_B a^{n-1}) = a^1 \otimes_B \cdots \otimes_B \Delta a^i \otimes_B \cdots \otimes_B a^{n-1}, \quad 1 \leq i \leq n-1;$$
$$\delta_n(a^1 \otimes_B \cdots \otimes_B a^{n-1}) = a^1 \otimes_B \cdots \otimes_B a^{n-1} \otimes_B 1;$$
$$\sigma_i(a^1 \otimes_B \cdots \otimes_B a^{n+1}) = a^1 \otimes_B \cdots \otimes_B a^i \otimes_B \epsilon(a^{i+1}) \otimes_B a^{i+2} \otimes_B \cdots \otimes_B a^{n+1}.$$

The cyclic operators are given by

$$\tau_n(a^1 \otimes_B \cdots \otimes_B a^n) = (\Delta^{n-1} S(a^1))(a^2 \otimes \ldots a^n \otimes 1).$$

The cyclic cohomology of $(A, B, \alpha, \beta, \Delta, \epsilon, S)$ is defined to be the cyclic cohomology of $A^{\natural}$.

2.4. Hopf cyclic cohomology of the Hopf algebroid $\mathcal{H}(G, M)$. In this section we explain the computation the Hopf cyclic cohomology of the Hopf algebroid $\mathcal{H}(G, M)$.

We review briefly the definition of differentiable cohomology of a Lie group. Let G be a Lie group acting on a manifold M. Consider E a G-equivariant bundle on M. An E-valued differentiable p-cochain is a smooth map c mapping $G^{\times p}$ to a smooth section of E, i.e., $C_d^p(G; E) = C^\infty(G^{\times p}; \Gamma(E))$. The differential d on $C_d^\bullet(G; E)$ is defined by

$$\begin{aligned}(dc)(g_1, \ldots, g_{p+1}) = g_1^*(c(g_2, \ldots, g_{p+1})) \\ + \sum_{i=1}^{p} (-1)^i c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{p+1}) \\ + (-1)^{p+1} c(g_1, \ldots, g_p).\end{aligned}$$

The differentiable cohomology $H_d^\bullet(G; E)$ of G with coefficient E is defined to be the cohomology of $(C_d^\bullet(G; E), d)$. We remark that the space $C_d^\bullet(G; E)$ has the structure of a cyclic simplicial space. We recall its definition below,

$$\delta_i(a)(g_1, \ldots, g_n, g_{n+1}) = \begin{cases} g_1^*(a(g_2, \ldots, g_{n+1})), & i = 0, \\ a(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}), & 1 \leq i \leq n, \\ a(g_1, \ldots, g_n), & i = n+1, \end{cases}$$

and

$$\sigma_i(a)(g_1, \ldots, g_n) = a(\ldots, g_{i-1}, 1, g_i, \ldots, g_n),$$

as well as

$$t(a)(g_1, \ldots, g_n) = (g_1 \ldots g_n)^* a((g_1 g_2 \ldots g_n)^{-1}, g_1, \ldots, g_{n-1}).$$

The cohomology of this simplicial complex $C_d^\bullet(G, E)$ is isomorphic to the differentiable cohomology of G with coefficient in E. We are now ready to present the computation of Hopf cyclic cohomology of the Hopf algebroid $\mathscr{H}(G, M)$.

Theorem 2.1. *Let G be a Lie group acting on a smooth manifold. We have*

$$\mathrm{HC}^\bullet(\mathscr{H}(G, M)) = \bigoplus_{k \geq 0} H^{\bullet - 2k}(G; (\Omega^*(M), d)).$$

Proof. We observe that a p-cochain on $\mathscr{H}(G, M)$ can be identified with $\Omega^*(M)$-valued functions on $G^{\times p}$, $p \geq 0$. This identification respects the cyclic simplicial structures on $C^\infty(G^{\times \bullet}, \Omega^*(M))$ and $\mathscr{H}(\mathscr{G})^\natural$. Therefore, we conclude that the Hochschild cohomology of $\mathscr{H}(G, M)$ is isomorphic to the differentiable cohomology $H^\bullet(G; (\Omega^*(M), d))$. By the SBI-sequence of cyclic cohomology, we have

$$\mathrm{HC}^\bullet(\mathscr{H}(G, M)) = \bigoplus_{k \geq 0} H^{\bullet - 2k}(G; (\Omega^*(M), d)). \qquad \square$$

Let $\Omega^*(M)^G$ be the space of G-invariant differential forms on M, which inherits a natural de Rham differential d. By [5], Section 2.1, Prop. 1, if G acts on M properly, then the differentiable cohomology $H^\bullet(G; (\Omega^*(M), d))$ is computed as follows:

$$H^\bullet(G; (\Omega^*(M), d)) = H^\bullet(\Omega^*(M)^G, d).$$

Proposition 2.2. *If G acts on M properly, then we have*

$$\mathrm{HC}^\bullet(\mathscr{H}(G, M)) = \bigoplus_{k \geq 0} H^{\bullet - 2k}(\Omega^*(M)^G, d)$$

and

$$\mathrm{HP}^\bullet(\mathscr{H}(G, M)) = \bigoplus_{k \in \mathbb{Z}} H^{\bullet + 2k}(\Omega^*(M)^G, d).$$

3. Generalized Hodge theory and a proof of Theorem 1.1

Now we prove Theorem 1.1. According to Proposition 2.2, all we need to prove is:

The cohomology groups $H^\bullet(\Omega^*(M)^G, d)$ are of finite dimension.

We adapt the proof of the finite dimensionality of the de Rham cohomology of compact manifolds (cf. [15], Ch. 6).

Without loss of generality, as G acts on M properly, we may assume that M is endowed with a G-invariant metric. And since M/G is compact, there exists a compact subset Y of M such that $G(Y) = \bigcup_{g\in G} gY = M$ (cf. [14], Lemma 2.3). With the usual de Rham Hodge $*$ operator, on $\Omega^*(M)$ we consider the following inner product

$$\langle \alpha, \beta\rangle_0 = \int_M \alpha \wedge *\beta. \tag{1}$$

As Y is a closed subset of M, there exist U, U', two open subsets of M, such that $Y \subset U$ and that the closures $\overline{U}$ and $\overline{U'}$ are both compact in M, and that $\overline{U} \subset U'$. It is easy to construct a smooth function $f : M \to [0,1]$ such that $f|_U = 1$ and $\mathrm{supp}(f) \subset U'$.

Let $\Gamma(\Omega^*(M))^G$ be the subspace of G-invariant sections of $\Omega^*(M)$. For an open set W of M, define

$$\|s\|^2_{W,0} = \int_W \langle s(x), s(x)\rangle dx, \quad \|s\|^2_{W,1} = \|s\|_{W,0} + \langle \Delta(s), s\rangle_{W,0}.$$

For any $s \in \Gamma(\Omega^*(M))^G$, we have

$$\|s\|_{U,0} \le \|fs\|_0 \le \|s\|_{U',0}.$$

As $G\cdot Y := \{gy,\ g\in G,\ y \in Y\} = M$, $G\cdot U = M$. Since $\overline{U'}$ is compact, there are finitely many elements $g_1, \dots, g_k$ of G such that $g_1U \cup \dots \cup g_kU$ covers U'. If s is a G-invariant section of $\Omega^*(M)$, it is easy to see that there exists a positive constant $C > 0$,

$$\|s\|_{U',0} \le C\|s\|_{U,0}.$$

Let $dg := dm(g)$ be the right invariant Haar measure on G. Define $\chi\colon G \to \mathbb{R}^+$ by $dm(g^{-1}) = \chi(g)dm(g)$. We define $\boldsymbol{H}^0_f(M, \Omega^*(M))^G$ to be the completion of the space $\{fs : s \in \Gamma(\Omega^*(M))^G\}$ under the norm $\|\cdot\|_0$ associated to the inner product (1). As indicated in the Appendix of [12] written by Bunke, we will first prove the following proposition.

Proposition 3.1. *For any $\mu \in L^2(M, \Omega^*(M))$, define*

$$\left(P_f\mu\right)(x) = \frac{f(x)}{(A(x))^2}\int_G \chi(g) f(gx)\mu(gx)\, dg, \tag{2}$$

where

$$A(x) = \Big(\int_G \chi(g)(f(gx))^2\, dg\Big)^{1/2}$$

is a G-equivariant function on M, i.e., $A(gx)^2 = \chi(g)^{-1}A(x)^2$, and is strictly positive. The operator P_f defines an orthogonal projection from $L^2(M, \Omega^*(M))$ onto $\boldsymbol{H}_f^0(M, \Omega^*(M))^G$.

Proof. It is straightforward to check that $A(x)$ is strictly positive and equivariant. In order to show that (2) defines actually an orthogonal projection, we need to prove the following properties.

- $P_f^2 = P_f$. For any $\mu \in L^2(M, \Omega^*(M))$,

$$\begin{aligned}(P_f^2\mu)(x) &= \frac{f(x)}{(A(x))^2}\int_G \chi(g)f(gx)\left(P_f\mu\right)(gx)\,dg\\ &= \frac{f(x)}{(A(x))^2}\int_G \chi(g)f(gx)\Big(\frac{f(gx)}{(A(gx))^2}\int_G \chi(h)f(hgx)\mu(hgx)\,dh\Big)\,dg\\ &= \frac{f(x)}{(A(x))^2}\int_G \frac{\chi(g)(f(gx))^2}{(A(x))^2}\,dg\Big(\int_G \chi(hg)f(hgx)\mu(hgx)\,d(hg)\Big)\\ &= \frac{f(x)}{(A(x))^2}\Big(\int_G \chi(hg)f(hgx)\mu(hgx)\,d(hg)\Big)\\ &= (P_f\mu)(x).\end{aligned}$$

- P_f is self-adjoint. For any $\mu, \nu \in L^2(M, \Omega^*(M))$,

$$\begin{aligned}\langle P_f\mu, \nu\rangle_0 &= \int_M \frac{f(x)}{(A(x))^2}\int_G \chi(g)f(gx)\langle\mu(gx), \nu(x)\rangle\,dg\,dx\\ &= \int_M\int_G \frac{f(g^{-1}x')}{(A(g^{-1}x'))^2}\chi(g)f(x')\langle\mu(x'), \nu(g^{-1}x')\rangle\,dg\,dx' \quad (x' = gx)\\ &= \int_M\int_G \frac{f(x')}{(A(x'))^2}f(g^{-1}x')\langle\mu(x'), \nu(g^{-1}x')\rangle\,dg\,dx'\\ &= \int_M\int_G \frac{f(x')}{(A(x'))^2}f(g^{-1}x')\langle\mu(x'), \nu(g^{-1}x')\rangle\chi(g^{-1})d(g^{-1})\,dx'\\ &= \langle\mu, P_f\nu\rangle_0.\end{aligned}$$

- It is also straightforward to check that for $\mu = f\alpha \in \boldsymbol{H}_f^0(M, \Omega^*(M))^G$ where $\alpha \in \Omega^*(M)^G$, $P_f\mu = \mu$.

The proposition is thus proved. □

Define $\boldsymbol{H}_f^1(M, \Omega^*(M))^G$ to be the completion of $\{fs \mid s \in \Gamma(\Omega^*(M))^G\}$ under a (fixed) first Sobolev norm associated to the inner product (1). And in general define $\boldsymbol{H}_f^k(M, \Omega^*(M))^G$ (and $\boldsymbol{H}_f^{-1}(M, \Omega^*(M))^G$) to be the completion of the space $\{fs \mid s \in \Gamma(\Omega^*(M))^G\}$ under the corresponding $\boldsymbol{H}^k$ (and $\boldsymbol{H}^{-1}$ norm) for $k \geq 2$. (For any open subset W of M and any compactly supported smooth differential form s on W, $\|s\|_{W,k}^2 = \|(1+\Delta)^{k/2}(s)\|_{W,0}^2$, for $k \geq 2$.)

This time we investigate the operator

$$d_f\colon \boldsymbol{H}^1_f \to \boldsymbol{H}^0_f, \quad f\alpha \mapsto f\,d\alpha. \tag{3}$$

We also consider its adjoint $d_f^*\colon \boldsymbol{H}^1_f \to \boldsymbol{H}^0_f$:

for $\alpha \in \Gamma(\Omega^p(M))^G$, $\beta \in \Gamma(\Omega^{p+1}(M))^G$,

$$\langle d_f(f\alpha), f\beta\rangle_0 = \langle f\alpha, d_f^*(f\beta)\rangle_0.$$

For $f \in C^\infty(M)$ denote by $\nabla(f)$ the gradient vector field associated to f with respect to the Riemannian metric on M. We can show easily that

$$d_f^*(f\beta) = P_f(-2i_{\nabla f}\beta) + f\delta\beta,$$

where $\delta = (-1)^{n(p+1)+n+1} * d * : \Omega^{p+1}(M) \to \Omega^p(M)$ and $i_V\alpha$ is the contraction of the form α with the vector field V.

Now we define a self-adjoint operator

$$\widetilde{\Delta} = d_f d_f^* + d_f^* d_f.$$

Proposition 3.2. $\widetilde{\Delta}\colon \boldsymbol{H}^2_f \to \boldsymbol{H}^0_f$ *is Fredholm.*

Proof. We will prove this fact by establishing a Gårding type inequality. Let $f\alpha \in \boldsymbol{H}^0_f$. Then

$$\begin{aligned}\widetilde{\Delta}(f\alpha) &= d_f(d_f^*(f\alpha)) + d_f^*(d_f(f\alpha))\\ &= d_f(-P_f(2i_{\nabla f}\alpha) + f\delta\alpha) + d_f^*(fd\alpha)\\ &= -d_f(P_f(2i_{\nabla f}\alpha)) + fd\delta\alpha - P_f(2i_{\nabla f}d\alpha) + f\delta d\alpha\\ &= f\Delta\alpha - d_f(P_f(2i_{\nabla f}\alpha)) - P_f(2i_{\nabla f}d\alpha).\end{aligned} \tag{4}$$

Now using (2), we have the following estimates:

$$\begin{aligned}&\|d_f(P_f(2i_{\nabla f}\alpha))\|_0\\ &\quad= \left\|2f(x)d\left(\frac{\int_G \chi(g)f(gx)(i_{\nabla f}\alpha)(gx)\,dg}{(A(x))^2}\right)\right\|_0\\ &\quad= 2\left\|f\int_G \chi(g)d\left(\frac{g^*f}{A^2}\right)(g^*i_{\nabla f}\alpha)\,dg + f\int_G \chi(g)\frac{g^*f}{A^2}d(g^*i_{\nabla f}\alpha)dg\right\|_0\\ &\quad\leqslant 2\left\|f\int_G \chi(g)d\left(\frac{g^*f}{A^2}\right)(g^*i_{\nabla f}\alpha)dg\right\|_0 + 2\left\|f\int_G \chi(g)\frac{g^*f}{A^2}d(g^*i_{\nabla f}\alpha)dg\right\|_0,\end{aligned}$$

where

$$\begin{aligned}\left\|f\int_G \chi(g)d\left(\frac{g^*f}{A^2}\right)(g^*i_{\nabla f}\alpha)dg\right\|_0 &\leqslant \left\|f\int_G \chi(g)\left|d\left(\frac{g^*f}{A^2}\right)\right||(g^*i_{\nabla f}\alpha)|dg\right\|_0\\ &\leqslant \left\|f\int_G \chi(g)\left|d\left(\frac{g^*f}{A^2}\right)\right||g^*(\nabla f)||g^*\alpha|dg\right\|_0\\ &\leqslant \left\|f\int_G \chi(g)\left|d\left(\frac{g^*f}{A^2}\right)\right||g^*(\nabla f)|dg|\alpha(x)|\right\|_0.\end{aligned}$$

As f has a compact support,

$$f\int_G \chi(g)\left|d\left(\frac{g^*f}{A^2}\right)\right||g^*(\nabla f)|\,dg$$

is finite and continuous everywhere, and therefore is bounded from above by a constant on $\overline{U}$. Hence we have

$$2\left\|f\int_G \chi(g)d\left(\frac{g^*f}{A^2}\right)(g^*i_{\nabla f}\alpha)\,dg\right\|_0 \leqslant \widetilde{C}_1\|\alpha\|_{U',0} \leqslant \widetilde{C}_2\|f\alpha\|_0.$$

Consider

$$2\left\|f\int_G \chi(g)\frac{g^*f}{A^2}d(g^*i_{\nabla f}\alpha)\,dg\right\|_0 = 2\|P_f(d\circ i_{\nabla f}\alpha)\|_0 \leq 2\|d\circ i_{\nabla f}\alpha\|_0. \tag{5}$$

Notice that f is supported inside U'. We choose a cut-off function c which is 1 on the support of f and 0 outside $U'' \subset \overline{U''} \subset U'$. We have

$$\|d\circ i_{\nabla f}\alpha\|_0 = \|d\circ i_{\nabla(f)}(c\alpha)\|_{U',0}. \tag{6}$$

We observe that $d\circ i_{\nabla f}$ is a differential operator on $\Omega^*(M)$ of order 1. As $c\alpha$ is a compactly supported smooth function in U', we have

$$\|d\circ i_{\nabla(f)}(c\alpha)\|_{U',0} \leq C\|c\alpha\|_{U',1}. \tag{7}$$

We compute $\|c\alpha\|_{U',1}$ to be

$$\begin{aligned}\|c\alpha\|^2_{U',1} &= \|c\alpha\|^2_{U',0} + \langle\Delta(c\alpha), c\alpha\rangle_{U'}\\ &= \|c\alpha\|^2_{U',0} + \langle d(c\alpha), d(c\alpha)\rangle_{U'} + \langle\delta(c\alpha),\delta(c\alpha)\rangle_{U'}\\ &= \|c\alpha\|^2_{U',0} + \|d(c\alpha)\|_{U',0} + \|\delta(c\alpha)\|_{U',0}.\end{aligned} \tag{8}$$

We discuss one by one the terms in the above line.

As c is bounded from above by 1, we have

$$\|c\alpha\|_{U',0} \leq \|\alpha\|_{U',0}.$$

As c and dc are both bounded,

$$\begin{aligned}\|d(c\alpha)\|_{U',0} &= \|dc\wedge\alpha + cd\alpha\|_{U',0}\\ &\leq \|dc\wedge\alpha\|_{U',0} + \|cd\alpha\|_{U',0}\\ &\leq C_1\|\alpha\|_{U',0} + C_2\|d\alpha\|_{U',0}.\end{aligned} \tag{9}$$

Similarly, as c and ∇c are compactly supported, they are both bounded. We have

$$\|\delta(c\alpha)\|_{U',0} = \|i_{\nabla c}\alpha + c\delta\alpha\|_{U',0} \leq C_3\|\alpha\|_{U',0} + \|\delta\alpha\|_{U',0}. \tag{10}$$

As $G \cdot U = M$ and $\overline{U'}$ is compact, there are finitely many $g_1, \dots, g_k$ such that $U' \subset g_1 U \cup \dots \cup g_k U$. While α, $d\alpha$, and $\delta\alpha$ are all G-invariant, we have

$$\begin{aligned} \|\alpha\|_{U',0} &\le C_4 \|\alpha\|_U \le C_4 \|f\alpha\|_0, \\ \|d\alpha\|_{U',0} &\le C_5 \|d\alpha\|_U \le C_5 \|d(f\alpha)\|_0, \\ \|\delta\alpha\|_{U',0} &\le C_6 \|\delta\alpha\|_U \le C_6 \|\delta(f\alpha)\|_0. \end{aligned} \tag{11}$$

Summarizing inequalities (5)–(11), we have

$$2\left\| f \int_G \chi(g) \frac{g^* f}{A^2} d(g^* i_{\nabla f} \alpha)\, dg \right\|_0 \le A \|f\alpha\|_1.$$

Similarly,

$$\|P_f(2 i_{\nabla f} d\alpha)\|_0 \leqslant 2 \|i_{\nabla f} d\alpha\|_0 \leqslant \widetilde{C}_7 \|d\alpha\|_{U',0} \leqslant \widetilde{C}_8 \|d(f\alpha)\|_0 \le B \|f\alpha\|_1.$$

By combining these inequalities, we have

$$\begin{aligned} \|\widetilde{\Delta}(f\alpha)\|_0 &\geqslant \|f\Delta\alpha\|_0 - \|d_f(P_f(2 i_{\nabla f}\alpha))\|_0 - \|P_f(2 i_{\nabla f} d\alpha)\|_0 \\ &\geqslant \|f\Delta\alpha\|_0 - (A+B)\|f\alpha\|_1 \\ &\geqslant \|\Delta\alpha\|_{U,0} - \widetilde{B}\|f\alpha\|_1. \end{aligned} \tag{12}$$

The standard elliptic inequality implies that

$$\|\Delta\alpha\|_{U,0} \ge \widetilde{A} \|\alpha\|_{U,2} - D\|\alpha\|_{U,0}. \tag{13}$$

By definition, we have

$$\|f\alpha\|_2^2 = \|f\alpha\|_0^2 + 2\|(d+\delta)(f\alpha)\|_0^2 + \|(d+\delta)^2(f\alpha)\|_0^2. \tag{14}$$

We can now compute

$$\begin{aligned} (d+\delta)^2(f\alpha) &= (d+\delta)(df \wedge \alpha + f d\alpha - i_{\nabla f}\alpha + f\delta(\alpha)) \\ &= \delta(df \wedge \alpha) - i_{\nabla f} d\alpha + f\delta d\alpha - d(i_{\nabla f}\alpha) + df \wedge \delta(\alpha) + f d\delta\alpha \\ &= f\Delta(\alpha) + \delta(df \wedge \alpha) - i_{\nabla f} d\alpha - d(i_{\nabla f}\alpha) + df \wedge \delta(\alpha). \end{aligned} \tag{15}$$

We notice that $\delta(df \wedge \alpha)$, $i_{\nabla f} d\alpha$, $d(i_{\nabla f}\alpha)$ and $df \wedge \delta(\alpha)$ are all differential operators of order less than or equal to 1 for α. So similar estimates as (6)–(11) show that every piece of them is bounded by a multiple of $\|f\alpha\|_1$.

Similar arguments as (11) that

$$\|f\Delta\alpha\|_0 \le \|\Delta\alpha\|_{U',0} \le D_1 \|\Delta\alpha\|_{U,0} \le D_1 \|\alpha\|_{U,2}. \tag{16}$$

By (14)–(16), we have

$$\|\alpha\|_{U,2} \ge D_2 \|f\alpha\|_2 - D_3 \|f\alpha\|_1.$$

With this estimate, from (12) and (13), we have

$$\|\widetilde{\Delta}(f\alpha)\|_0 \geq D_3\|f\alpha\|_2 - D_4\|f\alpha\|_1.$$

The so-called Peter–Paul inequality gives us

$$\|f\alpha\|_1 \leqslant \frac{1}{2}D_3/D_4\|f\alpha\|_2 + D_5\|f\alpha\|_0.$$

In summary, we have

$$\|\widetilde{\Delta}(f\alpha)\|_0 \geqslant \frac{1}{2}D_3\|f\alpha\|_2 - D_6\|f\alpha\|_0. \tag{17}$$

Due to the fact that the embedding of $\boldsymbol{H}_f^2(M, \Omega^*(M))^G$ in $\boldsymbol{H}_f^0(M, \Omega^*(M))^G$ is compact, the above Gårding type inequality implies that $\widetilde{\Delta}$ is Fredholm. □

Corollary 3.3. $\dim(\ker \widetilde{\Delta}) = \dim(\operatorname{coker} \widetilde{\Delta}) < +\infty$.

Lemma 3.4. $\ker \widetilde{\Delta} = (\operatorname{Im} \widetilde{\Delta})^\perp \cap \boldsymbol{H}_f^2$.

Proof. We have

$$\begin{aligned} f\alpha \in (\operatorname{Im} \widetilde{\Delta})^\perp \cap \boldsymbol{H}_f^2 &\iff \langle \widetilde{\Delta}(f\beta),\ f\alpha\rangle_0 = 0 \text{ for all } f\beta \in \boldsymbol{H}_f^2, \\ &\iff \langle f\beta,\ \widetilde{\Delta}(f\alpha)\rangle_0 = 0 \text{ for all } f\beta \in \boldsymbol{H}_f^2. \end{aligned}$$

As $\boldsymbol{H}_f^2$ is dense in $\boldsymbol{H}_f^0$, so $\widetilde{\Delta}(f\alpha) = 0$, which is equivalent to $f\alpha \in \ker \widetilde{\Delta}$. □

This lemma together with the previous Corollary 3.3 implies that

$$\ker \widetilde{\Delta} = (\operatorname{Im} \widetilde{\Delta})^\perp,$$

i.e., , we have the decomposition

$$\boldsymbol{H}_f^0 = \ker \widetilde{\Delta} \oplus \operatorname{Im} \widetilde{\Delta}. \tag{18}$$

Therefore, we can define the projection $H\colon \boldsymbol{H}_f^0 \to \ker \widetilde{\Delta}$. Let $f\alpha \in \boldsymbol{H}_f^0$, then $f\alpha - H(f\alpha) \in \operatorname{Im} \widetilde{\Delta}$. So there is a unique $f\beta \in \operatorname{Im} \widetilde{\Delta}$ such that

$$\widetilde{\Delta}(f\beta) = f\alpha - H(f\alpha).$$

We define in this way the Green operator $\mathfrak{G} : f\alpha \mapsto f\beta$.

We will need the following propositions to explore the properties of the Green operator.

Proposition 3.5. *Let $\{f\alpha_n\}$ be a sequence of smooth p-forms in $\boldsymbol{H}_f^2(M, \Omega^*(M))^G$ such that $\|f\alpha_n\|_0 \leqslant c$ and $\|\widetilde{\Delta}(f\alpha_n)\|_0 \leqslant c$ for all n and for some constant $c > 0$. Then it has a Cauchy subsequence.*

Proof. We prove that $f\alpha_n$ is a bounded sequence in $\boldsymbol{H}^1_f(M, \Omega^*(M))^G$. Then we conclude the proposition by the fact that $\boldsymbol{H}^1_f(M, \Omega^*(M))^G$ is compactly embedded in $\boldsymbol{H}^0_f(M, \Omega^*(M))^G$.

We have the following equation:

$$\|f\alpha_n\|_1^2 = \|f\alpha_n\|_0^2 + \langle \Delta(f\alpha_n), f\alpha_n\rangle_0.$$

By the Cauchy–Schwarz inequality, we have

$$\langle \Delta(f\alpha_n), f\alpha_n\rangle_0 \leq \|\Delta(f\alpha_n)\|_0 \|f\alpha_n\|_0.$$

By inequality (17), we have

$$\|\Delta(f\alpha_n)\|_0 \leq \|f\alpha_n\|_2 \leq A\|\widetilde{\Delta}(f\alpha_n)\|_0 + B\|f\alpha_n\|_0 \leq (A+B)c.$$

Therefore, $\|f\alpha_n\|_1$ is bounded by $c\sqrt{A+B+1}$. □

Now we prove the regularity for $\widetilde{\Delta}$:

Proposition 3.6. *If $f\beta \in \boldsymbol{H}^k_f(M, \Omega^*(M))^G$ and*

$$\widetilde{\Delta}(f\alpha) = f\beta$$

on M, then $f\alpha \in \boldsymbol{H}^{k+2}_f(M, \Omega^*(M))^G$ for any $k \geq 0$. In particular, if $f\beta$ is a smooth differential form, so is $f\alpha$.

Proof. As f is smooth and compactly supported, it is sufficient to prove the differentiability of α. This is a local statement. As α and β are both G-invariant and $G \cdot U = M$, we can restrict our analysis to U.

By (4) and (2),

$$\begin{aligned}
\widetilde{\Delta}(f\alpha) &= f\Delta\alpha - d_f(P_f(2i_{\nabla f}\alpha)) - P_f(2i_{\nabla f}d\alpha) \\
&= f\Delta\alpha - d_f\Big(2f\int_G \chi(g)\frac{g^*f}{A^2}g^*(i_{\nabla f}\alpha)dg\Big) \\
&\quad - 2f\int_G \chi(g)\frac{g^*f}{A^2}g^*(i_{\nabla f}d\alpha)dg.
\end{aligned}$$

Due to the G-invariance of α (thus of $d\alpha$), we can find two G-invariant smooth vector fields V_1, V_2 (which depend only on f) such that

$$\widetilde{\Delta}(f\alpha) = f\Delta\alpha + fd(i_{V_1}\alpha) + fi_{V_2}d\alpha. \tag{19}$$

Notice that on U, $f = 1$. Hence equation (19) implies that

$$\beta = \Delta\alpha + d(i_{V_1}\alpha) + i_{V_2}d\alpha.$$

on U. The last two terms are of lower order, so the regularity of $\widetilde{\Delta}$ is a consequence of that of Δ, the usual Laplace–Beltrami operator on M. □

The Green operator $\mathfrak{G}$ has the following properties:

1° $\mathfrak{G}$ is bounded. To this end we need to prove the existence of a constant $c > 0$ such that for any $f\beta \in \operatorname{Im}\widetilde{\Delta}$,

$$\|f\beta\|_0 \leqslant c\|\widetilde{\Delta}(f\beta)\|_0.$$

Suppose the contrary, then there exists a sequence $f\beta_j \in \operatorname{Im}\widetilde{\Delta}$ with

$$\|f\beta_j\|_0 = 1 \quad \text{and} \quad \|\widetilde{\Delta}(f\beta_j)\|_0 \to 0.$$

By Proposition 3.5, $\{f\beta_j\}$ has a Cauchy subsequence, which one can assume to be $\{f\beta_j\}$ itself without loss of generality. Hence $\lim_{j\to\infty}\langle f\beta_j,\ f\psi\rangle_0$ exists for each $f\psi \in \boldsymbol{H}_f^0(M, \Omega^*(M))^G$. It defines a linear functional l which is clearly bounded, and

$$l(\widetilde{\Delta}(f\varphi)) = \lim_{j\to\infty}\langle f\beta_j,\ \widetilde{\Delta}(f\varphi)\rangle_0 = \lim_{j\to\infty}\langle\widetilde{\Delta}(f\beta_j),\ f\varphi\rangle_0 = 0. \tag{20}$$

We obtain the existence of $f\beta \in (\operatorname{Im}\widetilde{\Delta})^{\perp} = \ker\widetilde{\Delta}$ such that

$$l(f\psi) = \langle f\beta,\ f\psi\rangle_0 \quad \text{with } f\beta_j \to f\beta \text{ in } \boldsymbol{H}_f^0(M, \Omega^*(M))^G.$$

From equation (20), we know that $f\beta$ is a weak solution of $\widetilde{\Delta}(\xi) = 0$. It follows from Proposition 3.6 that $f\beta$ is actually smooth and a strong solution of $\widetilde{\Delta}(\xi) = 0$. Now as $\|f\beta_j\|_0 = 1$ and $f\beta_j \in \operatorname{Im}\widetilde{\Delta}$, it follows that $\|f\beta\|_0 = 1$ and $f\beta \in \operatorname{Im}\widetilde{\Delta}$. Hence, $f\beta \in \operatorname{Im}\widetilde{\Delta} \cap \operatorname{Im}\widetilde{\Delta}^{\perp} = \{0\}$, which yields a contradiction.

2° $\mathfrak{G}$ is self-adjoint. In fact,

$$\begin{aligned}
\langle\mathfrak{G}(f\alpha),\ f\beta\rangle_0 &= \langle\mathfrak{G}(f\alpha),\ f\beta - H(f\beta)\rangle_0\\
&= \langle\mathfrak{G}(f\alpha),\ \widetilde{\Delta}(\mathfrak{G}(f\beta))\rangle_0\\
&= \langle\widetilde{\Delta}(\mathfrak{G}(f\alpha)),\ \mathfrak{G}(f\beta)\rangle_0\\
&= \langle f\alpha - H(f\alpha),\ \mathfrak{G}(f\beta)\rangle_0\\
&= \langle f\alpha,\ \mathfrak{G}(f\beta)\rangle_0.
\end{aligned}$$

3° G maps a bounded sequence into one with Cauchy subsequences, due to the fact that the embedding of $\boldsymbol{H}_f^2$ into $\boldsymbol{H}_f^0$ is compact.

Moreover, we have

Proposition 3.7. *The Green operator $\mathfrak{G}$ commutes with any linear operator that commutes with $\widetilde{\Delta}$.*

Proof. Suppose that $T\colon f\Gamma(M, \Omega^p(M))^G \to f\Gamma(M, \Omega^q(M))^G$ commutes with $\widetilde{\Delta}$. Let π_p denote the projection of $\boldsymbol{H}_f^0(M, \Omega^p(M))^G$ onto $\ker\widetilde{\Delta}$. By definition, on $\boldsymbol{H}_f^0(M, \Omega^p(M))^G$,

$$\mathfrak{G} = (\widetilde{\Delta}|_{\operatorname{Im}\widetilde{\Delta}})^{-1} \circ \pi_p.$$

Now $T\widetilde{\Delta} = \widetilde{\Delta}T$ implies that $T(\ker\widetilde{\Delta}) \subset \ker\widetilde{\Delta}$ and $T(\operatorname{Im}\widetilde{\Delta}) \subset \operatorname{Im}\widetilde{\Delta}$. Hence

$$T \circ \pi_p = \pi_p \circ T.$$

On the other hand,

$$\begin{aligned} T \circ (\widetilde{\Delta}|_{\operatorname{Im}\widetilde{\Delta}}) &= T \circ \widetilde{\Delta} \circ (1 - \pi_p) \\ &= \widetilde{\Delta} \circ T \circ (1 - \pi_p) \\ &= \widetilde{\Delta} \circ (1 - \pi_p) \circ T \\ &= (\widetilde{\Delta}|_{\operatorname{Im}\widetilde{\Delta}}) \circ T. \end{aligned}$$

So on $\operatorname{Im}\widetilde{\Delta}$,

$$T \circ (\widetilde{\Delta}|_{\operatorname{Im}\widetilde{\Delta}})^{-1} = (\widetilde{\Delta}|_{\operatorname{Im}\widetilde{\Delta}})^{-1} \circ T.$$

Therefore $\mathfrak{G}$ commutes with T. □

Finally we have

Proposition 3.8. *Let $\mathfrak{H}^*(M)^G$ denote the kernel of the operator $\widetilde{\Delta}$. The map H induces an isomorphism $H\colon H^p(\Omega^*(M)^G, d) \to \mathfrak{H}^*(M)^G$.*

Remark 3.9. We remark that every element in $\mathfrak{H}^*(M)^G$ is of the form $f\alpha$, where α is a G-invariant closed form. For $f\alpha \in \mathfrak{H}^*(M)^G$, as $\widetilde{\Delta}(f\alpha) = 0$,

$$0 = \langle \widetilde{\Delta}(f\alpha), f\alpha \rangle_0 = \langle d_f(f\alpha), d_f(f\alpha)\rangle_0 + \langle d_f^*(f\alpha), d_f^*(f\alpha)\rangle_0.$$

We conclude that $d_f(f\alpha) = fd\alpha = 0$ and $d_f^*(f\alpha) = 0$. As α is G-invariant, we conclude that $d\alpha = 0$.

Proof. Suppose that α is a G-invariant smooth closed p-form on M. Consider $f\alpha \in \boldsymbol{H}_f^0(M, \Omega^p(M))^G$. Since $d\alpha = 0$, it follows that $d_f(f\alpha) = 0$. We have the following decomposition (18):

$$f\alpha = d_f d_f^* \mathfrak{G}(f\alpha) + d_f^* d_f \mathfrak{G}(f\alpha) + H(f\alpha).$$

Since d_f commutes with $\widetilde{\Delta}$, it commutes also with $\mathfrak{G}$, so

$$f\alpha = d_f d_f^* \mathfrak{G}(f\alpha) + d_f^* \mathfrak{G}(d_f(f\alpha)) + H(f\alpha).$$

Thus if $f\alpha$ is closed for d_f (i.e., $d\alpha = 0$), then

$$f\alpha = d_f d_f^* \mathfrak{G}(f\alpha) + H(f\alpha). \tag{21}$$

We define $H(\alpha)$ to be $H(f\alpha)$.

If $\alpha = d\beta$, then we have

$$d_f(f\beta) = f\alpha.$$

As the Green operator $\mathfrak{G}$ commutes with d_f, it follows that $d_f d_f^* \mathfrak{G}(d_f(f\beta)) = d_f d_f^* d_f(\mathfrak{G}(f\beta)) = d_f(d_f^* d_f + d_f d_f^*)\mathfrak{G}(f\beta) = d_f(\widetilde{\Delta}\mathfrak{G}(f\beta)) = d_f(f\beta - H(f\beta))$. Notice that elements in $\ker \widetilde{\Delta}$ are d_f-closed. So we have $d_f d_f^* \mathfrak{G}(f\alpha) = d_f(f\beta) = f\alpha$, which shows that $H(f\alpha) = 0$. This means that H is a well-defined map from $H^p(\Omega^*(M)^G, d)$ to $\mathfrak{H}^*(M)^G$.

If $H(f\alpha) = 0$, then by equation (21), we have $f\alpha = d_f d_f^* \mathfrak{G}(f\alpha)$. By Proposition 3.6, we can write $d_f^* \mathfrak{G}(f\alpha) = f\beta$ for a G-invariant smooth form β. Then $f\alpha = f d\beta$ and $\alpha = d\beta$. This implies that H is injective.

By the regularity property for $\widetilde{\Delta}$ (Proposition 3.6), the elements in $\mathfrak{H}^*(M)^G$ are all smooth. Furthermore, all elements in $\ker \widetilde{\Delta}$ vanish under d_f. So every element in $\mathfrak{H}^*(M)^G$ can be written as $f\alpha$, where α is a G-invariant smooth closed form. Since the image under H of $f\alpha \in \ker \widetilde{\Delta}$ is $f\alpha$, we conclude that H is onto. □

Theorem 1.1 is a corollary of Proposition 2.2, 3.2, and 3.8.

4. Euler characteristic of a proper cocompact action

The finite dimensionality of the de Rham cohomology groups of G-invariant differential forms allows us to define the *Euler characteristic* of such a proper cocompact G-action:

$$\chi(M; G) := \sum_{i=0}^{n} (-1)^i \dim H^i(\Omega^*(M)^G, d).$$

Our main result in this section is the following

Theorem 4.1. *Let M be a n-dimensional manifold on which a Lie group G acts properly and cocompactly.*

(i) *We consider a family of twisted G-invariant de Rham differential operators defined on $\Omega^*(M)^G$:*

$$d_{A,k}(\alpha) = A^{-k} d(A^k\alpha) = (d + kA^{-1}dA\wedge)\alpha \text{ such that } d_{A,k}^2 = 0.$$

Define the twisted cohomlogies of G-invariant differential forms on M to be the cohomology of the differential $d_{A,k}$, which is denoted by $H^{p,k}(\Omega^*(M)^G; d)$. The cohomology group $H^{p,k}(\Omega^*(M)^G; d)$ is finite dimensional.

(ii) *The Poincaré duality theorem holds for $k = 1$, there is a non-degenerate pairing between $H^{p,1}(\Omega^*(M)^G; d)$ and $H^{n-p,1}(\Omega^*(M)^G; d)$. As a corollary, when n is odd, the Euler characteristic $\chi(M; G)$ of the proper cocompact G-action on M is* 0.

(iii) *When there is a nowhere vanishing G-invariant vector field on M, the Euler characteristic of the G-action on M is* 0.

We remark that, by Proposition 3.1, $A^{-1}dA = d\log(A)$ is G-invariant. Therefore $d_{A,k}$ is well defined on $\Omega^*(M)^G$.

Proof. Our proof of statement (i) is a copy of the proof of Theorem 1.1. We define an operator $d_{f,A,k}$ on $\boldsymbol{H}_f^1$ generalizing (3) by

$$d_{f,A,k}(f\alpha) = fA^{-k}d(A^k\alpha) = f(d + kA^{-1}dA\wedge)\alpha.$$

We compute the adjoint of $d_{f,A,k}$. For two G-invariant differential forms α and β:

$$\begin{aligned}
\langle d^*_{f,A,k}(f\alpha),\ f\beta\rangle_0 &= \langle f\alpha,\ d_{f,A,k}(f\beta)\rangle_0 = \langle f\alpha,\ fA^{-k}d(A^k\beta)\rangle_0\\
&= \langle f^2A^{-k}\alpha,\ d(A^k\beta)\rangle_0 = \langle \delta(f^2A^{-k}\alpha),\ A^k\beta\rangle_0\\
&= \langle f^2A^{-k}\delta\alpha - A^{-k}2f\,i_{\nabla f}\alpha - f^2(-k)A^{-k-1}i_{\nabla A}\alpha,\ A^k\beta\rangle_0\\
&= \langle f\delta\alpha - 2i_{\nabla f}\alpha + kfA^{-1}i_{\nabla A}\alpha,\ f\beta\rangle_0\\
&= \langle f\delta\alpha - P_f(2i_{\nabla f}\alpha) + kfA^{-1}i_{\nabla A}\alpha,\ f\beta\rangle_0,
\end{aligned}$$

and the G-invariance of α implies that

$$\begin{aligned}
P_f(2i_{\nabla f}\alpha)(x) &= \frac{f(x)}{(A(x))^2}\int_G \chi(g)f(gx)2i_{\nabla f}g^*(\alpha)(x)dg\\
&= \frac{f(x)}{(A(x))^2}(i_{\int_G \chi(g)2f(gx)\nabla f(gx)dg}\alpha)(x)\\
&= \frac{f(x)}{(A(x))^2}(i_{\nabla(\int_G \chi(g)(f(gx))^2dg)}\alpha)(x)\\
&= \frac{f(x)}{(A(x))^2}(i_{\nabla A^2}\alpha)(x)\\
&= (fA^{-2}i_{\nabla A^2}\alpha)(x)\\
&= 2(fA^{-1}i_{\nabla A}\alpha)(x).
\end{aligned}$$

Hence

$$d^*_{f,A,k}(f\alpha) = f(\delta + (k-2)A^{-1}i_{\nabla A})\alpha.$$

Now we define an operator from $\boldsymbol{H}_f^2$ to $\boldsymbol{H}_f^0$:

$$\widetilde{\Delta}_k = d_{f,A,k}d^*_{f,A,k} + d^*_{f,A,k}d_{f,A,k}.$$

The analogues of Propositions 3.2–3.8 for cohomology $H^{p,k}(\Omega^*(M)^G; d)$ and $\widetilde{\Delta}_k$ in the Section 3 easily generalize. Therefore any class in $H^{p,k}(\Omega^*(M)^G; d)$ has a unique generalized harmonic form representative $f\alpha$, i.e.,

$$d_{f,A,k}(f\alpha) = d^*_{f,A,k}(f\alpha) = 0.$$

This proves that the dimension of $H^{p,k}(\Omega^*(M)^G; d)$ is finite dimensional for any p, k.

For statement (ii), we prove that the pairing between $\boldsymbol{H}_f^2(M, \Omega^p(M))^G$ and $\boldsymbol{H}_f^2(M, \Omega^{n-p}(M))^G$ induces a non-degenerate pairing on the space of generalized harmonic forms of $\widetilde{\Delta}_1$, i.e.,

$$(f\alpha, f\beta) := \int_M f\alpha \wedge f\beta. \tag{22}$$

We prove that the Hodge star operator $*$ defines an isomorphism between the space of generalized harmonic p-forms to the space of generalized harmonic $(n-p)$-forms for the operator $\widetilde{\Delta}_1$, which implies that the non-degeneracy of the pairing (22).

We prove that $\widetilde{\Delta}_1 * = * \widetilde{\Delta}_1$, which implies that the Hodge star operator $*$ defines an isomorphism between the generalized harmonic forms.

Using the following equations, where α is a G-invariant p-form,

$$\begin{aligned} \delta\alpha &= (-1)^{np+n+1} * d * \alpha, \\ * * \alpha &= (-1)^{p(n-p)}\alpha, \\ i_{\nabla A} * (*\alpha) &= (-1)^{n-p} * (dA \wedge *\alpha), \end{aligned}$$

we can check that

$$\begin{aligned} d_f * (f\alpha) &= (-1)^p * f\delta\alpha, \\ f(\delta * \alpha) &= (-1)^{p+1} * fd\alpha, \\ f i_{\nabla A} * \alpha &= (-1)^p * (dA \wedge \alpha), \\ f dA \wedge (*\alpha) &= (-1)^{p+1} * (i_{\nabla A}\alpha). \end{aligned}$$

Combining the above equations, we have

$$d_{f,A,1}(*f\alpha) = (-1)^p * d_{f,A,1}^*(f\alpha), \quad d_{f,A,1}^*(*f\alpha) = (-1)^{p+1} * d_{f,A,1}(f\alpha).$$

In particular, we have

$$* \widetilde{\Delta}_1 = \widetilde{\Delta}_1 * . \tag{23}$$

Equation (23) shows that the Hodge star operator commutes with the generalized Laplace operator $\widetilde{\Delta}_1$. Therefore, the Hodge star operator $*$ defines an isomorphism between the kernels of $\widetilde{\Delta}_1$ on $\boldsymbol{H}_f^2(M, \Omega^p(M))^G$ and $\boldsymbol{H}_f^2(M, \Omega^{n-p}(M))^G$. Hence we have the Poincaré duality for the cohomology groups $H^{p,1}(\Omega^*(M)^G; d)$:

$$H^{p,1}(\Omega^*(M)^G; d) \cong H^{n-p,1}(\Omega^*(M)^G; d)^*.$$

For the statement about the Euler characteristic, we first notice that $\chi(M; G)$ is the index of the Fredholm operator

$$d_f + d_f^* : \boldsymbol{H}_f^0(M, \Omega^{\text{even}}(M))^G \to \boldsymbol{H}_f^0(M, \Omega^{\text{odd}}(M))^G.$$

Since $\{d_{f,A,k} + d^*_{f,A,k}\}$ is a continuous family of Fredholm operators with respect to $k \in \mathbf{R}$, their indices are all the same. This implies that

$$\begin{aligned}\chi(M;G) &= \operatorname{index}(d_{f,A,k} + d^*_{f,A,k} \colon \boldsymbol{H}^0_f(M,\Omega^{\mathrm{even}}(M))^G \to \boldsymbol{H}^0_f(M,\Omega^{\mathrm{odd}}(M))^G) \\ &= \sum_{i=0}^{n}(-1)^i \dim H^{p,k}(\Omega^*(M)^G, d).\end{aligned}$$

When the dimension n of the manifold M is odd, the Poincaré duality for $H^{p,1}(\Omega^*(M)^G; d)$ implies $\chi(M;G) = 0$. Statement (ii) is thus proved.

Remark 4.2. When the Lie group G is unimodular, then by replacing f by $\frac{f}{A}$, one can make all the differential operators $d_{f,A,k}$ as well as the cohomology groups $H^{p,k}$ independent of k, therefore we have actually the Poincaré duality for $H^p(\Omega^*(M)^G; d)$.

Now statement (iii), we suppose that there is a nowhere vanishing G-invariant vector field V on M. Without loss of generality, we may assume that $|V| \equiv 1$ everywhere on M. We assume that $\{e_i\}$ is an orthonormal basis of TM, and ∇^{TM} the Levi-Civita connection of the G-invariant Riemannian metric.

Following [1], we calculate (cf. [16], p. 73):

$$\begin{aligned}\hat{c}(V)(d_f + d_f^*)\hat{c}(V)(f\alpha) &= f(\hat{c}(V)(d+\delta)\hat{c}(V)\alpha - 2A^{-1}\hat{c}(V)i_{\nabla A}(\hat{c}(V)\alpha)) \\ &= f(-(d+\delta)\alpha + \hat{c}(V)\sum_{i=1}^{n} c(e_i)\hat{c}(\nabla^{TM}_{e_i}V)\alpha \\ &\quad - 2A^{-1}(V^*\wedge + i_V)(V(A)\alpha - V^*\wedge i_{\nabla A}\alpha + i_{\nabla A} i_V\alpha)) \\ &= f(-(d+\delta)\alpha + \hat{c}(V)\sum_{i=1}^{n} c(e_i)\hat{c}(\nabla^{TM}_{e_i}V)\alpha \\ &\quad + 2A^{-1}i_{\nabla A}\alpha - 2A^{-1}(V(A)\hat{c}(V)\alpha + \hat{c}(V)i_{\nabla A}i_V\alpha)) \\ &= -(d_f + d_f^*)(f\alpha) + f(\hat{c}(V)\sum_{i=1}^{n} c(e_i)\hat{c}(\nabla^{TM}_{e_i}V)\alpha \\ &\quad - 2A^{-1}(V(A)\hat{c}(V)\alpha + \hat{c}(V)i_{\nabla A}i_V\alpha)).\end{aligned} \tag{24}$$

In the above formula, $c(v)$ and $\hat{c}(V)$ are the Clifford operators of the vector field V on the spaces Ω^{odd} and Ω^{even}. More explicitly, if V^* is the 1-form dual to the vector field V with respect to the Riemannian metric, then

$$c(V)(\alpha) = V^*\wedge\alpha - i_V\alpha, \quad \hat{c}(V)(\alpha) = V^*\wedge\alpha + i_V\alpha, \quad \alpha \in \Omega^*(M).$$

The above computation (24) shows that the difference between $\hat{c}(V)(d_f + d_f^*)\hat{c}(V)$ and $-(d_f + d_f^*)$ is an operator of order 0. Proposition 3.2 generalizes directly to this operator and states that $\hat{c}(V)(d_f + d_f^*)\hat{c}(V)$ is a Fredholm operator from

$\boldsymbol{H}_f^0(M, \Omega^{\text{odd}}(M))^G$ to $\boldsymbol{H}_f^0(M, \Omega^{\text{even}}(M))^G$. Furthermore, as the difference between $\hat{c}(V)(d_f + d_f^*)\hat{c}(V)$ and $-(d_f + d_f^*)$ has an order less than the order of $-(d_f + d_f^*)$, one can also prove that the operator

$$\begin{gathered}-(d_f + d_f^*) + \epsilon(\hat{c}(V)(d_f + d_f^*)\hat{c}(V) + (d_f + d_f^*)): \\ \boldsymbol{H}_f^0(M, \Omega^{\text{odd}}(M))^G \to \boldsymbol{H}_f^0(M, \Omega^{\text{even}}(M))^G\end{gathered}$$

is a Fredholm operator for any $\epsilon \in \mathbb{R}$.

By the stability of index of Fredholm operators, we have

$$\begin{aligned}\chi(M; G) &= \operatorname{index}(d_f + d_f^* : \boldsymbol{H}_f^0(M, \Omega^{\text{even}}(M))^G \to \boldsymbol{H}_f^0(M, \Omega^{\text{odd}}(M))^G) \\ &= \operatorname{index}(\hat{c}(V)(d_f + d_f^*)\hat{c}(V): \\ &\qquad \boldsymbol{H}_f^0(M, \Omega^{\text{odd}}(M))^G \to \boldsymbol{H}_f^0(M, \Omega^{\text{even}}(M))^G) \\ &= \operatorname{index}(-(d_f + d_f^*) + \text{lower order terms}: \\ &\qquad \boldsymbol{H}_f^0(M, \Omega^{\text{odd}}(M))^G \to \boldsymbol{H}_f^0(M, \Omega^{\text{even}}(M))^G) \\ &= \operatorname{index}(-(d_f + d_f^*) : \boldsymbol{H}_f^0(M, \Omega^{\text{odd}}(M))^G \to \boldsymbol{H}_f^0(M, \Omega^{\text{even}}(M))^G) \\ &= -\chi(M; G).\end{aligned}$$

We conclude that $\chi(M; G) = 0$. □

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X. Tang, Department of Mathematics, Washington University, St. Louis, MO 63130-4899, U.S.A.

E-mail: xtang@math.wustl.edu

Y.-J. Yao, School of Mathematical Sciences, Fudan University, Shanghai 200433, P. R. China

E-mail: yaoyijun@fudan.edu.cn

W. Zhang, Chern Institute of Mathematics and LPMC, Nankai University, Tianjin 300071, P. R. China

E-mail: weiping@nankai.edu.cn