AN L^2–ALEXANDER–CONWAY INVARIANT FOR KNOTS AND THE VOLUME CONJECTURE *

WEIPING LI

Department of Mathematics, Oklahoma State University
Stillwater, Oklahoma 74078-0613
U. S. A
E-mail address: wli@math.okstate.edu

WEIPING ZHANG

Nankai Institute of Mathematics & LPMC
Nankai University, Tianjin 300071
P. R. China
E-mail address: weiping@nankai.edu.cn

Dedicated to the memory of Professor Shiing–Shen Chern

1. Introduction

In this paper, we focus on the L^2–Alexander invariant defined in [13.14] from the twisted Alexander invariant point of view. The Alexander polynomial is a knot invariant discovered by J. W. Alexander [1] in 1928. The Alexander polynomial remained the only known knot polynomial until the Jones polynomial was discovered by V. Jones [8] in 1984. It is well–known that the Alexander polynomial plays an important role in the theory of knots.

The paper is organized as follows. In §1, we review the twisted Alexander polynomials. The necessary background on the L^2–invariant is given in §2. An L^2–analogous of the Alexander–Conway invariant for knots is

*2000 *Mathematics Subject Classification.* Primary 57M25, 57M27; Secondary 58J52, 46L

Key words and phases. Braid, Knot, Fuglede–Kadison determinant, Alexander–Conway invariant, l^2–Reidemeister torsion, volume conjecture.

W.L is partially supported by NSF and W.Z is partially supported by MOEC and the 973 project.

presented in §3. A possible relation between our L^2–Alexander invariant and the volume conjecture is discussed in the last section.

Let L be a link in S^3 with $\mu(L)$–components and exterior $X = S^3 \setminus L$. Let P be a base point of X and $p : \tilde{X} \to X$ be the maximal Abelian covering space with $\pi_1(X) \xrightarrow{\alpha} H_1(X) \cong \mathbb{Z}^{\mu(L)}$. The module $H_1(\tilde{X}, \mathbb{Z})$ depends only on the fundamental group of X. Any generator of the ideal order $H_1(\tilde{X}, \mathbb{Z})$ is called the *Alexander polynomial* $\Delta_L(t)$ of $\pi_1(X)$ (see [1]).

A twisted version of the Alexander polynomial has been introduced and studied first by Lin [15] from the Seifert surface point of view. Wada defined twisted Alexander polynomial via the free calculus method for Wirtinger presentations of knots in [22]. Using the twisted homology of the maximal Abelian covering space, Kirk and Livingston [11] defined a version of twisted Alexander polynomial via the ideal order in certain module.

Let ρ be a representation of $\pi_1(X)$ on a finitely generated free module V over some unique factorization domain R. Choosing a basis for V with $\dim_R V = N$, ρ can be realized as a homomorphism $\rho : \pi_1(X) \to \mathrm{Aut}(V) = GL_N(R)$. The associated ring homomorphism is

$$\overline{\rho} : \mathbb{Z}\pi_1(X) \to \mathbb{Z}GL_N(R) = M_N(R),$$

where $M_N(R)$ is the matrix algebra.

The twisted version of Alexander polynomials defined in [22] is by working on the following group ring homomorphism

$$\mathbb{Z}F_n \xrightarrow{\psi} \mathbb{Z}\pi_1(X) \xrightarrow{\overline{\rho}\otimes\alpha} M_N(R) \otimes \mathbb{Z}G \cong M_N(R[t_1^{\pm 1}, \cdots, t_{\mu(L)}^{\pm 1}]). \tag{1}$$

Denote $\Phi = (\overline{\rho} \otimes \alpha) \circ \psi$ and $R[G] = R[t_1^{\pm 1}, \cdots, t_{\mu(L)}^{\pm 1}]$. The matrix $\Phi(\frac{\partial r_j}{\partial x_i})(1 \leq i \leq n, 1 \leq j \leq m)$ is called the Alexander matrix of $\pi_1(X)$ associated to the representation ρ. The matrix $\Phi(\frac{\partial r_j}{\partial x_i})$ is a presentation matrix of $H_1(\tilde{X}, \tilde{P})$ as $M_N(R[G])$–module. The twisted Alexander module of L associated to ρ is the $R[G]$–module $A(L, \rho) = H_1(\tilde{X}, \tilde{P}; R[G]^N)$.

For a Wirtinger presentation of $\pi_1(X)$ of the link complement in S^3, one has $\pi_1(X) = \{x_1, \cdots, x_n | r_1, \cdots, r_{n-1}\}$ and hence each matrix M_j is a square matrix. So

$$\Delta_{L,\rho}(t_1, \cdots, t_{\mu(L)}) = \frac{\det M_j}{\det(\Phi(x_j) - \mathrm{Id})}, \tag{2}$$

where the matrix M_j is a $(n-1) \times (n-1)$ minor of the Jacobian $\Phi(\frac{\partial r_j}{\partial x_i})_{n \times n}$ for the Wirtinger presentation of a knot group.

The twisted Alexander polynomial $\Delta_{L,\rho}$ is independent of the choice of the presentation of $\pi_1(X)$ by Theorem 1 and Theorem 2 of [22]. The definition works for any finitely presentable group (see [22]). In general, the twisted Alexander polynomial is a rational function.

Note that both the Kinoshita–Terasaka knot and the Conway's 11 crossing knot have the same trivial Alexander polynomial and different twisted Alexander polynomial by [22]. Kitano [12] interprets these twisted invariants in terms of Reidemeister torsions along the lines in [18].

2. L^2–invariants

Let Γ be a finitely generated discrete (infinite) group. Let $l^2(\Gamma)$ be the standard Hilbert space of squared summable formal sums over Γ with complex coefficients. An element in $l^2(\Gamma)$ can be written as $a = \sum_{\gamma\in\Gamma} a_\gamma \gamma$ with $a_\gamma \in \mathbf{C}$ and $\sum_{\gamma\in\Gamma} |a_\gamma|^2 < +\infty$. If $a = \sum_{\gamma\in\Gamma} a_\gamma\gamma$ and $b = \sum_{\gamma\in\Gamma} b_\gamma\gamma$ are two elements in $l^2(\Gamma)$, then their inner product is given by $\langle a, b\rangle = \sum_{\gamma\in\Gamma} a_\gamma \bar{b}_\gamma$.

The left multiplication with elements in Γ defines a natural unitary action of Γ on $l^2(\Gamma)$. The group von Neumann algebra $\mathcal{N}(\Gamma)$ is the algebra of Γ–equivariant bounded linear operators from $l^2(\Gamma)$ to $l^2(\Gamma)$. The von Neumann trace on $\mathcal{N}(\Gamma)$ is defined by

$$\mathrm{Tr}_\tau : \mathcal{N}(\Gamma) \to \mathbf{C}, \quad f \mapsto \langle f(e), e\rangle, \tag{3}$$

where $e \in \Gamma \subset l^2(\Gamma)$ is the unit element. The right multiplication induces a natural action of Γ on $l^2(\Gamma)$ commuting with the left multiplication of Γ. Thus $\Gamma \subset \mathcal{N}(\Gamma)$. Moreover, for any $\gamma \in \Gamma \subset \mathcal{N}(\Gamma)$, $\mathrm{Tr}_\tau[\gamma] = 1$ if $\gamma = e$ and $\mathrm{Tr}_\tau[\gamma] = 0$ if $\gamma \neq e$. For any positive integer n, set $l^2(\Gamma)^{[n]} = \underbrace{l^2(\Gamma) \oplus \cdots \oplus l^2(\Gamma)}_{n}$.

We call $l^2(\Gamma)^{[n]}$ a free $\mathcal{N}(\Gamma)$–Hilbert module of rank n. A morphism between two free $\mathcal{N}(\Gamma)$–Hilbert modules is a Γ–equivariant bounded linear map between them. Let $f : l^2(\Gamma)^{[n]} \to l^2(\Gamma)^{[n]}$ be such a morphism. Let e_i $(i = 1, \cdots, n)$ be the unit element in the i–th copy of $l^2(\Gamma)$ in $l^2(\Gamma)^{[n]}$. Then we can extend the von Neumann trace in (3) to define

$$\mathrm{Tr}_\tau[f] = \sum_{i=1}^{n} \langle f(e_i), e_i\rangle. \tag{4}$$

The Fuglede–Kadison determinant $\mathrm{Det}_\tau(f)$ of f can be defined as follows:

(i) If f is invertible and f^* is the adjoint of f, then define (cf. [4, Definition] and [16, Lemma 3.15 (2)])

$$\mathrm{Det}_\tau(f) = \exp\left(\frac{1}{2}\mathrm{Tr}_\tau\left[\log\left(f^*f\right)\right]\right); \tag{5}$$

(ii) If f is injective, then define (cf. [4, Lemma 5] and [16, Lemma 3.15 (4), (5)])

$$\mathrm{Det}_\tau(f) = \lim_{\varepsilon\to 0^+}\sqrt{\mathrm{Det}_\tau\left(f^*f+\varepsilon\right)} = \sqrt{\mathrm{Det}_\tau\left(f^*f\right)}. \tag{6}$$

(iii) If $f : l^2(\Gamma)^{[n]} \to l^2(\Gamma)^{[n]}$ is an invertible morphism, then there exists a C^1 path f_u, $u \in [0,1]$, of invertible morphisms such that $f_0 = f$, $f_1 = \mathrm{Id}$, and (cf. [4, Theorem 1 and Lemma 2]),

$$\log\left(\mathrm{Det}_\tau\left(f\right)\right) = -\mathrm{Re}\left(\int_0^1 \mathrm{Tr}_\tau\left[f_u^{-1}\frac{df_u}{du}\right]du\right). \tag{7}$$

Example: Let $\gamma \in \Gamma$ be of *infinite order*, and $|t| < 1$. It is clear that $\mathrm{Id} - t\gamma \in \mathcal{N}(\Gamma)$ is invertible and $\mathrm{Det}_\tau(\mathrm{Id} - t\gamma) = 1$ by (7) (cf. [13]).

Let (C_*, ∂) be a finite length $\mathcal{N}(\Gamma)$–chain complex

$$(C_*, \partial) : 0 \to C_n \overset{\partial_n}{\to} C_{n-1} \overset{\partial_{n-1}}{\to} \cdots \overset{\partial_1}{\to} C_0 \to 0, \tag{8}$$

where each C_i $(0 \le i \le n)$ is a (finite rank) $\mathcal{N}(\Gamma)$ free Hilbert module. Assume that (C_*, ∂) is weakly acyclic: $\ker(\partial_i) = \overline{\mathrm{Im}(\partial_{i-1})}$, $0 \le i \le n$. Let $\partial_i^* : C_{i-1} \to C_i$ be the adjoint of $\partial_i : C_i \to C_{i-1}$. Then $\partial_i\partial_i^* : \overline{\mathrm{Im}(\partial_i)} \to \overline{\mathrm{Im}(\partial_i)}$ is injective $(0 \le i \le n)$.

We call (C_*, ∂) is of *determinant class* if $\partial_i\partial_i^* : \overline{\mathrm{Im}(\partial_i)} \to \overline{\mathrm{Im}(\partial_i)}$ $(0 \le i \le n)$ is of *determinant class* (i.e. $\mathrm{Det}_\tau(\partial_i\partial_i^*|_{\overline{\mathrm{Im}(\partial_i)}}) > 0$). In this case, **the** L^2**–Reidemeister torsion** of (C_*, ∂) is defined to be a real number $T^{(2)}(C_*, \partial)$ given by (cf. [16, Definition 3.29])

$$\log T^{(2)}(C_*, \partial) = -\frac{1}{2}\sum_{i=0}^n (-1)^i \log \mathrm{Det}_\tau\left(\partial_i\partial_i^*|_{\overline{\mathrm{Im}(\partial_i)}}\right). \tag{9}$$

Let $\rho : \pi_1(X) \to GL(H)$ be an $\mathcal{N}(\Gamma)$–linear representation of $\Gamma = \pi_1(X)$ on a (finite rank) free $\mathcal{N}(\Gamma)$ Hilbert module, where X is a finite cell complex. Let $\widetilde{X}$ be the universal covering of X. Thus the chain complex $(C_*(\widetilde{X}) \otimes H, \widetilde{\partial})$ induces canonically a chain complex $(C_*(X, H_\rho), \partial_\rho)$ in the sense of (8) with $C_*(X, H_\rho) = (C_*(\widetilde{X}) \otimes_{\pi_1(X),\rho} H)$.

If $(C_*(X, H_\rho), \partial_\rho)$ is weakly acyclic and of *determinant class*, then its L^2–Reidemeister torsion $T^{(2)}(C_*(X, H_\rho), \partial_\rho)$ as in (9) is defined. If $\rho : \pi_1(X) \to GL(H)$ is unitary, then $T^{(2)}(C_*(X, H_\rho), \partial_\rho)$ is a well–defined piecewise linear invariant.

Note that the L^2–Reidemeister torsion detects the unknot by [16, Theorem 4.7 (2)].

3. An L^2–Alexander–Conway invariant for knots

Combining the methods in §1 and §2, we provide the construction of an L^2–Alexander–Conway invariant for knots in this section. See [13,14] for more details.

Let $K \subset S^3$ be a knot. Let $\{x_1, \cdots, x_k | r_1, \cdots, r_{k-1}\}$ be a Wirtinger presentation of $\Gamma = \pi_1(S^3 \setminus K)$.

Define α to be the canonical abelianization $\alpha : \Gamma \to U(1)$ with $\alpha(x_i) = t$ for $1 \leq i \leq k$. Let $GL(l^2(\Gamma))$ denote the set of invertible elements in $\mathcal{N}(\Gamma)$. Let $\rho_\Gamma : \Gamma \to GL(l^2(\Gamma))$ denote the fundamental representation of Γ, which is given by the right multiplication of the elements in Γ. The tensor product representation $\rho \otimes \alpha$ induces a ring homomorphism of the integral group rings

$$\widetilde{\rho_\Gamma \otimes \alpha} : \mathbf{Z}[\Gamma] \to \mathcal{N}(\Gamma) \otimes \mathbf{Z}[t^{\pm 1}] \subset \mathcal{N}(\Gamma). \tag{10}$$

Let $\Psi = (\widetilde{\rho_\Gamma \otimes \alpha}) \circ \widetilde{\phi} : \mathbf{Z}[F_k] \to \mathcal{N}(\Gamma)$ be the composition of the ring homomorphisms. Consider the morphism

$$A_{\rho_\Gamma \otimes \alpha} : l^2(\Gamma)^{[k-1]} \to l^2(\Gamma)^{[k]} \tag{11}$$

which when written in the $(k-1) \times k$ matrix form, the (i,j)–component is given by

$$A_{\rho_\Gamma \otimes \alpha, (i,j)} = \Psi \left(\frac{\partial r_i}{\partial x_j} \right) \in \mathcal{N}(\Gamma) \otimes \mathbf{Z}[t^{\pm 1}] \subset \mathcal{N}(\Gamma), \tag{12}$$

where $\frac{\partial r_i}{\partial x_j}$ is the standard Fox derivative.

We call $A_{\rho_\Gamma \otimes \alpha}$ the L^2–Alexander matrix of the presentation $P(\Gamma)$ associated to the fundamental representation ρ_Γ and the representation α. In [13], we proved the following proposition.

Proposition 3.1. *(1) $\Psi(x_j - 1) \in \mathcal{N}(\Gamma)$ is injective and has dense image for any $1 \leq j \leq k$.*

(2) If one of the $A^j_{\rho_\Gamma \otimes \alpha}$'s, $1 \leq j \leq k$, is injective, then every $A^j_{\rho_\Gamma \otimes \alpha}$, $1 \leq j \leq k$, is injective.

(3) For any $1 \leq j < j' \leq k$, one has

$$\mathrm{Det}_\tau \left(A^j_{\rho_\Gamma \otimes \alpha} \right) \mathrm{Det}_\tau \left(\Psi(x_{j'} - 1) \right) = \mathrm{Det}_\tau \left(A^{j'}_{\rho_\Gamma \otimes \alpha} \right) \mathrm{Det}_\tau \left(\Psi(x_j - 1) \right). \tag{13}$$

(4) $\mathrm{Det}_\tau \left(\Psi(x_j - 1) \right) = 1$ for $1 \leq j \leq k$

(5) $\Delta_K^{(2)}(t) = \mathrm{Det}_\tau\left(A^1_{\rho\Gamma\otimes\alpha}\right)$ *is independent of the choice of the Wirtinger presentation of the knot* K.

Thus we define $\Delta_K^{(2)}(t)$ to be the L^2–*Alexander invariant* of the knot K in S^3.

When $t = 1$, $\Delta_K^{(2)}(t)$ has been studied by Lück (see [16, Theorem 4.9]), who shows that $\Delta_K^{(2)}(1)$ is equivalent to the L^2–Reidemeister torsion of $S^3 \setminus K$. In [13], we identify $\Delta_K^{(2)}(t)$ with $t \in U(1)$ as certain twisted L^2–Reidemeister torsion of $S^3 \setminus K$ (see [13, Proposition 5.1]). In view of [22, Section 5], the above construction can also be applied to links. We prove a rigidity result for the $U(1)$ twisted L^2–torsion on a knot complement in [13, Theorem 6.1].

By considering $\alpha : H_1\left(S^3 \setminus K\right) \longrightarrow \mathbf{C}^*$ with $\alpha(h) = t$, we can prove that $\mathrm{Det}_\tau\left(A^1_{\rho\Gamma\otimes\alpha}\right)$ is well–defined up to the multiplicative group $\{|t|^p\}_{p\in\mathbf{Z}}$ (see [13]). However, one can resolve this $\{|t|^p\}$ ambiguity through the following theorem.

Theorem 3.2 (Li–Zhang 2005 [14]). *The quantity*

$$\Delta_K^{(2)}(t) = \sqrt{\frac{\mathrm{Det}_\tau\left(A^1_{\rho\Gamma\otimes\alpha}\right)}{\max\{1,|t|\}} \cdot \frac{\mathrm{Det}_\tau\left(A^1_{\rho\Gamma\otimes\alpha^{-1}}\right)}{\max\left\{1,|t|^{-1}\right\}}}$$

does not depend on the choice of the Wirtinger presentation of Γ. *Moreover, it depends only on* $|t|$.

Definition 3.3. The term $\Delta_K^{(2)}(t)$ in the above theorem is called an L^2–Alexander–Conway invariant of the knot K.

By the rigidity result in [14], this definition coincides with [13, Definition 3.5] for $t \in U(1)$.

Example. Let $K = 4_1$ be the figure eight knot with its Wirtinger presentation $P(\Gamma) = \left\langle x, y | zxz^{-1}y^{-1}\right\rangle$, where $z = x^{-1}yxy^{-1}x^{-1}$. Then one has

(i) If $|t| > 4$, then $\Delta_{4_1}^{(2)}(t) = \sqrt{t}$;

(ii) If $|t| = 1$, then $\Delta_{4_1}^{(2)}(t) = \exp\left(\frac{\mathrm{vol}\left(S^3\setminus 4_1\right)}{6\pi}\right) \sim \exp\left(\frac{1}{6\pi}\cdot 2.029\right) \neq 1.$

Thus $\Delta_{4_1}^{(2)}(t)$ is a non–trivial deformation of the hyperbolic volume of 4_1. It would be interesting to study the behavior of $\Delta_K^{(2)}(t)$ on $\mathbf{R}^*$.

Let $\beta \in \mathbf{B}_k$ be a braid representative of the knot K. By Artin's theorem [3], the knot group Γ admits a presentation

$$\left\langle x_1, \ldots, x_k | \beta(x_1)x_1^{-1} = \cdots = \beta(x_{k-1})x_{k-1}^{-1} = 1 \right\rangle .$$

By proceeding similarly as in the Wirtinger presentation case, one can define the L^2–Alexander matrix denoted now by $\widetilde{A}_{\rho_\Gamma \otimes \alpha}$, and define an L^2–Alexander–Conway invariant by, for $t \in \mathbf{C}^*$,

$$\widetilde{\Delta}_K^{(2)}(t) = \sqrt{\frac{\mathrm{Det}_\tau\left(\widetilde{A}^1_{\rho_\Gamma \otimes \alpha}\right)}{\max\{1, |t|\}} \cdot \frac{\mathrm{Det}_\tau\left(\widetilde{A}^1_{\rho_\Gamma \otimes \alpha^{-1}}\right)}{\max\{1, |t|^{-1}\}}}.$$

Theorem 3.4 (Li–Zhang 2005 [14]). *(i) The L^2–Alexander–Conway invariant $\widetilde{\Delta}_K^{(2)}(t)$ does not depend on the braid representative β for the knot K. So it defines an invariant for K.*

(ii) For $t \in U(1)$, $\widetilde{\Delta}_K^{(2)}(t) = \Delta_K^{(2)}(t) \quad \left(= \Delta_K^{(2)}(1)\right)$.

Theorem 3.4 indicates the interactive relation of our L^2–invariant on the braid representatives of the knot K. It can be viewed as an L^2–analogous of the Burau theorem [3, Theorem 3.11]. It is an interesting problem to answer our expectation $\widetilde{\Delta}_K^{(2)}(t) \equiv \Delta_K^{(2)}(t)$. Note that $\widetilde{\Delta}_{4_1}^{(2)}(t) = \Delta_{4_1}^{(2)}(t) = t^{1/2}$, $\widetilde{\Delta}_{5_1}^{(2)}(t) = \Delta_{5_1}^{(2)}(t) = t^{3/2}$ for $|t| > 4$ and $\widetilde{\Delta}_{5_1}^{(2)}(1) = \Delta_{5_1}^{(2)}(1) = 1$.

4. The volume conjecture

The volume conjecture given by Kashaev [9] is derived from the theory of quantum dilogarithm to build a possible relation between the combinatorial TQFT to quantum $2+1$ dimensional gravity. H. and J. Murakami in [19] reinterpreted the Kashaev invariant in [9] as a special case of the colored Jones polynomial associated with the quantum group $SU_q(2)$ evaluated at $q = e^{2\pi i/N}$. The volume conjecture for any knot K in S^3 can be stated as the following.

$$\lim_{N \to \infty} \frac{\log |J_N(K, q)|}{N} = \frac{1}{2\pi} \mathrm{Vol}(S^3 \setminus K),$$

where the volume is the simplicial volume. The volume conjecture is true for torus knots [10] and the figure eight knot [20]. See also [5,7] for related topics.

By [13, Proposition 5.1 and Theorem 6.1], the volume conjecture can be restated as follows (cf. [16, Conjecture 4.8]),

$$\lim_{N\to+\infty}\left|J_N\left(K,\exp\left(\frac{2\pi\sqrt{-1}}{N}\right)\right)\right|^{\frac{1}{3N}}=\Delta_K^{(2)}(1). \tag{14}$$

Using the 3–dimensional Chern–Simons theory with complex gauge groups $SL_2(\mathbb{C})$, Gukov [6] derived a generalized volume conjecture

$$\lim_{N\to\infty}\frac{\log J_N(K,q)}{N}=\frac{1}{2\pi}(\mathrm{Vol}(M)+i2\pi^2\mathrm{CS}(M)).$$

By comparing (14) with the Melvin–Morton conjecture (the Melvin–Morton conjecture was proved formally in [21] and rigorously in [2]), it seems plausible to view the volume conjecture as a kind of L^2–analogue of the Melvin–Morton conjecture. This fits with the picture outlined by Gukov in [6]. In particular, the rigidity property in [13, Theorem 6.1] fits with the form of the generalized Melvin–Morton conjecture stated in [6], where the hyperbolic torsion in the right hand side of [6, (6.30)] (which should play a role of the L^2–torsion (or the L^2–Alexander invariant here) does not contain a (unitary) deformed parameter.

We would like to end our article by listing some natural questions.

(Q1) Note that the generalized volume conjecture in (5.12) of [6] can be thought as a parametrized volume conjecture via the zero locus of the A-polynomial. Is our invariant $\Delta_K^{(2)}(t)$ related to the volume $\mathrm{Vol}(\rho)$ for $\rho:\Gamma\to SL_2(C)$ in the zero locus ?

(Q2) $\Delta_K^{(2)}(t)$ is upper semi–continuous with respect to $t\in C^*$. Whether it is a continuous function or with only first kind of discontinuity ? Whether $\Delta_K^{(2)}(t)\neq 0$ for all knots ?

(Q3) It would be interesting to give a topological proof of Lück–Schick's result in [17], identifying $\Delta_K^{(2)}(1)$ with the simplicial volume of $S^3\setminus K$, up to a constant scalar. Is there a direct proof by passing Lück–Schick's result ?

(Q4) Whether there is a knot polynomial whose Mähler measure equals to the L^2–Alexander invariant $\Delta_K^{(2)}(1)$ (or equivalently, the L^2–torsion of the knot complement) ? This is Question 8.1 of [13].

References

1. J. W. Alexander, *Topological Invariants of Knots and Links*, Trans. Amer. Math. Soc. 30, 275-306, 1928.
2. D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture*, Invent. Math. **1** (1996), 103-133.

3. J. S. Birman, *Braids, links and mapping class groups*, Ann. Math. Studies, No. **82**, Princeton University Press (1974).
4. B. Fuglede and R. V. Kadison, *Determinant theory in finite factors*, Ann. of Math. **55** (1952), 520-530.
5. S. Garoufalidis and T.Q. Le, *Asymptotics of the colored Jones function of a knot*, Preprint, arXiv: math.GT/0508100.
6. S. Gukov, *Three-dimensional quantum gravity, Chern-Simons theory, and the A-polynomial*, Commun. Math. Phys. **255** (2005), 577-627.
7. V. Huynh and T.Q. Le, *On the colored Jones polynomial and the Kashaev invariant*, Preprint, arXiv: math.GT/0503296.
8. V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Annals of Math, **126** (1987), 335-388.
9. R. M. Kashaev, *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. **39** (1997), 269-275.
10. R. M. Kashaev and O. Tirkkonen, *A proof of the volume conjecture for torus knots*, translation in J. Math. Sci. (N. Y.) **115** (2003), No. 1, 2033–2036.
11. P. Kirk and C. Livingston, *Twisted Alexander invariants, Reidemeister torsion, and Casson-Gordon invariants*, Topology **38** (1999), 635-661.
12. T. Kitano, *Twisted Alexander polynomial and Reidemeister torsion*, Pacific J. Math., **174** (1996), 431-442.
13. W. Li and W. Zhang, *An L^2–Alexander invariants for knots*, to appear in Commun. Contemp. Math.
14. W. Li and W. Zhang, *An L^2–Alexander–Conway invariants for knots*, in preparation.
15. X.-S. Lin, *Representations of knot groups and twisted Alexander polynomials*, Acta Math. Sinica (English Series) **17** (2001), 361-380.
16. W. Lück, *L^2-Invariants: Theory and Applications to Geometry and K-Theory*. Springer-Verlag, 2002.
17. W. Lück and T. Schick, *L^2-torsion of hyperbolic manifolds of finite volume*, Geom. Funct. Anal. **9** (1999), 518-567.
18. J. Milnor, *A duality theorem for Reidemeister torsion*, Ann. of Math., **76** (1962), 137-147.
19. H. Murakami and J. Murakami, *The colored Jones polynomials and the simplicial volume of a knot*, Acta. Math., **186** (2001), 85-104.
20. H. Murakami, *The colored Jones polynomials of the figure-eight knot and the volumes of three-manifolds obtained by Dehn surgeries*, Fund. Math. **184** (2004), 269–289.
21. L. Rozansky, *A contribution of the trivial connection to the Jones polynomial and Witten's invariant of 3-manifolds*, Commun. Math. Phys., **175** (1996), 275-318.
22. M. Wada, *Twisted Alexander polynomial for finitely presented groups*, Topology **33** (1994), 241-256.