LOCAL ATIYAH-SINGER INDEX THEOREM FOR FAMILIES OF DIRAC OPERATORS

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§0. Introduction

The first success of proving the Atiyah-Singer index theorem directly by heat kernel method was achieved by Patodi [10], who carried out the "fantastic cancellation" (cf. [9]) for the Laplace operators and for the first time proved a local version of the Gauss-Bonnet-Chern theorem. In recent years, several different direct heat kernel proofs of the Atiyah-Singer index theorem for Dirac operators have appeared independently: Bismut [3], Getzler [6], [7] and Yu [13] or [14], see also Berline- Vergne [2]. All the proofs have their own advantages.

Motivated by the problem of generalizing the heat kernel proofs of the index theorem to prove a local index theorem for families of elliptic operators, Quillen [12] introduced the concept of superconnections, and this was developed by Bismut to give a heat kernel representation for the Chern character of families of first order elliptic operators. Then using his probabilitistic method, Bismut [4] obtained a proof of the local index theorem for families of Dirac operators.

In this paper, we will use the method of Yu [14] to give another proof of the local index theorem for families of Dirac operators. The key point is Yu's idea of comparing the corresponding terms in the Taylor expansion series of functions, thus avoiding probability and some complicated estimates.

It seems that the method in [7] can be also generalized to give a proof of the local index theorem for families of Dirac operators.

Note also that Yu [15] has presented a direct proof of the local index theorem for signature operators explicitly in the same spirit as [13]. We found that our proof can be modified immediatly to give a proof of the local index theorem for families of signature operators.

For simplicity, we write out our proof only for classical Dirac operators, but our proof works also for twisted Dirac operators, as pointed out in [13]. For a brief account cf. the Appendix.

We take Yu [13] and Chap. I-III of Bismut [4] as our basic references.

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§1. Clifford module and supertrace

Let $V = V^0 \oplus V^1$ be a super (or Z_2-graded) complex vector space. As in [12], we use ε to denote the involution giving the grading: $\varepsilon(v) = (-1)^{\deg(v)} v$. Then the space End (V) of endomorphisms of V is a super algebra. The even (resp.odd) elements of End (V) commute (resp. anticommute) with ε.

Definition 1.1 The supertrace tr_s of $k \in \mathrm{End}(V)$ is defined by

$$\mathrm{tr}_s k = \mathrm{tr}\varepsilon k.$$

It is easy to verify that

$$(1.2) \qquad \begin{aligned} &\mathrm{tr}_s k = \mathrm{tr}(k|_{V^0}) - \mathrm{tr}(k|_{V^1}), \ k \text{ even} \\ &\mathrm{tr}_s k = 0, \ k \text{ odd} \end{aligned}$$

Now let H be a Grassmann algebra, then H is naturally Z_2- graded. Let K be the graded tensor product

$$K = \mathrm{End}(V) \hat{\otimes} H$$

then K possesses a nature Z_2-grading. The supertrace concept can be naturally extended to K and takes its value in H (cf. [8], [4]).

Definition 1.3 If $h \in H$, $k \in K$, define

$$\mathrm{tr}_s(hk) = h(\mathrm{tr}_s k).$$

Now let $N = 2n$ be an even integer. $c(R^{2n})$ denotes the Clifford algebra of R^{2n} generated by $1, e_1, \cdots, e_{2n}$ with the following commutative relations:

$$e_i^2 = -1, \quad e_i e_j + e_j e_i = 0, \quad i \neq j.$$

As well known, Spin $(2n)$ acts unitarily on the 2^n-dimensional complex vector space of spinors S and that $c(R^{2n}) \otimes_R C$ can be identified with End(S), so that Spin $(2n) \subset \mathrm{End}(S)$. Set

$$\tau = (\sqrt{-1})^n e_1 \cdots e_{2n} \tag{1.4}$$

Then τ acts unitarily on S and moreover

$$\tau^2 = 1 \tag{1.5}$$

Set

$$S_+ = \{s \in S : \tau s = s\}, S_- = \{s \in S : \tau s = -s\}. \tag{1.6}$$

Then S_+, S_- are 2^{n-1}-dimensional vector subspaces of S such that

$$S = S_+ \oplus S_- \tag{1.7}$$

Spin(2n) acts unitarily and irreducibly on S_+ and S_-.

S can be naturally regarded as a super vector space with obvious grading and involution τ. Let W be a usual vector space, then $S \otimes W$ also carries a naural grading.

Lemma 1.8

$$\mathrm{tr}_s e_{i_1} \cdots e_{i_p} = 0, \quad p < 2n$$
$$\mathrm{tr}_s e_1 \cdots e_{2n} = (\frac{2}{\sqrt{-1}})^n.$$

Let m be another integer, $f_1, \cdots, f_\alpha, \cdots, f_m$ denote the canonical oriented Euclidean basis, and $dy^1, \cdots, dy^\alpha, \cdots, dy^m$ the basis of the corresponding dual space.

Definition 1.9 ϵ denote the graded tensor product of the Z_2-graded algbra $c(R^{2n})$ and $\Lambda(R^m)$, i .e .

$$\epsilon = c(R^{2n}) \hat{\otimes} \Lambda(R^m).$$

For simplicity, we will use no "$\hat{\otimes}$" sign to indicate the product in ϵ. e .g. we have

$$e_i dy^\alpha + dy^\alpha e_i = 0. \tag{1.10}$$

§2. Fibration of manifolds

A large portion of this section, which is adapted from [4], is included here only for completeness.

Let B be an m-dimensional connected compact Riemannian manifold. Denote its Riemannian metric by g_B.

X is a connected compact orientable manifold of even dimension 2n. We assume that X is a spin manifold, so that $w_2(X) = 0$.

M is a $2n + m$ dimensional compact connected manifold. We assume that $\pi : M \to B$ is a submersion of M onto B, which defines a fibration of M with fiber X. Namely, we assume that there is an open covering _U of B such that for every U∈_U, there exists a C^∞-diffeomorphism $\varphi_U : \pi^{-1}(U) \to U \times X$, and if $U \bigcap V \neq \phi$, $\varphi_U \circ \varphi_V^{-1} : (U \bigcap V) \times X \to (U \bigcap V) \times X$ is given by $(y, x) \mapsto (y, f_{U,V}(y, x))$ where $f_{U,V}(y, \cdot)$ is a C^∞-diffeomorphism of X which is C^∞ in both variables y and x.

For $y \in B$, $\pi^{-1}(y)$ is then a submanifold G_y of M, and π defines a fibration G of M. TG denote the 2n- dimensional vector bundle on M whose fiber T_xG is the tangent space at x to the fiber $G_{\pi(x)}$. And we assume that M is oriented.

By using any Riemannian structure on M, we can obtain an m-dimensional smooth sub-vector bundle T^HM of TM such that

$$TM = T^HM \oplus TG \tag{2.1}$$

In particular, $\forall x \in M$, T_x^HM and $T_{\pi(x)}B$ are isomorphic under π.

Recall that B is Riemannian, so we can lift the Euclidean scalar product g_B of TB to T^HM. And we assume that TG is endowed with a scalar product g_G. Thus we can introduce in TM a new scalar product $g_B \oplus g_G$, and denote by ∇^L the Levi-Civita connection on TM with respect to this metric.

Let O be the SO(2n) bundle of oriented orthonormal frames in TG.

We now do the assumption that the bundle TG is spin. Namely we assume that the $SO(2n)$ bundle $O \to M$ can be lifted to a Spin(2n) bundle O' so that the projection $O' \to O$ induces the covering projection Spin(2n)→SO(2n) on each fiber.

Let F, F_+, F_- be the bundle of spinors:

$$\begin{aligned} F &= O' \times_{\mathrm{Spin}(2n)} S \\ F_\pm &= O' \times_{\mathrm{Spin}(2n)} S_\pm \end{aligned} \tag{2.2}$$

Recall that if $e \in TG$, $\|e\| = 1$, e acts unitarily by Clifford multiplication on F, and interchanges F_+ and F_-.

Definition 2.3 ϵ is the bundle defined by

$$\epsilon_x = c(T_xG)\hat{\otimes}\Lambda_{\pi(x)}(B).$$

The supertrace construction of §1 can be extended obviously to give a supertrace on the superalgebra bundle ϵ.

Next we construct a connection on TG.

Definition 2.4 ∇ denotes the connection on TG defined by the following relation:

$$\nabla_Y Z = P_G(\nabla_Y^L Z), \quad Y \in TM, \quad Z \in TG$$

where P_G denote the orthogonal projection of TM on TG. ∇ obviously preserves the scalar product in TG.

∇ can be lifted to give a connection on F, F_+ and F_- respectively. Let $H^\infty, H_+^\infty, H_-^\infty$ denote the set of C^∞ sections of F, F_+, F_- over M.

Clearly, we may regard $H^\infty, H_\pm^\infty$, as the set of C^∞ sections over B of infinite dimensional vector bundles, which we still denote by $H^\infty, H_\pm^\infty$. The fibers $H_y^\infty, H_{\pm,y}^\infty$ are the sets of C^∞ sections over G_y of $F, F_\pm$.

Also note that since F is an Hermitian bundle, and since the fibers G_y carry a natural volume element dx, if $h, h' \in H_y$, we can define the scalar product

$$(2.5) \qquad < h, h' >= \int_{G_y} < h(x), h'(x) > dx.$$

We now define a connection on $H_{\pm}$.

Definition 2.6 $\tilde{\nabla}$ denotes the connection on $H_{\pm}$ such that if $Y \in TB$, $h \in H_{\pm}$, then

$$\tilde{\nabla}_Y h = \nabla_{Y^H} h$$

where Y^H is the (unique) lifting of $Y \in TB$ in $T^H M$.

§3. Dirac operators and a heat kernel formula for the index

Definition 3.1 D denotes the operator acting on H

$$D = \sum_1^{2n} e_i \nabla e_i$$

Of course the operator D acts fiberwise in the fibers G_y.

For $y \in B, D_y$ denotes the restriction of D to the fiber H_y^∞. D interchanges H_+^∞ and H_-^∞. D_+, D_- are the restrictions of D to H_+^∞, H_-^∞.

By Atiyah and Singer [1], the difference bundle over B

$$(3.2) \qquad \ker D_{+,y} - \ker D_{-,y}$$

is well defined in the sense of K-theory.

The aim of this paper is to present a calculation of the Chern character of (3.2) as a differential form over B explicitly.

First, we will give a brief review of the heat kernel representation of this Chern character given by Bismut [4].

In what follows, to simplify the notations, we will use the following conventions:

(1) All the summation signs will be omitted;

(2) The subscripts α, β will be used for horizontal variables and the subscripts i, j for vertical ones (i.e. the variables in TG);

(3) We omit H in $(f_\alpha)^H$, i.e., we identify the orthonormal basis $f_1, \cdots, f_m$ of $T_y B$ with their lift in $T_x^H M$ (for $x \in G_y$). Also, $dy^1, \cdots, dy^m$ are now considered as differential forms on M;

(4) We omit the exterior product sign $\wedge$;

(5) We omit the "$\sim$" in $\tilde{\nabla}$.

We extend $e_1, \cdots, e_{2n}; f_1, \cdots, f_m$ to give an orthonormal frame $E_1, \cdots, E_{2n}; F_1, \cdots, F_m$ in the way of [13] and pick a fixed spin frame as in [13].

Denote

$$\Gamma_{IJ}^K =< \nabla_{Z_I}^L Z_J, Z_K >$$

where Z_I is the total notation for E_i, F_α, etc.

Definition 3.3 Define

$$\begin{aligned} H =& e_i(E_i + \frac{1}{4}\Gamma_{ij}^k e_j e_k + \frac{1}{2}\Gamma_{ij}^\alpha e_j dy^\alpha + \frac{1}{4}\Gamma_{i\alpha}^\beta dy^\alpha dy^\beta) + \\ & dy^\alpha(F_\alpha + \frac{1}{4}\Gamma_{\alpha i}^j e_i e_j + \frac{1}{2}\Gamma_{\alpha i}^\beta e_i dy^\beta + \frac{1}{4}\Gamma_{\alpha\beta}^\gamma dy^\beta dy^\gamma), \\ I =& H^2. \end{aligned}$$

Let I^y be the restriction of I to G_y. For a given $y \in B$, the operator I^y acts on $H_y \hat{\otimes} \Lambda_y(B)$ in the following sense: if $h \in \mathrm{End}(F), \eta, \eta' \in \Lambda(B)$, $e \in F$, then the action of $h\eta \in \mathrm{End}(F)\hat{\otimes}\Lambda(B)$ on $e\eta' \in F\hat{\otimes}\Lambda(B)$ is given by

$$(h\eta)(e\eta') = (-1)^{\deg\eta \cdot \deg e} h(e) \cdot \eta\eta' \tag{3.4}$$

further more, if $h \in H_y^{\infty}$, $\eta \in \Lambda_y(B)$,

$$I^y(h\eta) = (I^y h)\eta. \tag{3.5}$$

As indicated in [4], using standard results on elliptic equations, we can construct the "heat kernel" semi-group e^{-tI^y} which also acts in the fiber. For any $t > 0, e^{-tI^y}$ is given by a kernel $P_t^y(x,x')$ (for $x, x' \in G_y$) which is C^∞ in $(t,x,x') \in (0,+\infty) \times G_y \times G_y$.

Since the fibration $M \to B$ is locally trivial, there is an open neighborhood of y in B such that $\pi^{-1}(U)$ is diffeomorphic to $U \times X$. In what follows, we will not distinguish $\pi^{-1}(U)$ and $U \times X$.

In particular, since I^y is a smooth family of second order elliptic differential operators, it is not difficult to prove that $P_t^y(x,x')$ is C^∞ in $(t,x,x',y) \in (0,\infty) \times X \times X \times U$, cf. [4], Proposition 2.8.

For $x, x' \in G_y$, $P_t^y(x,x')$ is a linear mapping from F_x, into $F_x \hat{\otimes} \Lambda_y(B)$.

Let τ_x be the involution defining the grading in $F_{x'}$ then $\mathrm{Hom}(F_{x'}, F_x)$ has a nature grading. The even (resp.odd) elements commute (resp. anticommute) with τ_x. Thus, $P_t^y(x,x')$ is an even element of the graded tensor product $\mathrm{Hom}(F_{x'},F_x)\hat{\otimes}\Lambda_y(B)$. In particular, $P_t^y(x,x)$ is an even element in the graded algebra $\mathrm{End}(F_x)\hat{\otimes}\Lambda_y(B)$, and $\mathrm{tr}_s P_t^y(x,x)$ is an even element in $\Lambda_y(B)$.

As in [12], we change the normalization constant in the definition of the Chern character. Namely, for a vector bundle V with connection form μ and curvature C, we set

$$\mathrm{Ch}(V) = \mathrm{Tr}exp(-C). \tag{3.6}$$

In [4], Bismut proved the following fundamental result:

Theorem 3.7 (Bismut [4]). Let H^t be given by

$$\begin{aligned} H^t =& e_i(E_i + \frac{1}{4}\Gamma_{ij}^k e_i e_j + \frac{1}{2}\Gamma_{ij}^\alpha e_j \frac{dy^\alpha}{\sqrt{t}} + \frac{1}{4t}\Gamma_{i\alpha}^\beta dy^\alpha dy^\beta) \\ &+ \frac{dy^\alpha}{\sqrt{t}}(F_\alpha + \frac{1}{4}\Gamma_{\alpha i}^j e_i e_j + \frac{1}{2}\Gamma_{\alpha i}^\beta e_i \frac{dy^\beta}{\sqrt{t}} + \frac{1}{4}\Gamma_{\alpha\beta}^\gamma \frac{dy^\beta dy^\gamma}{t}) \\ I^t =& t(H^t)^2 \end{aligned} \tag{3.8}$$

Then

$$\int_{G_y} \mathrm{tr}_s(P_1^{L,t,y}(x,x))dx \tag{3.9}$$

is a C^∞ form over B which is a representative of $\mathrm{Ch}(\ker D_{+,y} - \ker D_{-,y})$, where $P_1^{L,t,y}(x,x')$ is the C^∞ kernel over G_y of $e^{-I_1^t}$.

The goal of this paper is to calculate out

$$\mathrm{tr}_s(P_1^{L,t,y}(x,x))dx \tag{3.10}$$

§4. A local parametrix and Minakshsundaram-Pleijel equations

In this section, we shall deduce (3.10) to a calculable form, and in the next section we will carry out the explicit calculation.

In all what follows, we may keep in mind that we are fixing a typical G_y, so that the subscript y will be omitted unless necessary.

First, as in [5], $\forall t > 0$, let φ_t be the homomorphism

$$\begin{aligned} &\varphi_t : \mathrm{Hom}(F_{x'}, F_x)\hat{\otimes}\Lambda_y(B) \to \mathrm{Hom}(F_{x'}, F_x)\hat{\otimes}\Lambda_y(B) \\ &\varphi_t : hdy^\alpha \mapsto \frac{1}{\sqrt{t}} hdy^\alpha, h \in \mathrm{Hom}(F_{x'}, F_x) \end{aligned} \tag{4.0}$$

then clearly,

$$H^t = \varphi_t(H).$$

Hence

$$\begin{aligned} \mathrm{tr}_s e^{-I^t} &= \int_{G_y} \mathrm{tr}_s P_1^{L,t,y}(x,x)dx \\ &= \mathrm{tr}_s e^{-t(\varphi_t(H))^2} = \mathrm{tr}_s e^{-t\varphi_t H^2} \\ &= \mathrm{tr}_s \varphi_t e^{-tH^2} \\ &= \int_{G_y} \varphi_t \mathrm{tr}_s P_t^y(x,x)dx \end{aligned}$$

Thus we get a corollary of Theorem 3.7:

<u>Proposition 4.1</u> $\forall t > 0$,

$$\varphi_t \int_{G_y} \mathrm{tr}_s P_t^y(x,x)dx$$

is a representative of $\mathrm{Ch}(\ker D_{+,y} - \ker D_{-,y})$.

We wish to calculate out

$$\lim_{t\to 0} \int_{G_y} \mathrm{tr}_s(\varphi_t P_t^y(x,x))dx \tag{4.2}$$

Now we note that $P_t(x,x')$ is uniquely characterized by the following properties:

$$\lim_{t\to 0}(\frac{\partial}{\partial t} + I_{x'})P_t(x,x') = 0 \tag{4.3}$$

and $\forall V(x') \in H_y$,

$$\lim_{t\to 0} \int_{G_y} P_t(x,x')V(x')dx' = V(x) \tag{4.4}$$

<u>Proof</u>: Recalling that

$$e^{-tI}V(x) = \int_{G_y} P_t(x,x')V(x')dx'$$

so (4.3) and (4.4) are clearly hold. Now let $G_t(x,x')$ be another C^∞ function satisfying (4.3) and (4.4), we have

$$(\frac{\partial}{\partial t} + I_{x'})(P_t(x,x') - G_t(x,x')) = 0 \tag{4.5a}$$

and $\forall V \in H_y$,

$$\lim_{t\to 0}\int_{G_y}(P_t(x,x')-G_t(x,x'))V(x')dx'=0 \tag{4.5b}$$

From (4.5a,b) it is obvious that we should have

$$P_t(x,x')=G_t(x,x').$$

Definition 4.6 $\forall t>0$, set

$$H_N(x,x';t)=\frac{e^{-\rho^2/4t}}{(4\pi t)^n}\sum_{i=1}^{N}t^iU^i(x,x'), \tag{4.7}$$

where $N\geq n+[\frac{1}{2}m]$ and $\rho=d(x,x')$, $(x,x')\in\Delta(\varepsilon)=\{(x,x')\in M\times M|d(x,x')<\varepsilon\}$ for some sufficiently small $\varepsilon>0$, and each $U^{(i)}$ is a .

$$U^{(i)}(x,x'):F_{x'}\to F_x\hat{\otimes}\Lambda_y(B) \tag{4.8}$$

If $H_N(x,x';t)$ satisfies the following two conditions:
(1)

$$(\frac{\partial}{\partial t}+I_{x'})H_N(x,x';t)=\frac{e^{-\rho^2/4t}}{(4\pi t)^n}t^Nh(x,x';t) \tag{4.9}$$

(2)

$$U^{(0)}(x,x)=Id:F_x\to F_x$$

where h is a continuous function, then we call H_N a local parametrix for I.

We now show that this H_N does exist.

Lemma 4.10 (Compare Yu [13])

If $\phi\in C^\infty(M)$, $S\in H^\infty$, then

$$\begin{aligned}I(\phi S)=&-\phi_{ii}S-2\phi_i\nabla_{E_i}S+\phi I(S)\\&+(a_i\phi_i\Gamma^k_{ij}e_je_k+b_i\phi_i\Gamma^\alpha_{ij}e_jdy^\alpha+c_i\phi_i\Gamma^\beta_{i\alpha}dy^\alpha dy^\beta)\end{aligned}$$

for some constants a_i,b_i,c_i.

Proof. First, as in [13], we easily deduced that

$$H(\phi S)=e_i\phi_iS+dy^\alpha\phi_\alpha S+\phi HS \tag{4.11}$$

$$\begin{aligned}H(e_i\phi_iS)=&e_i(e_i\phi_l)e_lS+e_ie_l\phi_l(E_iS)+\frac{1}{4}\Gamma^k_{ij}e_ie_je_k\phi_le_lS\\&+\frac{1}{2}\Gamma^\alpha_{ij}e_ie_jdy^\alpha\phi_le_lS+\frac{1}{4}\Gamma^\beta_{i\alpha}e_idy^\alpha dy^\beta e_l\phi_lS\\&+dy^\alpha(F_\alpha\phi_l)e_lS+dy^\alpha\phi_le_l(F_\alpha S)+\frac{1}{4}\Gamma^j_{\alpha i}dy^\alpha e_ie_j\phi_lS\\&+\frac{1}{2}\Gamma^\beta_{\alpha i}dy^\alpha e_idy^\beta e_l\phi_lS+\frac{1}{4}\Gamma^\gamma_{\alpha\beta}dy^\alpha dy^\beta dy^\gamma e_l\phi_lS\\=&-\phi_{ii}-\phi_ie_iHS+\phi_{\alpha i}dy^\alpha e_iS-2\phi_i\nabla_{E_i}S\\&+a_i\phi_i\Gamma^k_{ij}e_je_k+b_i\phi_i\Gamma^\alpha_{ij}e_jdy^\alpha+c_i\phi_i\Gamma^\beta_{i\alpha}dy^\alpha dy^\beta\end{aligned} \tag{4.12}$$

$$(4.13)\qquad H(\phi_\alpha dy^\alpha S) = \phi_{i\alpha} e_i dy^\alpha S - \phi_\alpha dy^\alpha (HS)$$

and from (4.11) it follows that

$$(4.14)\qquad H(\phi HS) = e_i\phi_i(HS) + dy^\alpha \phi_\alpha (HS) + \phi(H^2 S)$$

Now from (4.11) to (4.14), we obtain the Lemma by summation.

Set $\phi = \frac{e^{-\rho^2/4t}}{(4\pi t)^n}$ as in (4.9). Recall from [13] that in the local normal coordinate system, if $\rho^2(x,x') = x_1^2 + \cdots + x_{2n}^2$, we have

$$(4.15)\qquad \phi_i = -\phi\frac{x_i}{2t}$$

$$(4.16)\qquad \phi_{ii} = \phi(\frac{\rho^2}{4t} - \frac{n}{t} - \frac{1}{t}\sum_i x_i B_i)$$

for some functions B_i.

Set $S = \sum_{i=0}^N t^i U^{(i)}(x,x')$, then from (4.15), (4.16) and (4.10) we have

$$\frac{\partial}{\partial t} H_N(x';t) = \phi\sum_{i=0}^N (\frac{\rho^2}{4t^2} - \frac{n}{t} + \frac{i}{t}) t^i U^{(i)}(x')$$

$$\begin{aligned} I(H_N(x';t)) =& -\phi(\frac{\rho^2}{4t^2} - \frac{n}{t} - \frac{x_iB_i}{t})S - 2\phi(\frac{-x_i}{2t})\nabla_{E_i}S + \phi IS \\ &+ (a_i\phi_i\Gamma_{ij}^k e_j e_k + b_i\phi_i\Gamma_{ij}^\alpha e_j dy^\alpha + c_i\phi_i\Gamma_{i\alpha}^\beta dy^\alpha dy^\beta)S \\ =& -\sum_{i=0}^N (\frac{\rho^2}{4t^2} - \frac{n}{t} - \frac{x_iB_i}{t}) t^i U^{(i)} + \sum_{i=0}^n t^{i-1}\hat{d}U^{(i)} \\ &+ \sum_{i=0}^N I(U^{(i)})t^i - \frac{1}{2}(a_i x_i \Gamma_{ij}^k e_j e_k + b_i x_i \Gamma_{ij}^\alpha e_j dy^\alpha \\ &+ c_i x_i \Gamma_{i\alpha}^\beta dy^\alpha dy^\beta) t^{i-1} U^{(i)}. \end{aligned}$$

Hence we obtain from (4.9) the following analogue of the Minakshsundaram-Pleijel equations given by Yu [13]:

$$(4.17)\qquad \begin{aligned} &(\hat{d} + x_i(a_i\Gamma_{ij}^k e_j e_k + b_i\Gamma_{ij}^\alpha e_j dy^\alpha + c_i\Gamma_{i\alpha}^\beta dy^\alpha dy^\beta))U^{(i)} + (x_iB_i + i)U^{(i)} \\ &= -IU^{(i-1)}, \qquad i \le N \\ &U^{(0)}(x,x) = Id : F_x \to F_x. \end{aligned}$$

Proposition 4.18 The local parametrix H_N exists iff $\forall i \le N$, $U^{(i)}$ satisfies the equations in (4.17).

In the next section, we will calculate out the local index throughout these equations.

§5. The local index theorem

First we make it explicit what to be calculate. Recall from [14] that if H_N is the local parametrix, we have

$$(5.1)\qquad P_t(x,x') - H_N(x,x';t) = O(t^{1+N-n})$$

while from (4.0) it is clear that

(5.2)
$$\lim_{t\to 0} t^{[\frac{1}{2}m]+\frac{2}{8}}\varphi_t = 0$$

so when $N \geq [\frac{1}{2}m] + n$, we have

(5.3)
$$\lim_{t\to 0} \varphi_t(O(t^{1+N-n})) = 0$$

From (5.1) and (5.3), it follow that

(5.4)
$$\lim_{t\to 0} \varphi_t P_t(x,x) = \lim_{t\to 0} \varphi_t H_N(x,x;t)$$

The supertrace of the right hand of (5.4) is precisely what we procceed to calculate out.

Now we take a convension similar to that in [13]:

Let φ be a C^∞ function defined locally in a neighborhood of x, denote the degree of zero of φ at x by $\nu(\varphi)$, to every

(5.5a)
$$\begin{aligned}\alpha(x') =& \varphi_{l_1}(x')\frac{\partial}{\partial x_{i_1}}\varphi_{l_2}(x')\cdots\frac{\partial}{\partial x_{i_m}}\varphi_{l_{m+1}}(x')dy^{\alpha_1}\cdots dy^{\alpha_p}\cdot \\ &\cdot e_{j_1}\cdots e_{j_s} : F'_x \to F_x; \alpha_i \neq \alpha_j\ (i\neq j); j_a \neq j_t\ (a \neq t)\end{aligned}$$

We define

(5.5b)
$$\chi(\alpha) = m + p + s - \nu(\varphi_1\cdots\varphi_{m+1})$$

and we denote $\{\chi < m\}$ the linear space generated by all the elements α for which $\chi(\alpha) < m$, etc. and denote $(\chi < m)$ an element of $\{\chi < m\}$, e.g. $\omega = \eta + (\chi < m)$ means that there exists a $\beta \in \{\alpha < m\}$ such that $\omega = \eta + \beta$, we can also write it as

(5.6)
$$\omega \equiv \eta \mod\{\chi < m\}$$

Lemma 5.7

$$\Gamma_{ij}^k = \frac{1}{2}\sum_l R_{iljk}x_l + (\chi < -1),$$
$$\Gamma_{ij}^\alpha = \frac{1}{2}\sum_l R_{ilj\alpha}x_l + (\chi < -1),$$
$$\Gamma_{i\alpha}^\beta = \frac{1}{2}\sum_l R_{il\alpha\beta}x_l + (\chi < -1).$$

Proof. Comparing [13], we only need to note that we are working on a fixed G_y.

Proposition 5.8

$$\begin{aligned}I = &-\frac{\partial^2}{\partial x_i^2} + \frac{1}{4}R_{ijst}x_i\frac{\partial}{\partial x_j}e_se_t + \frac{1}{2}R_{ijs\alpha}x_i\frac{\partial}{\partial x_j}e_sdy^\alpha + \frac{1}{4}R_{ij\alpha\beta}x_i\frac{\partial}{\partial x_j}dy^\alpha dy^\beta \\ &+ \frac{1}{64}x_ix_jR_{irlk}R_{rjst}e_le_ke_se_t + \frac{1}{16}x_ix_jR_{irk\alpha}R_{rjst}e_ke_se_tdy^\alpha + \\ &+ \frac{1}{32}x_ix_jR_{irlk}R_{rj\alpha\beta}e_le_kdy^\alpha dy^\beta + \frac{1}{16}x_ix_jR_{irk\alpha}R_{rjl\beta}e_kdy^\alpha e_ldy^\beta + \\ &+ \frac{1}{16}x_ix_jR_{irl\alpha}R_{rj\lambda\mu}e_ldy^\alpha dy^\lambda dy^\mu + \\ &+ \frac{1}{64}x_ix_jR_{ir\alpha\beta}R_{rj\lambda\mu}dy^\alpha dy^\beta dy^\lambda dy^\mu + (\chi < 2).\end{aligned}$$

Proof. It follows directly from Lemma 5.7 and the generalized Lichnerowicz formula given by Bismut [4], Theorem 3.5.

As in [13], we denote

$$\begin{aligned}
a_0 &= -\frac{1}{4}R_{ijst}x_i\frac{\partial}{\partial x_j}e_s e_t - \frac{1}{2}R_{ijs\alpha}x_i\frac{\partial}{\partial x_j}e_s dy^\alpha - \frac{1}{4}R_{ij\alpha\beta}x_i\frac{\partial}{\partial x_j}dy^\alpha dy^\beta;\\
a_2 &= \sum_i \frac{\partial^2}{\partial x_i^2};\\
a_{-2} &= -\Big(\frac{1}{64}x_i x_j R_{irkl}R_{rjst}e_l e_k e_s e_t + \frac{1}{16}x_i x_j R_{irk\alpha}R_{rjst}e_k e_s e_t dy^\alpha +\\
&\quad + \frac{1}{32}x_i x_j R_{irlk}R_{rj\alpha\beta}e_l e_k dy^\alpha dy^\beta + \frac{1}{16}x_i x_j R_{irk\alpha}R_{rjl\beta}e_k dy^\alpha e_l dy^\beta +\\
&\quad \frac{1}{16}x_i x_j R_{irl\alpha}R_{rj\lambda\mu}e_l dy^\alpha dy^\lambda dy^\mu +\\
&\quad + \frac{1}{64}x_i x_j R_{ir\alpha\beta}R_{rj\lambda\mu}dy^\alpha dy^\beta dy^\lambda dy^\mu\Big).
\end{aligned}$$

And we set

$$A_i = a_i x_l R_{iljk}e_j e_k + b_i x_l R_{ilj\alpha}e_j dy^\alpha + c_i x_l R_{il\alpha\beta}dy^\alpha dy^\beta.$$

Obviously,

$$x_i\left(a_i\Gamma_{ij}^k e_j e_k + b_i\Gamma_{ij}^\alpha e_j dy^\alpha + c_i\Gamma_{i\alpha}^\beta dy^\alpha dy^\beta\right) = x_i A_i + (\chi < 0). \tag{5.9}$$

Lemma 5.10 $\forall 1 \le j \le 2n$,

$$\frac{\partial}{\partial x_j}(x_i A_i) = 0.$$

Proof.

$$\frac{\partial}{\partial x_j}(x_i x_l R_{ilst}e_s e_t) = x_i(R_{jist} + R_{ijst})e_s e_t = 0,$$

the other two can be proved in the same way.

Corollary 5.11 For $S \in (Hom(F,F)\hat{\otimes} \wedge_y (B))$,

$$\begin{aligned}
a_0((\sum_i x_i A_i)S) &=(\sum_i x_i A_i)a_0 S + (\chi < \chi(a_0 S)),\\
a_2((\sum x_i A_i)S) &=(\sum x_i A_i)a_2 S.
\end{aligned}$$

Lemma 5.12

$$a_{-2}A_i \equiv A_i a_{-2}, \quad \mathrm{mod}\{\chi < 3\}.$$

Proof. Direct calculations.

Now we recall the basic idea of Yu[13] of comparing the corresponding terms of the Taylor expansion series: let f be a function in a neighborhood of x, $f: U \to R$ or F. We expand f by its Taylor series:

$$f = \sum_{m=0}^{\infty} \hat{f}(m) \tag{5.13}$$

where $\hat{f}(m)$ is the m-th degree homogeneous polynomial in $x_1, \cdots, x_{2n}$. We know that, for $V \in F_{x'}, U^{(i)}(x,x') : F_{x'} \to F_x, U^{(i)}V$ can be viewed as a spinor field, which under the fixed spin frame, can be viewed as a function with values in F which we still denote by $U^{(i)}$.

Notice that $\hat{d}\hat{U}(m) = m\hat{U}(m)$, and denote $\sum_i x_i B_i = h$ for some h, comparing the corresponding terms of Taylor series in (4.17), we get

$$\begin{aligned}&(m+i)\hat{U}^{(i)}(m) + x_i A_i \hat{U}^{(i)}(m-2)\\ &+ \sum_{\substack{m_1+m_2=m\\ m_1>0}} \hat{h}(m_1)\hat{U}^{(i)}(m_2)\\ =&a_2\hat{U}^{(i-1)}(m+2) + a_0\hat{U}^{(i-1)}(m) + a_{-2}\hat{U}^{(i-1)}(m-2)\\ &+\sum f_j \hat{U}^{(i-1)}(m_j)\end{aligned}$$

where $\chi(f_j) < 2$. Rewrite it as

$$\begin{aligned}\hat{U}^{(i)}(m) =&\frac{1}{(m+i)}\sum_\alpha \hat{U}^{(i-1)}(m+\alpha) + \sum_j g_j \hat{U}^{(i-1)}(m_j)+\\ &+(\sum_i x_i A_i)\sum_j s_{m_j}\hat{U}^{(i-1)}(m_j)\end{aligned} \tag{5.14}$$

with $\chi(g_j) < 2$, $\chi(S_{m_j}) \leq 2$. From (5.14), it can be easily deduced that

$$\begin{aligned}\hat{U}^{(i)}(m) = &\sum_{\alpha_1,\cdots,\alpha_i} \frac{a_{\alpha_1}\cdots a_{\alpha_i}}{\Gamma(\alpha_1,\cdots,\alpha_i;m)}(\hat{U}^{(0)}(m+\alpha_1+\cdots+\alpha_i))+\\ &\sum_j f_j \hat{U}^{(0)}(m_j) + (\sum_i x_i A_i)\sum_j \tilde{S}_j \hat{U}^{(0)}(m_j)\end{aligned} \tag{5.15}$$

with $\chi(f_j) < 2i$, $\chi(\tilde{S}_j) < 2i$. Note that in the deducing, Lemma 5.12 and Corollary 5.11 are freely used.

Proposition 5.16 For $i < n$,

$$\mathrm{tr}_s U^{(i)}(x) = 0.$$

Proof. c.f. [13] or compare with the following proof of Lemma 5.20.

Corollary 5.17

$$\lim_{t\to 0}\varphi_t(\sum_{i=0}^{n-1}\mathrm{tr}_s U^{(i)}\frac{t^i}{(4\pi t)^n}) = 0$$

So what we really ought to calculate out is

$$\begin{aligned}&\lim_{t\to 0}\varphi_t(\frac{1}{(4\pi t)^n}\sum_{k=0}^{[\frac{1}{2}m]}\mathrm{tr}_s U^{(n+k)}(x,x)t^{n+k})\\ =&\lim_{t\to 0}\frac{1}{(4\pi)^n}\sum_{k=0}^{[\frac{1}{2}m]} t^k\varphi_t(\mathrm{tr}_s U^{(n+k)}(x,x)).\end{aligned} \tag{5.18}$$

Let us take a look at the one

$$\lim_{t\to 0} t^k \varphi_t(\mathrm{tr}_s U^{(n+k)}(x,x)), \quad 0 \leq k \leq [\frac{1}{2}m] \tag{5.19}$$

Lemma 5.20 If $\chi(\alpha) < 2n + 2k$, then

$$\lim_{t\to 0} t^k\varphi_t(\mathrm{tr}_s\alpha) = 0, \quad 0 \leq k \leq [\frac{1}{2}m].$$

Proof. We can assume that α can be written as

$$\alpha = \varphi(x')dy^{\alpha_1}\cdots dy^{\alpha_p}e_{j_1}\cdots e_{j_s}$$

If $\chi(\alpha) < 2n+2k$, then either $\varphi(x) = 0$ or $p+s < 2n+2k$. In the former case, $\mathrm{tr}_s\alpha = 0$ is trivial and in the latter case, if $s < 2n$, then $\mathrm{tr}_s(\varphi(x)dy^{\alpha_1}\cdots dy^{\alpha_p}e_{j_1}\cdots e_{j_s}) = \varphi(x)dy^{\alpha_1}\cdots dy^{\alpha_p}\cdot \mathrm{tr}_s(e_{j_1}\cdots e_{j_s}) = 0$ and if $s \geq 2n$, then $p < 2k$ so

$$\begin{aligned}\lim_{t\to 0} t^k\varphi_t(\mathrm{tr}_s\alpha) &= \lim_{t\to 0}\varphi(x)\varphi_t(dy^{\alpha_1}\cdots dy^{\alpha_p})t^k\mathrm{tr}_s(e_{i_1}\cdots e_{i_s})\\ &= \lim_{t\to 0}\varphi(x)\mathrm{tr}_s(e_{i_1}\cdots e_{i_s})dy^{\alpha_1}\cdots dy^{\alpha_p}t^{k-\frac{1}{2}p} = 0\end{aligned}$$

Lemma 5.21 If $\varphi(x,x') \in \mathrm{Hom}(F_{x'},F_x)\hat{\otimes}\wedge_y(B)$, and some $\alpha_k = 0$, then

$$(a_{\alpha_1}\cdots a_{\alpha_l}\varphi) < 2l + \chi(\varphi)$$

Proof. cf. [13].

Now we can easily see from (4.17), (5.15), (5.19), (5.20) and the above Lemma 5.21 that $\sum_i x_iA_i$, a_0, $h = \sum_i x_iB_i$ and the term $(\chi < 2)$ in Propotion 5.18 are really irrelavent for the calculation of the supertrace (5.19). We write this result as follows:

Proposition 5.22 If $V^{(i)}(x,x') \in \mathrm{Hom}(F_{x'},F_x)\hat{\otimes}\wedge_y(B)$ satisfies the following equations:

$$x_i\frac{\partial}{\partial x_i}V^{(i)} + iV^{(i)} = (\sum_i\frac{\partial^2}{\partial x_i^2} + a_{-2})V^{(i-1)} \tag{5.23}$$

$$V^{(0)}(x,x) = Id : F_x \to F_x$$

then

$$\lim_{t\to 0}\varphi_t(\mathrm{tr}_sV^{(n+k)}(x,x))t^k = \lim_{t\to 0}\varphi_t(\mathrm{tr}_sU^{(n+k)}(x,x))t^k \tag{5.24}$$

where $0 \leq k \leq [\frac{1}{2}m]$.

Now the analogue of (5.15) is

$$V^{(i)}(m) = \sum_{\alpha_1,\cdots,\alpha_i}\frac{a_{\alpha_1}\cdots a_{\alpha_i}}{\Gamma(\alpha_1,\cdots,\alpha_i;0)}(V^{(0)}(m+\alpha_1+\cdots+\alpha_i)) \tag{5.25}$$

with each $\alpha_i \neq 0$. So

$$\chi(V^{(n+k)}) \leq 2n+2k. \tag{5.26}$$

$V^{(n+k)}$ can be expressed as a sum of the following terms:

$$\alpha = \varphi(x')dy^{\alpha_1}\cdots dy^{\alpha_p}e_{i_1}\cdots e_{i_s} \tag{5.27}$$

where in the process, we only take the interchanging between dy^{α}'s and e_i's, e.g., $e_idy^{\alpha} = -dy^{\alpha}e_i$, and has not interchanged the order of e_i's. It may be happened to α the following cases:

(1) $\varphi(x) = 0$, then $\alpha(x) = 0$;

(2) $\varphi(x) \neq 0$, but $\exists r \neq q \ni \alpha_r = \alpha_q$, then $\alpha(x) = 0$;

(3) $\varphi(x) \neq 0$, and $\forall r \neq q$, $\alpha_r \neq \alpha_q$, then $p+s = 2k+2n$, if $p < 2k$, then $\lim_{t\to 0}\varphi_t(\varphi(x)dy^{\alpha_1}\cdots$ 0, and if $p > 2k$, then $s < 2n, \mathrm{tr}_s(e_{i_1}\cdots e_{i_s}) = 0$. So in this case we must have $p = 2k$ and $s = 2n$ for a possible non-zero contribution to the supertrace, but if we have some $i_r = i_q$, $r \neq q$, then $\chi(e_{i_1}\cdots e_{i_{2n}}) < 2n$ which implies that $\mathrm{tr}_s(e_{i_1}\cdots e_{i_s}) = 0$.

Summarizing these, we get

Proposition 5.28 The only terms having nontrivial contributions to (5.24) are thouse of α's such that

$$\alpha = \varphi(x')dy^{\alpha_1}\cdots dy^{\alpha_p}e_{i_1}\cdots e_{i_s}$$

where $\varphi(x) \neq 0$, $\alpha_r \neq \alpha_q$ $(r \neq q)$, $e_{i_t} \neq e_{i_l}$ $(t \neq l)$ and $p + s = 2n + 2k$. In particular, if in the original expression of α, there have some $e_i'^s$ resppeared, then $\lim_{t\to 0} t^k \varphi_t(\mathrm{tr}_s \alpha) = 0$.

Proof. All that we need to notice is the following:

$$e_i e_j \equiv -e_j e_i \qquad (\mathrm{mod}\{\chi < 2\}).$$

From this proposition and Lemma 1.8, we immediately have:

Proposition 5.29 If $W^{(i)}(x, x') \in \wedge_{x'}(TG)\hat{\otimes} \wedge_y (B)$ satisfies the equations

$$\begin{aligned} & x_i \frac{\partial}{\partial x_i} W^{(i)} + iW^{(i)} \\ (5.30) \qquad & = (\sum_i \frac{\partial^2}{\partial x_i^2} - \frac{1}{64} x_i x_j R_{irIJ} R_{rjST} dz^I dz^J dz^S dz^T) W^{(i-1)}, \\ & W^{(0)}(x, x) = 1 \end{aligned}$$

where we denote $z = (x, y)$ and use I, J etc., to denote the total subscripts i, j, α, etc. Then

$$(5.31) \qquad \lim_{t\to 0} \varphi_t(\mathrm{tr}_s(P_t(x,x)))dx = (\frac{1}{2\pi\sqrt{-1}})^n \sum_{k=0}^{[\frac{1}{2}m]} (W^{(n+k)}(x))_{2n}$$

where $(\cdot)_{2n}$ stands for the term which is a multiple of $dx_1 \cdots dx_{2n}$.

Proof. This follows directly from (5.4), (5.18), (5.19), (5.22), (5.28) and (1.8). Notice that $dx \cdot dx = 0$.

Denote by $\Omega = -\frac{1}{2} R_{ijIJ} dz^I dz^J$ the matrix of two forms over M. Clearly, Ω is the curvature matrix for the connection ∇ in (2.4) of the vector bundle TG over M. As in the usual computations of characteristic classes, we take the identification of Ω to its Chern root matrix:

$$(5.32) \qquad \begin{pmatrix} 0 & u_1 & & & \\ -u_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & u_n \\ & & & -u_n & 0 \end{pmatrix}$$

then

$$(5.33) \qquad x_i x_j \Omega_{ir} \Omega_{rj} = -\sum_{l=1}^{n} (x_{2l-1}^2 + x_{2l}^2) u_l^2$$

and (5.30) becomes

$$(5.34) \qquad x_i \frac{\partial}{\partial x_i} W^{(i)} + iW^{(i)} = (\sum_{i=0}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{1}{16} \sum_{l=1}^{n} (x_{2l-1}^2 + x_{2l}^2) u_l^2) W^{(i-1)}$$

$$W^{(0)}(x, x) = 1$$

As in [13], set

$$H(x_1,\cdots,x_{2n};t)=\frac{e^{-\rho^2/4t}}{(4\pi t)^n}\sum_{i=0}^{\infty}W^{(i)}(x_1,\cdots,x_{2n})t^i \tag{5.35}$$

By (5.34) and (5.35), we have:

$$\begin{aligned}&\frac{\partial H}{\partial t}=\sum_i\frac{\partial^2}{\partial x_i^2}H+\frac{1}{16}\sum_{l=1}^{n}(x_{2l-1}^2+x_{2l}^2)u_l^2H\\&\lim_{t\to 0}(4\pi t)^nH(0,\cdots,0;t)=1\end{aligned} \tag{5.36}$$

Solving this equation as in [13], we find

$$H(x_1,\cdots,x_{2n};t)=(\frac{1}{4\pi})^n\prod_{l=1}^{n}(\frac{\sqrt{-1}u_l/2}{\sinh\sqrt{-1}u_lt/2}e^{(x_{2l-1}^2+x_{2l}^2)\frac{\sqrt{-1}u_l}{8}\coth\frac{\sqrt{-1}u_lt}{2}}) \tag{5.37}$$

So

$$H(0,\cdots,0;t)=\frac{1}{(4\pi)^n}\prod_{l=1}^{n}\frac{\sqrt{-1}u_l/2}{\sinh\sqrt{-1}u_lt/2} \tag{5.38}$$

Combining with (5.35), we get

$$\sum_{i=0}^{\infty}W^{(i)}(0)t^i=\prod_{l=1}^{n}\frac{\sqrt{-1}u_lt/2}{\sinh\sqrt{-1}u_lt/2} \tag{5.39}$$

Thus

$$\sum_{i=0}^{\infty}W^{(i)}(0)=\prod_{l=1}^{n}\frac{\sqrt{-1}u_l/2}{\sinh\sqrt{-1}u_l/2}=\hat{A}(\sqrt{-1}\Omega) \tag{5.40}$$

Notice that when $N>n+[\frac{1}{2}m]$, $W^{(N)}(0)=0$ and when $N<n$, $W^{(N)}(0)$ is not a multipul of $dx^1\cdots dx^{2n}$. Hence from (5.31) and (5.40) we finally get

$$\lim_{t\to 0}\varphi_t(\mathrm{tr}_sP_t(x,x))dx=(\frac{1}{2\pi\sqrt{-1}})^n(\hat{A}(\sqrt{-1}\Omega))_{2n} \tag{5.41}$$

This is what we call the local Atiyah-Singer index theorem for families of Dirac operators. As a direct corollary, we get

Theorem (Atiyah-Singer [1]):

$$(\frac{1}{2\pi\sqrt{-1}})^n\int_{G_y}\hat{A}(\sqrt{-1}\Omega) \tag{5.42}$$

is a representative of $\mathrm{Ch}(\ker D_{+,y}-\ker D_{-,y})$.

APPENDIX

In this appendix, we will briefly outline how our method works for twisted Dirac operators. For simplicity (and without loss of generality), we only carry out the single operator case. Now, by Lichnerowicz formula, we can deduce that (Compare [13])

$$\begin{aligned}D^2=&-\frac{\partial^2}{\partial x_i^2}+\frac{1}{4}R_{ijst}x_i\frac{\partial}{\partial x_j}e_se_t+\frac{1}{64}x_ix_jR_{irst}R_{rjpq}e_se_te_pe_q\\&+\frac{1}{2}e_ie_j\otimes F(e_i,e_j)+(\chi<2)\end{aligned} \tag{a.1}$$

where F is the curvature matrix of the connection of the vector bundle ξ over G. Now doing similarly as in §5, we see that if $V^{(i)}$ is the solution of the following equations

$$
\begin{aligned}
&x_i \frac{\partial}{\partial x_i} V^{(i)} + iV^{(i)} = [\sum_i \frac{\partial^2}{\partial x_i^2} - \frac{1}{16} x_i x_j \Omega_{ir} \Omega_{rj} - \frac{1}{2} dx^s dx^t \otimes F(e_s, e_t)] V^{(i-1)} \\
&V^{(-1)} \equiv 0, \quad V^{(0)}(0) = Id : (F \otimes \xi)_x \to (F \otimes \xi)_x
\end{aligned}
\tag{a.2}
$$

then the local index is

$$
\tau(D) = \frac{1}{2\pi(\sqrt{-1})^n} \mathrm{tr}_s V^{(n)}(0)
\tag{a.3}
$$

Aside (5.32), we pick another identification

$$
-\frac{1}{2} dx^i dx^j \otimes F(e_i, e_j) \leftrightarrow 1 \otimes \begin{pmatrix} v_1 & & \\ & \ddots & \\ & & v_N \end{pmatrix}
\tag{a.4}
$$

where $N = \dim\xi$. Set $H = \frac{e^{-\rho^2/4t}}{(4\pi t)^n} \sum_i t^i V^{(i)}$ then by solving an equation similar to (5.36), we find

$$
\begin{aligned}
H(x_1, \cdots, x_{2n}; t) = &\prod_{l=1}^{n} \left(\frac{\sqrt{-1} u_l}{8\pi \sinh(\sqrt{-1} u_l t/2)} e^{(x_{2l-1}^2 + x_{2l}^2) \frac{\sqrt{-1} u_l}{8} \coth \frac{\sqrt{-1} u_l t}{2}} \right) \cdot \\
&\cdot \begin{pmatrix} e^{v_1 t} & & \\ & \ddots & \\ & & e^{v_N t} \end{pmatrix}
\end{aligned}
$$

So

$$
\begin{aligned}
&\tau(D) \\
=&\lim_{t \to 0} \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \frac{1}{n!} \cdot \frac{\partial}{\partial t^n} \left((4\pi t)^n \left(\prod_1^n \frac{\sqrt{-1} u_l}{8\pi \sinh(\sqrt{-1} u_l t/2)} \right) (e^{v_1 t} + \cdots + e^{v_N t}) \right) \\
=&\left(\frac{1}{2\pi} \right)^n \left(\hat{A}(\Omega) \mathrm{ch}\left(\frac{F}{\sqrt{-1}} \right) \right)_{2n}
\end{aligned}
$$

this is the local Atiyah-Singer index theorem for twisted Dirac operators.

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Note added in proof: We learned after completing this work that Berline-Vergne, in a preprint dating Sept. 1986, had also given a differential geometric proof of this Bismut local index theorem. (Topology 26 No.4 (1987))