TOEPLITZ QUANTIZATION AND SYMPLECTIC REDUCTION

XIAONAN MA

Centre de Mathématiques Laurent Schwartz,
UMR 7640 du CNRS,
Ecole Polytechnique,
91128 Palaiseau Cedex, France
E-mail: ma@math.polytechnique.fr

WEIPING ZHANG

Chern Institute of Mathematics & LPMC,
Nankai University,
Tianjin 300071, P.R. China
E-mail: weiping@nankai.edu.cn

In Ref. 9, we announced the asymptotic expansion of the G-invariant Bergman kernel of the spin^c Dirac operator associated with high tensor powers of a positive line bundle on a symplectic manifold. In this note, we describe several consequences of our asymptotic expansion of the G-invariant Bergman kernel in the Kähler case, especially, we study the Toeplitz quantization in the framework of the symplectic reduction. The full details can be found in Ref. 10.

1. Toeplitz quantization

Let (X,ω) be a compact Kähler manifold with Kähler form ω, and $\dim_{\mathbb{C}} X = n$. Let J be the almost complex structure on the real tangent bundle TX. Let $g^{TX}(v,w) := \omega(v,Jw)$ be the corresponding Riemannian metric on TX.

Let L be a holomorphic line bundle over X with Hermitian metric h^L. Let ∇^L be the holomorphic Hermitian connection on (L,h^L) with curvature $R^L := (\nabla^L)^2$. We suppose that (L,h^L) is a pre-quatum line bundle of (X,ω), i.e.

$$\frac{\sqrt{-1}}{2\pi} R^L = \omega. \qquad (1)$$

According the geometric quantization introduced by Kostant and

Souriau, the Kähler manifold (X, ω) is the classical phase space and $H^0(X, L)$, the space of holomorphic sections of L on X, is the quantum space. The set of classical observables is the Poisson algebra $\mathscr{C}^\infty(X)$, the quantum observables are the linear operators on $H^0(X, L)$. The semi-classical limit is a way to relate the classical and quantum observables, basically, for any $p \in \mathbb{N}$, we replace L by L^p, then we obtain a sequence of spaces $H^0(X, L^p)$, the semi-classical limit is the process of $p \to \infty$. In this note, we will restrict ourself to a family of quantum observables : Toeplitz operators.

Let $\{\, , \,\}$ be the Poisson bracket on $(X, 2\pi\omega)$: for $f_1, f_2 \in \mathscr{C}^\infty(X)$, if ξ_{f_2} is the Hamiltonian vector field generated by f_2 which is defined by $2\pi i_{\xi_{f_2}}\omega = df_2$, then

$$\{f_1, f_2\}(x) = (\xi_{f_2}(df_1))(x). \tag{2}$$

Let dv_X be the Riemannian volume form of (X, g^{TX}), then $dv_X = \omega^n/n!$. We define the L^2-scalar product $\langle \quad \rangle$ on $\mathscr{C}^\infty(X, L^p)$ by

$$\langle s_1, s_2 \rangle = \int_X \langle s_1, s_2 \rangle_{L^p}(x)\, dv_X(x)\,. \tag{3}$$

Let Π_p denote the orthogonal projection from $(L^2(X, L^p), \langle \quad \rangle)$, the space of L^2 sections of L^p on X, to $H^0(X, L^p)$, the space of holomorphic sections of L^p on X.

For any $f \in \mathscr{C}^\infty(X)$, consider the Toeplitz operators

$$T_p(f) = \Pi_p f \Pi_p : H^0(X, L^p) \to H^0(X, L^p). \tag{4}$$

We denote by $\|T_p(f)\|$ the operator norm of $T_p(f)$ with respect to the scalar product $\langle \quad \rangle$.

We now state two results of Bordemann-Meinrenken-Schlichenmaier[2], concerning the asymptotic behavior of $T_p(f)$ as $p \to +\infty$.

Theorem 1.1. *As $p \to +\infty$, one has*

$$\lim_{p\to+\infty} \|T_p(f)\| = \|f\|_\infty, \tag{5a}$$

$$[T_p(f), T_p(g)] = \frac{1}{\sqrt{-1}p} T_p(\{f, g\}) + O(p^{-2}). \tag{5b}$$

2. Hamiltonian action and symplectic reduction

Let E be a holomorphic vector bundle on X with Hermitian metric h^E. Let ∇^E be the holomorphic Hermitian connection on (E, h^E). Let G be a compact connected Lie group. Let $\mathfrak{g}$ be the Lie algebra of G.

Suppose that G acts holomorphically on X, and the action of G lifts holomorphically on L, E and preserves the metrics h^L, h^E. Then the action of G preserves ω, the connections ∇^L, ∇^E.

For $K \in \mathfrak{g}$, we denote by K^X the vector field on X generated by K, and by L_K the infinitesimal action induced by K on the corresponding vector bundles. Let $\mu : X \to \mathfrak{g}^*$ be defined by

$$2\pi\sqrt{-1}\mu(K) := \nabla^L_{K^X} - L_K, \ K \in \mathfrak{g}. \tag{6}$$

Then μ is the corresponding **moment map**, i.e. for any $K \in \mathfrak{g}$,

$$d\mu(K) = i_{K^X}\omega. \tag{7}$$

Definition 2.1. The Marsden-Weinstein **symplectic reduction** space X_G is defined to be

$$X_G = \mu^{-1}(0)/G. \tag{8}$$

Basic assumption: $0 \in \mathfrak{g}^*$ is a regular value of the moment map $\mu : X \to \mathfrak{g}^*$.

Then $\mu^{-1}(0)$ is a closed manifold. For simplicity, also assume that G acts on $\mu^{-1}(0)$ freely, then X_G is a compact smooth manifold and carries an induced symplectic form ω_G.

Moreover, J induces a complex structure J_G on TX_G such that $\omega_G(\cdot, J_G\cdot)$ determines a Riemannian metric g^{TX_G} on TX_G. Thus (X_G, ω_G, J_G) is also Kähler.

The line bundle (L, h^L) induces a Hermitian line bundle (L_G, h^{L_G}) on X_G by identifying G-invariant sections of L on $\mu^{-1}(0)$. In fact (L_G, h^{L_G}) is a pre-quantized holomorphic line bundle over (X_G, ω_G), cf. Ref. 5.

In the same way, (E, h^E) induces a holomorphic Hermitian vector bundle (E_G, h^{E_G}) on X_G.

3. Toeplitz quantization and symplectic reduction

We now assume that a connected compact Lie group acts on (X, ω, J, L) in a Hamiltonian way as before.

Let $i : \mu^{-1}(0) \hookrightarrow X$ denote the canonical embedding. We assume as before that 0 is a regular value of μ and G acts on $\mu^{-1}(0)$ freely. Then

$$\pi : \mu^{-1}(0) \to X_G$$

is a principal fibration with fiber G.

Let $H^0(X, L^p \otimes E)^G$ be the G-invariant part of $H^0(X, L^p \otimes E)$, the space of holomorphic sections of $L^p \otimes E$ on X. Let $\mathscr{C}^\infty(X, L^p \otimes E)^G$ (resp.

$\mathscr{C}^\infty(\mu^{-1}(0), L^p \otimes E)^G)$ be the G-invariant smooth sections of $L^p \otimes E$ on X (resp. $\mu^{-1}(0)$). Let $\pi_G : \mathscr{C}^\infty(\mu^{-1}(0), L^p \otimes E)^G \to \mathscr{C}^\infty(X_G, L^p_G \otimes E_G)$ be the natural identification. By a result of Zhang[13], for p large enough, the map

$$\pi_G \circ i^* : \mathscr{C}^\infty(X, L^p \otimes E)^G \to \mathscr{C}^\infty(X_G, L^p_G \otimes E_G)$$

induces a natural isomorphism

$$\sigma_p = \pi_G \circ i^* : H^0(X, L^p \otimes E)^G \to H^0(X_G, L^p_G \otimes E_G). \tag{9}$$

(When $E = \mathbb{C}$, this result was first proved by Guillemin-Sternberg[5].)

Let dv_{X_G} be the Riemannian volume form on (X_G, g^{TX_G}). Let $\Pi_{G,p}$ be the orthogonal projection from $\mathscr{C}^\infty(X_G, L^p_G \otimes E_G)$ (with the scalar product $\langle\ \ \rangle$ induced by h^{L_G}, h^{E_G} and dv_{X_G} as in (3)), onto $H^0(X_G, L^p_G \otimes E_G)$.

Definition 3.1. A family of operators $T_p : H^0(X_G, L^p_G \otimes E_G) \to H^0(X_G, L^p_G \otimes E_G)$ is a Toeplitz operator if there exists a sequence of sections $g_l \in \mathscr{C}^\infty(X_G, \mathrm{End}(E_G))$ with an asymptotic expansion $g(\cdot, p)$ of the form $\sum_{l=0}^\infty p^{-l} g_l(x) + \mathscr{O}(p^{-\infty})$ in the $\mathscr{C}^\infty$ topology such that

$$T_p = \Pi_{G,p} g(\cdot, p) \Pi_{G,p} + \mathscr{O}(p^{-\infty}). \tag{10}$$

We call $g_0(x)$ the principal symbol of T_p.

For any $x \in X_G$, let $\mathrm{vol}(\pi^{-1}(x))$ be the volume of the orbit $\pi^{-1}(x)$ equipped with the metric induced by g^{TX}. We define the potential function

$$h(x) = \sqrt{\mathrm{vol}(\pi^{-1}(x))}. \tag{11}$$

For any $p > 0$, let P^G_p denote the orthogonal projection from $(\mathscr{C}^\infty(X, L^p \otimes E), \langle\ \ \rangle)$ to $H^0(X, L^p \otimes E)^G$. Set

$$\sigma^G_p = \sigma_p P^G_p : \mathscr{C}^\infty(X, L^p \otimes E) \to H^0(X_G, L^p_G \otimes E_G). \tag{12}$$

Let

$$(\sigma^G_p)^* : H^0(X_G, L^p_G \otimes E_G) \to \mathscr{C}^\infty(X, L^p \otimes E)$$

denote the adjoint of σ_p.

Theorem 3.1. *For any $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, let $f^G \in \mathscr{C}^\infty(X_G, \mathrm{End}(E_G))$ denote the associated G-invariant section defined by $f^G(x) = \int_G g f(g^{-1}x) dg$, here dg is a Haar measure on G. Then*

$$\mathcal{T}_p(f) = p^{-\frac{\dim G}{2}} \sigma^G_p f (\sigma^G_p)^* : H^0(X_G, L^p_G \otimes E_G) \to H^0(X_G, L^p_G \otimes E_G) \tag{13}$$

is a Toeplitz operator with principal symbol $2^{\frac{\dim G}{2}} \frac{f^G}{h^2}(x)$. *Especially,*

$$\mathcal{T}_p(f) = \Pi_{G,p} 2^{\frac{\dim G}{2}} \frac{f^G}{h^2} \Pi_{G,p} + \mathscr{O}(1/p) \tag{14}$$

as $p \to +\infty$. *In particular,* $p^{-\dim G/2} \sigma_p^G (\sigma_p^G)^*$ *is a Toeplitz operator with principal symbol* $2^{\dim G/2}/h^2$.

Corollary 3.1. *For any* $f_1, f_2 \in \mathscr{C}^\infty(X)$, *we identify them as sections of* $\mathrm{End}(E)$ *by multiplications, then one has*

$$[\mathcal{T}_p(f_1), \mathcal{T}_p(f_2)] = \frac{2^{\dim G}}{\sqrt{-1}p} \Pi_{G,p} \left\{ \frac{f_1^G}{h^2}, \frac{f_2^G}{h^2} \right\} \Pi_{G,p} + \mathscr{O}(p^{-2}). \tag{15}$$

One can view this corollary as a generalization of the Bordemann-Meinrenken-Schlichenmaier theorem, Theorem 1.1, in the framework of geometric quantization. If $E = \mathbb{C}$ and $G = \{1\}$, Corollary 3.1 is (5b). If $G = \{1\}$ and general E, Corollary 3.1 was obtained in Ref. 7, 8.

On the other hand, if one defines the unitary operator

$$\Sigma_p = (\sigma_p^G)^* (\sigma_p^G (\sigma_p^G)^*)^{-1/2} : H^0(X_G, L_G^p \otimes E_G) \to \mathscr{C}^\infty(X, L^p \otimes E), \tag{16}$$

then one has the following result:

Theorem 3.2. *For any* $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$,

$$T_p^G(f) = \Sigma_p^* f \Sigma_p : H^0(X_G, L_G^p \otimes E_G) \to H^0(X_G, L_G^p \otimes E_G) \tag{17}$$

is a Toeplitz operator on X_G *with principal symbol* f^G.

Remark 3.1. If $E = \mathbb{C}$, Paoletti[11] also claimed that $p^{-\frac{\dim G}{2}} \sigma_p^G (\sigma_p^G)^*$ is a Toeplitz operator. When $G = T^k$ is a torus, and $E = \mathbb{C}$, Theorem 3.2 was first proved by Charles[3].

Let $\langle\, , \rangle_{L_G^p \otimes E_G}$ be the metric on $L_G^p \otimes E_G$ induced by h^{L_G} and h^{E_G}. In view of Tian and Zhang's analytic approach (cf. Ref. 12. (3.54)) of geometric quantization conjecture of Guillemin-Sternberg, the natural Hermitian product on $\mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$ is the following weighted Hermitian product $\langle\, , \rangle_h$:

$$\langle s_1, s_2 \rangle_h = \int_{X_G} \langle s_1, s_2 \rangle_{L_G^p \otimes E_G}(x_0) h^2(x_0)\, dv_{X_G}(x_0). \tag{18}$$

Theorem 3.3. *The isomorphism* $(2p)^{-\frac{\dim G}{4}} \sigma_p$ *is an asymptotic isometry from* $(H^0(X, L^p \otimes E)^G, \langle\, , \rangle)$ *onto* $(H^0(X_G, L_G^p \otimes E_G), \langle\, , \rangle_h)$: *i.e. if* $\{s_i^p\}_{i=1}^{d_p}$

is an orthonormal basis of $(H^0(X, L^p \otimes E)^G, \langle\ ,\ \rangle)$*, then*

$$(2p)^{-\frac{\dim G}{2}} \langle \sigma_p s_i^p, \sigma_p s_j^p \rangle_h = \delta_{ij} + \mathscr{O}(\frac{1}{p}). \tag{19}$$

4. The asymptotic expansion of the *G*-invariant Bergman kernel

Definition 4.1. The G-invariant Bergman kernel $P_p^G(x, x')$ with $x,\ x' \in X$ is the smooth kernel of the orthogonal projection $P_p^G : \mathscr{C}^\infty(X, L^p \otimes E) \to H^0(X, L^p \otimes E)^G$ with respect to $dv_X(x')$.

Our proof of the results in Section 3 relies on the asymptotic behavior as $p \to +\infty$ of the G-invariant Bergman kernel $P_p^G(x, x')$. We now describe some behavior of $P_p^G(x, y)$, as $p \to +\infty$.

Let U be an arbitrary (fixed) small open G-invariant neighborhood of $\mu^{-1}(0)$. At first, we have that for any $x,\ x' \in X \setminus U$, as $p \to +\infty$,

$$|P_p^G(x, x')|_{\mathscr{C}^\infty} = \mathscr{O}(p^{-\infty}). \tag{20}$$

This result shows that when $p \to +\infty$, $P_p^G(x, x')$ "localizes" near $\mu^{-1}(0)$ (and thus close to X_G). The main technical result of Ref. 9. Theorem 2.2, and 10. Theorem 0.2 is the asymptotic expansion of $P_p^G(x, x')$ for $x,\ x' \in U$ when $p \to \infty$ whose proofs use techniques adapting from the works of Bismut-Lebeau [1], Dai-Liu-Ma[4] and Ma-Marinescu[6]. One key step is to deform the Laplacian of the spin^c Dirac operator by a Casimir type operator. We refer the readers to Ref. 9, 10 for the details.

References

1. J.-M. Bismut and G. Lebeau, *Complex immersions and Quillen metrics*, Inst. Hautes Études Sci. Publ. Math. (1991), no. 74, ii+298 pp. (1992).
2. M. Bordemann and E. Meinrenken and M. Schlichenmaier, *Toeplitz quantization of Kähler manifolds and* gl(N)*,* $N \to \infty$ *limits*, Comm. Math. Phys. **165** (1994), no. 2, 281–296.
3. L. Charles, *Toeplitz operators and hamiltonian torus action*, Preprint (2004), math.SG/0405128.
4. X. Dai, K. Liu, and X. Ma, *On the asymptotic expansion of Bergman kernel*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 3, 193–198. The full version: J. Differential Geom. to appear, math.DG/0404494.
5. V. Guillemin and S. Sternberg, *Geometric quantization and multiplicities of group representations*, Invent. Math. **67** (1982), no. 3, 515–538.
6. X. Ma and G. Marinescu, *Generalized Bergman kernels on symplectic manifolds*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 7, 493–498. The full version: math.DG/0411559.

7. X. Ma and G. Marinescu, *Toeplitz operators on symplectic manifolds*, Preprint.
8. X. Ma and G. Marinescu, *Holomorphic Morse inequalities and Bergman kernels*, book in preparation, (2005).
9. X. Ma and W. Zhang, *Bergman kernels and symplectic reduction*, C. R. Math. Acad. Sci. Paris **341** (2005), 297-302.
10. X. Ma and W. Zhang, *Bergman kernels and symplectic reduction*, Preprint 2005.
11. R. Paoletti, *The Szegö kernel of a symplectic quotient*, Adv. Math. **197** (2005), 523-553.
12. Y. Tian and W. Zhang, *An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg*, Invent. Math. **132** (1998), no. 2, 229–259.
13. W. Zhang, *Holomorphic quantization formula in singular reduction*, Commun. Contemp. Math. **1** (1999), no. 3, 281–293.