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η-INVARIANT AND MODULAR FORMS

by FEI HAN[†]

(*Department of Mathematics, National University of Singapore, Block S17, 10 Lower Kent Ridge Road, Singapore 119076*)

and WEIPING ZHANG[‡]

(*Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, P.R. China*)

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Abstract

We show that the Atiyah–Patodi–Singer reduced η-invariant of the twisted Dirac operator on a closed $4m-1$-dimensional spin manifold, with the twisted bundle being the Witten bundle appearing in the theory of elliptic genus, is a meromorphic modular form of weight $2m$ up to an integral q-series. We prove this result by combining our construction of certain modular characteristic forms associated to a generalized Witten bundle on spin^c-manifolds with a deep topological theorem due to Hopkins.

1. Introduction and statement of results

Let X be a smooth manifold. Let $T_{\mathbf{C}}X$ be the complexification of the tangent bundle TX. One defines the Witten bundle on X [13] as follows:

$$\Theta_q(TX) = \bigotimes_{u=1}^{\infty} S_{q^u}(T_{\mathbf{C}}X - \mathbf{C}^{\dim X}) \otimes \bigotimes_{v=1}^{\infty} \Lambda_{-q^{v-1/2}}(T_{\mathbf{C}}X - \mathbf{C}^{\dim X}), \tag{1.1}$$

where $S_t(\cdot)$ (respectively, $\Lambda_t(\cdot)$) denotes the symmetric (respectively, exterior) power and $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in \mathbf{H}$, the upper half-plane.

Let g^{TX} be a Riemannian metric on TX and ∇^{TX} be the associated Levi-Civita connection. If we write

$$\Theta_q(TX) = B_0(TX) + B_1(TX)q^{1/2} + B_2(TX)q + \cdots, \tag{1.2}$$

then each $B_i(TX)$ carries a Hermitian metric as well as a Hermitian connection $\nabla^{B_i(TX)}$ canonically induced from g^{TX} and ∇^{TX}. In this way, ∇^{TX} induces a Hermitian connection $\nabla^{\Theta_q(TX)}$ on the Witten bundle $\Theta_q(TX)$.

[†]Corresponding author. E-mail: mathanf@nus.edu.sg

[‡]E-mail: weiping@nankai.edu.cn

Now assume that X is closed, spin and of dimension $4m$. Let $S(TX) = S_+(TX) \oplus S_-(TX)$ be the corresponding Hermitian bundle of spinors. For each i, let $D_{X,+}^{B_i(TX)} : \Gamma(S_+(TX) \otimes B_i(TX)) \to \Gamma(S_-(TX) \otimes B_i(TX))$ be the corresponding twisted Dirac operator. It is an important and well-known fact (cf. [14]) that the q-series

$$\mathrm{Ind}(D_{X,+}^{\Theta_q(TX)}) = \sum_{i=0}^{\infty} \mathrm{Ind}(D_{X,+}^{B_i(TX)}) q^{i/2}, \tag{1.3}$$

which by the Atiyah–Singer index theorem [1] equals to the elliptic genus (We refer the reader to [9, Section 2.1; 16, Chapter 1] for the notation of the corresponding characteristic forms appearing below.)

$$\int_X \hat{A}(TX, \nabla^{TX}) \mathrm{ch}(\Theta_q(TX), \nabla^{\Theta_q(TX)}) = \sum_{i=0}^{\infty} q^{i/2} \int_X \hat{A}(TX, \nabla^{TX}) \mathrm{ch}(B_i(TX), \nabla^{B_i(TX)}) \tag{1.4}$$

is an integral modular form of weight $2m$ over $\Gamma^0(2)$, where $\Gamma^0(2)$ is the index 2 modular subgroup of $\mathrm{SL}_2(\mathbf{Z})$ defined by

$$\Gamma^0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \,\middle|\, b \equiv 0 \ (\mathrm{mod}\ 2) \right\}.$$

It is natural to look at what would happen if X is a $4m-1$-dimensional closed spin manifold. In this case, let E be a Hermitian vector bundle over X carrying a Hermitian connection ∇^E. Let $D_X^E : \Gamma(S(TX) \otimes E) \to \Gamma(S(TX) \otimes E)$ be the associated twisted Dirac operator, which is formally self-adjoint.

Following [2], for any $\mathrm{Re}(s) \gg 0$, set

$$\eta(D_X^E, s) = \sum_{\lambda \in \mathrm{Spec}(D_X^E) \setminus \{0\}} \frac{\mathrm{Sgn}(\lambda)}{|\lambda|^s}. \tag{1.5}$$

By [2], one knows that $\eta(D_X^E, s)$ is a holomorphic function in s with $\mathrm{Re}(s) > \dim X/2$. Moreover, it extends to a meromorphic function over $\mathbf{C}$, which is holomorphic at $s = 0$. The η invariant of D_X^E, in the sense of Atiyah–Patodi–Singer [2], is defined by

$$\eta(D_X^E) = \eta(D_X^E, 0),$$

while the reduced η invariant is defined and denoted by

$$\bar{\eta}(D_X^E) = \frac{\dim(\ker D_X^E) + \eta(D_X^E)}{2}.$$

It is the aim of this paper to study the modularity of the q-series

$$\bar{\eta}(D_X^{\Theta_q(TX)}) = \sum_{i=0}^{\infty} \bar{\eta}(D_X^{B_i(TX)}) q^{i/2}, \tag{1.6}$$

which is a spectral invariant depending on g^{TX}.

Assume temporarily that X is the boundary of a $4m$-dimensional spin manifold Y. Let g^{TY} be a Riemannian metric on TY, which is of product structure near $\partial Y = X$ and restricts to g^{TX} on X. By the Atiyah–Patodi–Singer index theorem established in [2], one has

$$\int_Y \hat{A}(TY, \nabla^{TY}) \mathrm{ch}(\Theta_q(TY), \nabla^{\Theta_q(TY)}) - \bar{\eta}(D_X^{\Theta_q(TX)}) \in \mathbf{Z}[[q^{1/2}]]. \tag{1.7}$$

The term of integration over Y in (1.7) is a modular form of weight $2m$ over $\Gamma^0(2)$ (similar to the modularity mentioned above for the elliptic genus in (1.4), cf. [11]), although it is not necessary to be an integral modular form anymore. Therefore, from (1.7), one sees that if X bounds a spin manifold, then for any Riemannian metric on TX, $\bar{\eta}(D_X^{\Theta_q(TX)})$, up to an integral q-series, is a modular form of weight $2m$ over $\Gamma^0(2)$.

Now the natural question is whether this modularity property for the reduced η-invariants holds for any $4m-1$-dimensional closed spin manifold. The main difficulty of this problem lies in the fact that, given a $4m-1$-dimensional closed spin manifold, it may happen that it does not bound a spin manifold.

Indeed, it is a well-known fact in cobordism theory that there is a positive integer k such that k disjoint copies of X bound a spin manifold $\tilde{Y}$. In this case, one has the following analogue of (1.7):

$$\int_{\tilde{Y}} \hat{A}(T\tilde{Y}, \nabla^{T\tilde{Y}})\, \mathrm{ch}(\Theta_q(T\tilde{Y}), \nabla^{\Theta_q(T\tilde{Y})}) - k\, \bar{\eta}(D_X^{\Theta_q(TX)}) \in \mathbf{Z}[[q^{1/2}]]. \tag{1.8}$$

From (1.8), one sees that $\bar{\eta}(D_X^{\Theta_q(TX)})$ is a modular form up to an element in $\mathbf{Z}[[q^{1/2}]]/k$. Thus, the natural classical method gives the conclusion that $\bar{\eta}\,(D_X^{\Theta_q(TX)})$ is a modular form up to an element in $\mathbf{Q}[[q^{1/2}]]$ instead of $\mathbf{Z}[[q^{1/2}]]$.

On the other hand, if $\tilde{g}$ is another Riemannian metric on TX with $\tilde{\nabla}^{TX}$ being its Levi-Civita connection and $\tilde{D}_X^{\Theta_q(TX)}$ being the corresponding twisted Dirac operator, then by the variation formula for the reduced η invariant (cf. [2, 3]), one has

$$\bar{\eta}(D_X^{\Theta_q(TX)}) - \bar{\eta}(\tilde{D}_X^{\Theta_q(TX)}) = \int_X \mathrm{CS}_\Phi(\nabla^{TX}, \tilde{\nabla}^{TX}, \tau) \bmod \mathbf{Z}[[q^{1/2}]], \tag{1.9}$$

where $\mathrm{CS}_\Phi(\nabla^{TX}, \tilde{\nabla}^{TX}, \tau)$ is the Chern–Simons transgression form associated to $\Phi(\nabla^{TX}, \tau) = \{\hat{A}(TX, \nabla^{TX})\mathrm{ch}(\Theta_q(TX), \nabla^{\Theta_q(TX)})\}^{(4m)}$. It is easy to see that $\int_X \mathrm{CS}_\Phi(\nabla^{TX}, \tilde{\nabla}^{TX}, \tau)$ is a modular form of weight $2m$ over $\Gamma^0(2)$ (cf. [7]). Thus, the variation of $\bar{\eta}(D_X^{\Theta_q(TX)})$ has mod $\mathbf{Z}$ modularity property. It turns out to be an interesting open problem that whether $\bar{\eta}(D_X^{\Theta_q(TX)})$ is by itself a modular form of weight $2m$ over $\Gamma^0(2)$ up to an element in $\mathbf{Z}[[q^{1/2}]]$.

The purpose of this short note is to give an answer to this question. Our main result can be stated as follows.

THEOREM 1.1 *Let X be a $4m-1$-dimensional closed spin Riemannian manifold. Then the reduced η-invariant $\bar{\eta}(D_X^{\Theta_q(TX)})$ of the twisted Dirac operator $D_X^{\Theta_q(TX)}$ is a meromorphic modular form of weight $2m$ over $\Gamma^0(2)$, up to an element in $\mathbf{Z}[[q^{1/2}]]$.*

Here meromorphic modular form is a weaker notion than modular form without requiring holomorphicity, but only meromorphicty on the upper half-plane.

To prove Theorem 1.1, instead of using the cobordism result as above, we make use of a result due to Hopkins (cf. [10, Section 8]) which asserts that for any complex vector bundle V over X, there is a non-negative integer s such that $X \times \mathbf{C}P^1 \times \cdots \times \mathbf{C}P^1$ (s-copies of $\mathbf{C}P^1$) bounds a spin manifold Y and $V \boxtimes H^s$ on $X \times \mathbf{C}P^1 \times \cdots \times \mathbf{C}P^1$ extends to Y, where H denotes the Hopf hyperplane bundle on $\mathbf{C}P^1$. We then apply the modular characteristic forms, which is associated to a generalized Witten bundle we have constructed in [9], on the bounding manifold, as well as the Atiyah–Patodi–Singer index theorem [2] to get the modularity of the reduced η-invariant in question.

It remains a challenge to find a purely analytic proof of Theorem 1.1 without using the deep topological results as the above-mentioned Hopkins' theorem.

Theorem 1.1 immediately implies that the quantity in (1.8) is a meromorphic modular form up to an element in $k\mathbf{Z}[[q^{1/2}]]$, where k is the positive integer such that k disjoint copies of X bounds $\tilde{Y}$ as explained before (1.8). Observe that in (1.8), each q-coefficient mod k is a mod k index studied by Freed and Melrose [8]. It is a topological invariant and the main result in [8] provides a topological interpretation of it. Therefore, as an application of Theorem 1.1, we have the following corollary.

COROLLARY 1.1 *Let Y be a $4m$-dimensional spin $\mathbf{Z}/k$-manifold in the sense of Sullivan (cf. [8]). Then the mod k index associated to the Witten bundle $\Theta_q(TY)$ can be represented by a meromorphic modular form of weight $2m$ over $\Gamma^0(2)$.*

On the other hand, in view of [5, (25)], which corresponds to the case of $k = 1$ in (1.8) for the category of stable almost complex manifolds, Theorem 1.1 might become a starting point of a kind of tertiary index theory, in the sense of [5, Theorem 4.2], for spin manifolds. Recently, Bunke informed us that Theorem 1.1 can be given an alternative proof by using the theory of the universal η invariant [4, Lemma 3.1], and a spin version of the f-invariant has also been constructed in [4, Definition 13.2].

For completeness, we would like to point out what happens in dimension $4m + 1$. Actually, when X is an $8n + 5$-dimensional closed spin manifold, since for each i, $\eta(D_X^{B_i(TX)}) = 0$ and $\dim(\ker D_X^{B_i(TX)})$ is even (cf. [2]), we have $\bar{\eta}(D_X^{\Theta_q(TX)}) = 0 \bmod \mathbf{Z}[[q^{1/2}]]$. In dimension $8n + 1$, since $\eta(D_X^{B_i(TX)}) = 0$ for each i (cf. [2]), we have $\bar{\eta}(D_X^{\Theta_q(TX)}) = \dim(\ker D_X^{\Theta_q(TX)})/2$. Therefore, in view of the Atiyah–Singer mod 2 index theorem, $\bar{\eta}(D_X^{\Theta_q(TX)})$ can be identified with Ochanine's beta invariant $\beta_q(X)$, the modularity of which has been shown in [12].

This paper is organized as follows. In Section 2, we briefly recall our construction (in [9]) of the modular form associated to a generalized Witten bundle involving a complex line bundle. In Section 3, we combine our modular form and the Hopkins boundary theorem to prove Theorem 1.1. In Section 4, we propose a possible refinement of Theorem 1.1 in $8n + 3$ dimension.

2. Complex line bundles and modular forms

In this section, we briefly review our construction (in [9]) of a modular form, which is associated to a generalized Witten bundle involving a complex line bundle.

Let M be a $4l$-dimensional Riemannian manifold. Let ∇^{TM} be the associated Levi-Civita connection.

Let ξ be a complex line bundle over M. Equivalently, one can view ξ as a rank 2 real oriented vector bundle over M. Let ξ carry a Euclidean metric and also a Euclidean connection ∇^{ξ}, let $c = e(\xi, \nabla^{\xi})$ be the Euler form associated to ∇^{ξ} (cf. [16, Section 3.4]). Let $\xi_{\mathbf{C}}$ be the complexification of ξ.

If E is a complex vector bundle over M, set $\tilde{E} = E - \dim E \in K(M)$.

Following [9, (2.5)], set

$$\Theta_q(TM,\xi)=\bigotimes_{u=1}^{\infty}S_{q^u}(\widetilde{T_{\mathbf{C}}M})\otimes\bigotimes_{v=1}^{\infty}\Lambda_{-q^{v-1/2}}(\widetilde{T_{\mathbf{C}}M}-2\widetilde{\xi_{\mathbf{C}}})$$
$$\otimes\bigotimes_{r=1}^{\infty}\Lambda_{q^{r-1/2}}(\widetilde{\xi_{\mathbf{C}}})\otimes\bigotimes_{t=1}^{\infty}\Lambda_{q^t}(\widetilde{\xi_{\mathbf{C}}}), \tag{2.1}$$

which is an element in $K(M)[[q^{1/2}]]$. As before, ∇^{TM} and ∇^{ξ} induce a Hermitian connection $\nabla^{\Theta_q(TM,\xi)}$ on $\Theta_q(TM,\xi)$.

Let $P(TM,\xi,\tau)\in\Omega^{4l}(M)$ be the characteristic form defined by

$$P(TM,\xi,\tau):=\left\{\hat{A}(TM,\nabla^{TM})\cosh\left(\frac{c}{2}\right)\operatorname{ch}(\Theta_q(TM,\xi),\nabla^{\Theta_q(TM,\xi)})\right\}^{(4l)}. \tag{2.2}$$

It is shown in [9] that $P(TM,\xi,\tau)$ can be expressed by using the formal Chern roots of $(T_{\mathbf{C}}M,\nabla^{T_{\mathbf{C}}M})$ and c through the Jacobi theta functions, which are defined as follows (cf. [6; 9, Section 2.3]):

$$\theta(v,\tau)=2q^{1/8}\sin(\pi v)\prod_{j=1}^{\infty}[(1-q^j)(1-\mathrm{e}^{2\pi\sqrt{-1}v}q^j)(1-\mathrm{e}^{-2\pi\sqrt{-1}v}q^j)],$$

$$\theta_1(v,\tau)=2q^{1/8}\cos(\pi v)\prod_{j=1}^{\infty}[(1-q^j)(1+\mathrm{e}^{2\pi\sqrt{-1}v}q^j)(1+\mathrm{e}^{-2\pi\sqrt{-1}v}q^j)],$$

$$\theta_2(v,\tau)=\prod_{j=1}^{\infty}[(1-q^j)(1-\mathrm{e}^{2\pi\sqrt{-1}v}q^{j-1/2})(1-\mathrm{e}^{-2\pi\sqrt{-1}v}q^{j-1/2})],$$

$$\theta_3(v,\tau)=\prod_{j=1}^{\infty}[(1-q^j)(1+\mathrm{e}^{2\pi\sqrt{-1}v}q^{j-1/2})(1+\mathrm{e}^{-2\pi\sqrt{-1}v}q^{j-1/2})].$$

The theta functions are all holomorphic functions for $(v,\tau)\in\mathbf{C}\times\mathbf{H}$, where $\mathbf{C}$ is the complex plane and $\mathbf{H}$ is the upper half-plane. Let $\{\pm 2\pi\sqrt{-1}x_i\}$ be the formal Chern roots for $(T_{\mathbf{C}}M,\nabla^{T_{\mathbf{C}}M})$ and $c=2\pi\sqrt{-1}u$, we have

$$P(TM,\xi,\tau)=\left\{\left(\prod_{i=1}^{2l}x_i\frac{\theta'(0,\tau)}{\theta(x_i,\tau)}\frac{\theta_2(x_i,\tau)}{\theta_2(0,\tau)}\right)\frac{\theta_1(u,\tau)}{\theta_1(0,\tau)}\frac{\theta_2^2(0,\tau)}{\theta_2^2(u,\tau)}\frac{\theta_3(u,\tau)}{\theta_3(0,\tau)}\right\}^{(4l)}. \tag{2.3}$$

By using the transformation laws of theta functions (cf. [6; 9, Section 2.3]), one sees as in [9, Proposition 2.6] that $P(TM,\xi,\tau)$ is a modular form of weight $2l$ over $\Gamma^0(2)$.

3. Proof of the main theorem

In this section, we will prove our main result Theorem 1.1.

The topological tool we will use is the following boundary theorem of Hopkins (cf. [10, Section 8]).

THEOREM 3.1 (Hopkins) *Let X be a compact, odd-dimensional spin manifold and $V \to X$ be a complex vector bundle over X. Then there is an integer s such that the vector bundle $V \boxtimes (\boxtimes_{j=1}^{s} H_j) \to X \times (\boldsymbol{C}P^1)^s$ is a boundary, where H_j denotes the Hopf hyperplane bundle on the jth copy of $\boldsymbol{C}P^1$. In other words, there is a spin manifold Y with a complex vector bundle W on Y such that $W|_{\partial Y} = V \boxtimes (\boxtimes_{j=1}^{s} H_j)$.*

In what follows, we will combine this Hopkins boundary theorem with the modular characteristic form constructed in Section 2 to give a proof of Theorem 1.1.

Proof of Theorem 1.1 Without loss of generality, for the $4m-1$-dimensional closed spin manifold X, in view of the Hopkins boundary theorem, we take an even integer s so that the complex line bundle

$$p^*(\boxtimes_{j=1}^{s} H_j) \to X \times (\mathbf{C}P^1)^s$$

bounds, where $p : X \times (\mathbf{C}P^1)^s \to (\mathbf{C}P^1)^s$ is the natural projection. This means that there is a spin manifold Y and a complex line bundle ζ over Y such that $\partial Y = X \times (\mathbf{C}P^1)^s$ and $\zeta|_{X\times(\mathbf{C}P^1)^s} = p^*(\boxtimes_{j=1}^{s} H_j)$.

Let g^{TX} be any Riemannian metric on X. Equip $(\mathbf{C}P^1)^s$ with arbitrary Riemannian metrics and the H_js with arbitrary Euclidean metrics and Euclidean connections.

Let g^{TY} be a metric on TY such that it is of product structure near $X \times (\mathbf{C}P^1)^s$ and restricts to the product metric on $X \times (\mathbf{C}P^1)^s$. Let ∇^{TY} be the Levi-Civita connection associated to g^{TY}.

Let g^{ζ} be an Euclidean metric on ζ (viewed as an oriented real plane bundle) such that g^{ζ} is of product structure near $X \times (\mathbf{C}P^1)^s$ and restricts to the Euclidean metric on $p^*(\boxtimes_{j=1}^{s} H_j)$ on $X \times (\mathbf{C}P^1)^s$. Let ∇^{ζ} be an Euclidean connection of g^{ζ} which is of product structure near $X \times (\mathbf{C}P^1)^s$ and restricts to the canonically induced Euclidean connection on $p^*(\boxtimes_{j=1}^{s} H_j)$ on $X \times (\mathbf{C}P^1)^s$.

Let $c = e(\zeta)$ and $z_j = c_1(H_j)/\pi\sqrt{-1}$, $1 \le j \le s$.

By applying the Atiyah–Patodi–Singer index theorem [2] to the twisted Dirac operator $D_Y^{\Theta_q(TY,\zeta^2)\otimes\zeta}$, in noting that

$$(\Theta_q(TY,\zeta^2)\otimes\zeta)|_{X\times(\mathbf{C}P^1)^s} = \Theta_q(T(X\times(\mathbf{C}P^1)^s),(p^*(\boxtimes_{j=1}^{s} H_j))^2)\otimes p^*(\boxtimes_{j=1}^{s} H_j),$$

one finds that there exist integers a_is such that

$$\begin{aligned}
&\bar{\eta}(D_{X\times(\mathbf{C}P^1)^s}^{\Theta_q(T(X\times(\mathbf{C}P^1)^s),(p^*(\boxtimes_{j=1}^{s} H_j))^2)\otimes p^*(\boxtimes_{j=1}^{s} H_j)}) \\
&\quad = \int_Y \hat{A}(TY,\nabla^{TY})\,\mathrm{ch}(\Theta_q(TY,\zeta^2)\otimes\zeta,\nabla^{\Theta_q(TY,\zeta^2)\otimes\zeta}) - \sum_{i=0}^{\infty} a_i q^{i/2} \\
&\quad = \int_Y \hat{A}(TY,\nabla^{TY})\,\mathrm{e}^{c}\,\mathrm{ch}(\Theta_q(TY,\zeta^2),\nabla^{\Theta_q(TY,\zeta^2)}) - \sum_{i=0}^{\infty} a_i q^{i/2} \\
&\quad = \int_Y \hat{A}(TY,\nabla^{TY})\cosh(c)\,\mathrm{ch}(\Theta_q(TY,\zeta^2),\nabla^{\Theta_q(TY,\zeta^2)}) - \sum_{i=0}^{\infty} a_i q^{i/2}, \qquad (3.1)
\end{aligned}$$

where the last equality follows from the fact that s is an even integer.

Let $r : X \times (\mathbf{C}P^1)^s \to X$ be the natural projection. For bundles $E \to X$ and $F \to (\mathbf{C}P^1)^s$, by separation of variables, we have

$$\eta(D_{X\times(\mathbf{C}P^1)^s}^{(r^*E)\otimes(p^*F)}) = \eta(D_X^E)\cdot \mathrm{Ind}(D^F_{(\mathbf{C}P^1)^s,+}).$$

So we have

$$\bar\eta(D_{X\times(\mathbf{C}P^1)^s}^{(r^*E)\otimes(p^*F)}) = \bar\eta(D_X^E)\cdot \mathrm{Ind}(D^F_{(\mathbf{C}P^1)^s,+}) + \dim(\ker D_X^E)\dim(\ker(D^F_{(\mathbf{C}P^1)^s,-})).$$

From the above formula, we can see that there are integers b_is such that

$$\begin{aligned}
&\bar\eta(D_{X\times(\mathbf{C}P^1)^s}^{\Theta_q(T(X\times(\mathbf{C}P^1)^s),(p^*(\boxtimes_{j=1}^s H_j))^2)\otimes p^*(\boxtimes_{j=1}^s H_j)}) - \sum_{i=0}^\infty b_i q^{i/2}\\
&= \bar\eta(D_{X\times(\mathbf{C}P^1)^s}^{\Theta_q(r^*TX\oplus p^*T(\mathbf{C}P^1)^s,(p^*(\boxtimes_{j=1}^s H_j))^2)\otimes p^*(\boxtimes_{j=1}^s H_j)}) - \sum_{i=0}^\infty b_i q^{i/2}\\
&= \bar\eta(D_{X\times(\mathbf{C}P^1)^s}^{r^*\Theta_q(TX)\otimes p^*(\Theta_q(T(\mathbf{C}P^1)^s,(\boxtimes_{j=1}^s H_j)^2)\otimes\boxtimes_{j=1}^s H_j)}) - \sum_{i=0}^\infty b_i q^{i/2}\\
&= \bar\eta(D_X^{\Theta_q(TX)})\cdot \mathrm{Ind}(D_{(\mathbf{C}P^1)^s,+}^{\Theta_q(T(\mathbf{C}P^1)^s,(\boxtimes_{j=1}^s H_j)^2)\otimes\boxtimes_{j=1}^s H_j})\\
&= \bar\eta(D_X^{\Theta_q(TX)})\cdot\int_{(\mathbf{C}P^1)^s}\hat A(T(\mathbf{C}P^1)^s,\nabla^{T(\mathbf{C}P^1)^s})\,\mathrm{e}^{c_1(H_1)+\cdots+c_1(H_s)}\,\mathrm{ch}(\Theta_q(T(\mathbf{C}P^1)^s,(\boxtimes_{j=1}^s H_j)^2))\\
&= \bar\eta(D_X^{\Theta_q(TX)})\cdot\int_{(\mathbf{C}P^1)^s}\left(\prod_{j=1}^s z_j\frac{\theta'(0,\tau)}{\theta(z_j,\tau)}\times\frac{\theta_2(z_j,\tau)}{\theta_2(0,\tau)}\right)\\
&\quad\frac{\theta_1(\sum_{j=1}^s z_j,\tau)}{\theta_1(0,\tau)}\frac{\theta_2^2(0,\tau)}{\theta_2^2(\sum_{j=1}^s z_j,\tau)}\frac{\theta_3(\sum_{j=1}^s z_j,\tau)}{\theta_3(0,\tau)}\\
&= \bar\eta(D_X^{\Theta_q(TX)})\cdot\int_{(\mathbf{C}P^1)^s}\frac{\theta_1(\sum_{j=1}^s z_j,\tau)}{\theta_1(0,\tau)}\frac{\theta_2^2(0,\tau)}{\theta_2^2(\sum_{j=1}^s z_j,\tau)}\frac{\theta_3(\sum_{j=1}^s z_j,\tau)}{\theta_3(0,\tau)},
\end{aligned}\tag{3.2}$$

where the last equality holds due to the fact that $x/\theta(x,\tau)$ and $\theta_2(x,\tau)$ are both even functions about x and $\int_{\mathbf{C}P^1} z_j^n = 0$ if $n > 1$.

Since s is an even integer, from the knowledge about the modular form $P(TM,\xi,\tau)$ constructed in Section 2, we know that

$$f_s(\tau) := \int_{(\mathbf{C}P^1)^s}\frac{\theta_1(\sum_{j=1}^s z_j,\tau)}{\theta_1(0,\tau)}\frac{\theta_2^2(0,\tau)}{\theta_2^2(\sum_{j=1}^s z_j,\tau)}\frac{\theta_3(\sum_{j=1}^s z_j,\tau)}{\theta_3(0,\tau)}$$

is an integral modular form of weight s over $\Gamma^0(2)$. Moreover, since

$$\int_{(\mathbf{C}P^1)^s}\hat A(T(\mathbf{C}P^1)^s,\nabla^{T(\mathbf{C}P^1)^s})\,\mathrm{e}^{c_1(H_1)+\cdots+c_1(H_s)} = 1,$$

we see that $f_s(\tau)$ has constant term 1. Therefore, $f_s^{-1}(\tau)\in\mathbf{Z}[[q^{1/2}]]$.

From (3.1) and (3.2), we have

$$\bar\eta(D_X^{\Theta_q(TX)}) = f_s^{-1}(\tau)\cdot\int_Y \hat A(TY,\nabla^{TY})\cosh(c)\,\mathrm{ch}(\Theta_q(TY,\zeta^2),\nabla^{\Theta_q(TY,\zeta^2)}) - f_s^{-1}(\tau)\cdot\left(\sum_{i=0}^{\infty}(a_i+b_i)q^{i/2}\right). \tag{3.3}$$

Still by the modularity of $P(TM,\xi,\tau)$ constructed in Section 2, we know that

$$\int_Y \hat A(TY,\nabla^{TY})\cosh(c)\,\mathrm{ch}(\Theta_q(TY,\zeta^2),\nabla^{\Theta_q(TY,\zeta^2)})$$

is a modular form of weight $2m+s$ over $\Gamma^0(2)$. So

$$f_s^{-1}(\tau)\cdot\int_Y \hat A(TY,\nabla^{TY})\cosh(c)\,\mathrm{ch}(\Theta_q(TY,\zeta^2),\nabla^{\Theta_q(TY,\zeta^2)})$$

is a meromorphic modular form of weight $2m$ over $\Gamma^0(2)$.

Therefore, from (3.3), we see that

$$\bar\eta(D_X^{\Theta_q(TX)}) = f_s^{-1}(\tau)\cdot\int_Y \hat A(TY,\nabla^{TY})\cosh(c)\,\mathrm{ch}(\Theta_q(TY,\zeta^2),\nabla^{\Theta_q(TY,\zeta^2)}) \bmod \mathbf{Z}[[q^{1/2}]],$$

a meromorphic modular form of weight $2m$ over $\Gamma^0(2)$. The proof of Theorem 1.1 is complete. □

REMARK 3.1 The modular form $f_s(\tau)$ in the above proof can be explicitly expressed by theta functions and their derivatives. For example, we have

$$f_2(\tau) = -\frac{1}{\pi^2}\left(\frac{\theta_1''(0,\tau)}{\theta_1(0,\tau)} - 2\frac{\theta_2''(0,\tau)}{\theta_2(0,\tau)} + \frac{\theta_3''(0,\tau)}{\theta_3(0,\tau)}\right) \tag{3.4}$$

and

$$f_4(\tau) = \frac{1}{\pi^4}\left(\frac{\theta_1^{(4)}(0,\tau)}{\theta_1(0,\tau)} - 2\frac{\theta_2^{(4)}(0,\tau)}{\theta_2(0,\tau)} + \frac{\theta_3^{(4)}(0,\tau)}{\theta_3(0,\tau)} + 18\left(\frac{\theta_2''(0,\tau)}{\theta_2(0,\tau)}\right)^2 - 12\frac{\theta_1''(0,\tau)}{\theta_1(0,\tau)}\frac{\theta_2''(0,\tau)}{\theta_2(0,\tau)} - 12\frac{\theta_3''(0,\tau)}{\theta_3(0,\tau)}\frac{\theta_2''(0,\tau)}{\theta_2(0,\tau)} + 6\frac{\theta_1''(0,\tau)}{\theta_1(0,\tau)}\frac{\theta_3''(0,\tau)}{\theta_3(0,\tau)}\right). \tag{3.5}$$

REMARK 3.2 Let X be a compact, odd-dimensional spin manifold. Define

$$H(X) := \{h\in\mathbf{Z} : \text{the line bundle } p^*(\boxtimes_{j=1}^h H_j)\to X\times(\mathbf{C}P^1)^h \text{ bounds}\},$$

where $p : X\times(\mathbf{C}P^1)^h \to (\mathbf{C}P^1)^h$ is the natural projection and H_j denotes the Hopf hyperplane bundle on the jth copy of $\mathbf{C}P^1$. Define the Hopkins' index of X, $h(X) := \min H(X)$. Obviously, when X is a boundary by itself, $h(X)=0$. It is clear that $H(X)=\{s\in\mathbf{Z} : s\geq h(X)\}$.

In the proof of Theorem 1.1, we may take any even number $s \in H(X)$ and denote the corresponding Y and ζ by Y_s and ζ_s. Then the proof of Theorem 1.1 tells us that, up to an element in $\mathbf{Z}[[q^{1/2}]]$,

$$\bar{\eta}(D_X^{\Theta_q(TX)}) = f_s^{-1}(\tau) \cdot \int_{Y_s} \hat{A}(TY_s, \nabla^{TY_s}) \cosh(e(\zeta_s)) \operatorname{ch}(\Theta_q(TY_s, \zeta_s^2), \nabla^{\Theta_q(TY_s, \zeta_s^2)}).$$

Clearly, if $h(X) = 0$, one gets (1.7). Therefore, for every even number $s \geq 2[(h(X)+1)/2]$, one can construct a meromorphic modular form of weight $2m$ over $\Gamma^0(2)$ of above form, that is equal to $\bar{\eta}(D_X^{\Theta_q(TX)})$ up to an element in $\mathbf{Z}[[q^{1/2}]]$. The poles of these meromorphic modular forms are just the zeros of the modular forms $f_s(\tau)$. We hope that further study of the modular forms $f_s(\tau)$ will bring better understanding of modularity of $\bar{\eta}(D_X^{\Theta_q(TX)})$.

REMARK 3.3 We refer the reader to [4] for an alternative approach to the modularity of $\bar{\eta}(D_X^{\Theta_q(TX)})$, which is shown to be not only a meromorphic modular form, but also a modular form using the theory of universal η-invariant.

4. The cases of dimension $8n+3$

In this section, we discuss the case of dimension $8n+3$. In this dimension, it is known that $\bar{\eta}(D_X^{\Theta_q(TX)})$ is mod $2\mathbf{Z}[[q^{1/2}]]$ smooth. That is, in the right-hand side of (1.9), the term mod $\mathbf{Z}[[q^{1/2}]]$ can be replaced by mod $2\mathbf{Z}[[q^{1/2}]]$. Therefore, it is natural to propose the following conjecture whose statement refines Theorem 1.1 in this case.

CONJECTURE 4.1 *Let X be an $8n+3$-dimensional closed spin Riemannian manifold. Then the reduced η-invariant $\bar{\eta}(D_X^{\Theta_q(TX)})$ of the twisted Dirac operator $D_X^{\Theta_q(TX)}$ is a meromorphic modular form of weight $4n+2$ over $\Gamma^0(2)$, up to an element in $2\mathbf{Z}[[q^{1/2}]]$.*

Recall that a mod $2k$ refinement of the Freed–Melrose mod k index for real vector bundles over $8n+4$-dimensional manifolds has been defined in [15, Section 3]. In view of this, one can propose a refinement of Corollary 1.1, in the case of $\dim Y = 8n+4$, as follows.

CONJECTURE 4.2 *Let Y be an $8n+4$-dimensional spin $\mathbf{Z}/k$-manifold in the sense of Sullivan (cf. [8]). Then the mod $2k$ index associated to the Witten bundle $\Theta_q(TY)$ can be represented by a meromorphic modular form of weight $4n+2$ over $\Gamma^0(2)$.*

By the method of this paper, in order to prove Conjectures 4.1 and 4.2, one perhaps needs a kind of Hopkins boundary theorem for real vector bundles. Or, one may try to develop a direct analytic approach, which, even for Theorem 1.1, is a challenging problem as we indicated in Section 1.

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