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A mod 2 index theorem for the twisted Signature operator *

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Abstract A mod 2 index theorem for the twisted Signature operator on $4q+1$ dimensional manifolds is established. This result generalizes a result of Farber and Turaev, which was proved for the case of orthogonal flat bundles, to arbitrary real vector bundles. It also provides an analytic interpretation of the sign of the Poincaré-Reidemeister scalar product defined by Farber and Turaev.

Keywords: twisted Signature operator, index theorem.

1 Main result

Let M be a $4q+1$ dimensional smooth oriented closed manifold and let g^{TM} be a metric on TM. Let ∇^{TM} be the Levi-Civita connection of g^{TM}. Then ∇^{TM} lifts canonically to a Euclidean connection $\nabla^{\Lambda^*(T^*M)}$ on the (real) exterior algebra bundle $\Lambda^*(T^*M)$. For any $e\in TM$, let $e^*\in T^*M$ correspond to e via g^{TM}. Let $c(e)$, $\hat{c}(e)$ be the Clifford operators acting on $\Lambda^*(T^*M)$ given by $c(e)=e^*\wedge - i_e$, $\hat{c}(e)=e^*\wedge + i_e$, where $e^*\wedge$ and i_e are the standard notation for exterior and inner multiplications.

Let E be a Euclidean vector bundle over M. Let ∇^E be a Euclidean connection on E. Let $\nabla^{\Lambda^*(T^*M)\otimes E}$ be the tensor product connection on $\Lambda^*(T^*M)\otimes E$ obtained from $\nabla^{\Lambda^*(T^*M)}$ and ∇^E.

If $e\in TM$, let $c(e)$ and $\hat{c}(e)$ act on $\Lambda^*(T^*M)\otimes E$ as $c(e)\otimes \mathrm{Id}_E$ and $\hat{c}(e)\otimes \mathrm{Id}_E$ respectively. Let $e_1,\cdots,e_{4q+1}$ be an oriented orthonormal base of TM.

The twisted Signature operator D^E (that is, twisted by E) can be defined as

$$D^E=\hat{c}(e_1)\cdots\hat{c}(e_{4q+1})\Big(\sum_{i=1}^{4q+1}c(e_i)\nabla_{e_i}^{\Lambda^*(T^*M)\otimes E}\Big):$$

$$\Gamma(\Lambda^{\mathrm{even}}(T^*M)\otimes E)\to\Gamma(\Lambda^{\mathrm{even}}(T^*M)\otimes E). \tag{1.1}$$

Clearly, D^E is a well-defined first order elliptic differential operator. Furthermore, one verifies easily that D^E is formally skew-adjoint. Thus, according to Atiyah and Singer[1], D^E admits a mod 2 index defined by

$$\mathrm{ind}_2 D^E\equiv\dim(\ker D^E)\ \mathrm{mod}\ 2, \tag{1.2}$$

which does not depend on the metrics and connections involved in its definition. Furthermore, the

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Atiyah-Singer mod 2 index theorem[1] provides a topological interpretation of $\mathrm{ind}_2 D^E$.

As a particular example, when $E=\mathbb{R}$, the trivial line bundle with its trivial connection and metric, one recovers the famous Kervaire semi-characteristic $k(M)$ defined by

$$k(M) = \sum_{i=0}^{2q} \dim H^{2i}(M;\mathbb{R}) \bmod 2. \tag{1.3}$$

In what follows, we will also denote $\mathrm{ind}_2 D^E$ by $k(M;E)$.

The main result of this paper can be stated as follows.

Theorem 1.1. *The following identity holds,*

$$k(M;E) \equiv k(M)\cdot \dim E + \langle w_1(E) w_{4q}(M), [M]\rangle \bmod 2, \tag{1.4}$$

where w_i is the notation for the i-th Stiefel-Whitney class.

When (E, g^E, ∇^E) is an orthogonal flat vector bundle, Theorem 1.1 has been proved by Farber and Turaev1). In some sense one may view Theorem 1.1 as a mod 2 analogue of the usual twisted Signature theorem on even dimensional manifolds (cf. (13.25) of reference [2]).

On the other hand, when E is a real flat vector bundle over M, Farber and Turaev1) defined a Poincaré-Reidemeister scalar product which is either positive or negative definite. In particular, they showed that whether this scalar product is positive or negative definite depends exactly on the right hand side of (1.4). Thus, Theorem 1.1 also provides an analytic interpretation of the sign of this scalar product.

Our proof of Theorem 1.1 is analytic and is thus different from the topological proof given by Farber and Turaev1) for the case of orthogonal flat vector bundles. In fact, we recall that in a previous paper2), we have proved a topological counting formula for the Kervaire semi-characteristic $k(M)$. In the next section, we will first show that a similar localization formula holds also for $k(M;E)$. Theorem 1.1 will then follow by comparing these two localization formulas.

2 A localization formula for $k(M;E)$

In this section, we prove a localization formula which extends the main result in footnote 2) to general twisted vector bundles. Since the proof of this localization formula is more or less the same as what in footnote 2), we will only explain the basic ideas and leave the details to the interested readers.

This section is organized as follows. In 2.1, inspired by ref.[3] we construct a skew-adjoint elliptic operator which gives an alternative analytic interpretation of $k(M;E)$. In 2.2, we state the main result of this section. In 2.3, inspired by footnote 2) we introduce a deformation for the skew-adjoint operator constructed in 2.1. This deformation will then be used in 2.4 to complete the proof of the main result of this section.

We make the same assumption and use the same notation as in Section 1.

2.1 An alternative analytic interpretation of $k(M;E)$

Since $\dim M$ is odd, by a theorem of Hopf (cf. ref.[4]), there always exists a nowhere zero vector field $V\in \Gamma(TM)$. Without loss of generality, we assume that

1) Farber, M., Turaev, V., Poincaré-Reidemeister metric, Euler structures and torsions, *Preprint*, math./9803137.

2) Zhang, W., A counting formula for the Kervaire semi-characteristic, To appear in *Topology*.

$$|V|^2_{g^{TM}} = 1. \tag{2.1}$$

Let $e_1, \cdots, e_{4q+1}$ be an oriented orthonormal base of TM.

Definition 2.1. The operator D_V^E is the operator acting on $\Gamma(\Lambda^{\mathrm{even}}(T^*M)\otimes E)$ defined by

$$D_V^E = \frac{1}{2}\Big(\hat{c}(V)\Big(\sum_{i=1}^{4q+1} c(e_i)\nabla_{e_i}^{\Lambda^*(T^*M)\otimes E}\Big) - \Big(\sum_{i=1}^{4q+1} c(e_i)\nabla_{e_i}^{\Lambda^*(T^*M)\otimes E}\Big)c(V)\Big). \tag{2.2}$$

Remark 2.1. When $E = \mathbb{R}$, the trivial line bundle with its trivial connection and metric, the above operator was first defined in Definition 2.1 of reference [3].

Since the Clifford actions c and $\hat{c}$ anticommute with each other, one verifies easily that D_V^E is a first order (real) skew-adjoint elliptic operator. We denote its mod 2 index in the sense of Atiyah and Singer[1] by $\mathrm{ind}_2 D_V^E$.

The following extension of Theorem 2.5 of ref. [3] provides an alternative analytic interpretation of $k(M;E)$.

Theorem 2.1. *The following identity holds,*

$$\mathrm{ind}_2 D_V^E = k(M;E). \tag{2.3}$$

Proof. The proof is almost the same as that of Theorem 2.5 of ref. [3]. One first verifies as in Prop. 2.2 of ref. [3] that

$$D_V^E = \hat{c}(V)\Big(\sum_{i=1}^{4q+1} c(e_i)\nabla_{e_i}^{\Lambda^*(T^*M)\otimes E}\Big) - \frac{1}{2}\sum_{i=1}^{4q+1} c(e_i)\hat{c}(\nabla_{e_i}^{TM}V). \tag{2.4}$$

Now, by using (2.1), one verifies that the elliptic operator

$$\widetilde{D}^E = D^E - \frac{1}{2}c(V)\hat{c}(e_1)\cdots\hat{c}(e_{4q+1})\sum_{i=1}^{4q+1} c(e_i)\hat{c}(\nabla_{e_i}^{TM}V) \tag{2.5}$$

is also skew-adjoint.

From (2.4) and (2.5), one verifies directly that

$$\dim(\ker D_V^E) = \dim(\ker \widetilde{D}^E). \tag{2.6}$$

One the other hand, from (2.5) and the homotopy invariance of the mod 2 index (ref. [1]), one sees that

$$\dim(\ker \widetilde{D}^E) \equiv \dim(\ker D^E) \mod 2. \tag{2.7}$$

From (2.6) and (2.7), one gets (2.3). Q.E.D.

2.2 A localization formula for $k(M;E)$

Let γ_V denote the oriented line bundle generated and oriented by V. Let $\gamma_V^\perp$ be the orthogonal complement to γ_V in TM. Then $\gamma_V^\perp$ carries an induced orientation from those of TM and γ_V. Let $g^{\gamma_V^\perp}$ be the metric on $\gamma_V^\perp$ induced from g^{TM}.

Let X be a transversal section of $\gamma_V^\perp$. Let F be the zero set of X. Then F consists of a union of disjoint circles. Let $i: F\to M$ denote the obvious embedding. Without loss of generality one may well assume that $V|_F$ is tangent to F and that $i^*\gamma_V^\perp$ is the normal bundle to F in M. (In fact, let f denote a unit tangent vector field of F. Then since $\dim\gamma_V^\perp$ is of codimension one, one verifies easily from the transversality assumption that $\langle V|_F, f\rangle$ is nowhere zero on F (as $V|_F$ should be transversal to the normal bundle to F). One can then deform V easily through nowhere zero vector fields to a nowhere zero vector field V', which is still transversal to $\gamma_V^\perp$, such that $V'|_F = \mathrm{sign}(\langle V|_F, f\rangle)f$. One can then start with V' and, by the homotopy invariance of the mod 2 index (ref. [1]), this does not affect the final result.)

For any $x \in F$, let $e_0 = V$, $e_1, \cdots e_{4q}$ be an oriented orthonormal base near x, and let $y_0, \cdots, y_{4q}$ be the normal coordinate system near x associated to $e_0(x), \cdots, e_{4q}(x)$. Then near x, X can be written as

$$X = \sum_{i=1}^{4q} f_i(y) e_i. \tag{2.8}$$

By the transversality of X, on sees that the following endomorphism of $\gamma_V^{\perp}|_x$ is invertible:

$$C(x) = \{c_{ij}(x)\}_{1 \leqslant i,j \leqslant 4q} \quad \text{with} \quad c_{ij}(x) = \frac{\partial f_i}{\partial y_j}(0), \tag{2.9}$$

where the matrix is with respect to the base $e_1(x), \cdots, e_{4q}(x)$.

Let $C^*(x)$ be the adjoint of $C(x)$ with respect to the fiber metric $g^{\gamma_V^{\perp}|}$, and $|C(x)| = \sqrt{C^*(x)C(x)}$. Let $K(x)$ be the endomorphism of $\Lambda^*(\gamma_V^{\perp,*}|_x)$ defined by

$$K(x) = \mathrm{Tr}[|C(x)|] + \sum_{i,j=1}^{4q} c_{ij}(x) c(e_j(x)) \hat{c}(e_i(x)). \tag{2.10}$$

One verifies easily that $K(x)$ does not depend on the choice of the orthonormal base $e_1(x), \cdots, e_{4q}(x)$ of $\gamma_V^{\perp}|_x$. Thus it defines an endomorphism K of the exterior algebra bundle $\Lambda^*(\gamma_V^{\perp,*})|_F$ over F.

Now by Prop. 2.21 of ref. [5], for any $x \in F$, $\dim(\ker K(x)) = 1$. One then deduces easily that $\ker K(x)$, $x \in F$, forms a real line bundle $o_F(X)$ over F. It admits a nature Euclidean metric as well as a natural Euclidean connection.

Let $D^{i^* E \otimes o_F(X)}$ be the twisted Signature operator on F in the sense of (1.1).

We can now state the main result of this section as follows.

Theorem 2.2. *The following identity holds*,

$$k(M;E) = \mathrm{ind}_2 D^{i^* E \otimes o_F(X)}, \tag{2.11}$$

where $\mathrm{ind}_2\, D^{i^* E \otimes o_F(X)}$ *is the* mod 2 *index of* $D^{i^* E \otimes o_F(X)}$ *in the sense of Atiyah and Singer*[1].

As an immediate consequence, if X has no zero, one gets the following partial extension of a result of Atiyah (Theorem 1.2 of reference [6]).

Corollary 2.1. *If there exist two linearly independent vector fields on* M, *then for any real vector bundle* E *over* M, *one has* $k(M;E) = 0$.

2.3 A deformation of the skew-adjoint operator D_V^E

We introduce the following deformation of D_V^E, extending Definition 2.1 of footnote 1).

Definition 2.2. For any $T \in \mathbb{R}$, let $D_{V,T}^E$ be the operator defined by

$$D_{V,T}^E = D_V^E + T\hat{c}(V)\hat{c}(X) : \Gamma(\Lambda^{\text{even}}(T^*M) \otimes E) \to \Gamma(\Lambda^{\text{even}}(T^*M) \otimes E). \tag{2.12}$$

As X is perpendicular to V, one verifies that $D_{V,T}^E$ is also skew-adjoint. Thus, by the homotopy invariance of the mod 2 index (ref. [1]), one has that for any $T \in \mathbb{R}$,

$$\dim(\ker D_{V,T}^E) \equiv \dim(\ker D_V^E) \quad \text{mod } 2. \tag{2.13}$$

1) Zhang, W., A counting formula for the Kervaire semi-characteristic, To appear in *Topology*.

We now state an important Bochner-type formula for $-D_{V,T}^{E,2}$ which can be proved in exactly the same way as in (2.3)—(2.6) of footnote 1):

$$-D_{V,T}^{E,2}=-D_{V}^{E,2}+T\sum_{i=0}^{4q}\left(c(e_i)\hat{c}(\nabla_{e_i}^{TM}X)-\langle\nabla_{e_i}^{TM}X,V\rangle c(e_i)\hat{c}(V)\right)+T^2|X|^2, \tag{2.14}$$

where $e_0,\cdots,e_{4q}$ is an oriented orthonormal base of TM.

2.4 Proof of Theorem 2.2

First of all, by using the Bochner type formula (2.14), one has the following direct analogue of Prop. 2.2 of footnote 1).

Proposition 2.1. *For any open neighborhood U of F, there exist constants $C'>0$, $b>0$ such that for any $T\geqslant 1$ and any $s\in\Gamma(\wedge^{\mathrm{even}}(T^*M)\otimes E)$ with* Supp $s\subset M\setminus U$, *one has the following estimate of Sobolev norms:*

$$\|D_{V,T}^{E}s\|_0^2\geqslant C'(\|s\|_1^2+(T-b)\|s\|_0^2). \tag{2.15}$$

By Proposition 2.1, we need only to concentrate on the analysis near F, which can be done in exactly the same way as what in Sec. 2b) and Appendix of footnote 1), which in turn relies on the methods and techniques developed in Secs. 8 and 9 of ref. [7]. The only place which needs a specific modification for our new situation is for the analogue of Prop. 2.4 of footnote 1), which we describe now.

Recall that $o_F(X)\subset\wedge^*(E^*|_F)$ denote the Euclidean line bundle formed by $\ker K(x)$, $x\in F$. Then $\wedge^*(T^*F)\otimes i^*E\otimes o_F(X)$ is a sub-bundle of $(\wedge^*(T^*M)\otimes E)|_F$. Let p denote the canonical orthogonal projection from $\Gamma((\wedge^*(T^*M)\otimes E)|_F)$ onto $\Gamma(\wedge^*(T^*F)\otimes i^*E\otimes o_F(X))$. Let $\hat{D}^{i^*E\otimes o_F(X)}$ be the operator acting on $\Gamma(\wedge^*(T^*F)\otimes i^*E\otimes o_F(X))$ defined by

$$\hat{D}^{i^*E\otimes o_F(X)}=c(V)\nabla_V^{\wedge^*(T^*F)\otimes i^*E\otimes o_F(X)}, \tag{2.16}$$

where $\nabla^{\wedge^*(T^*F)\otimes i^*E\otimes o_F(X)}$ is the natural tensor production connection obtained from $\nabla^{\wedge^*(T^*F)}$, $i^*\nabla^E$ and the canonically induced Euclidean connection on $o_F(X)$.

Now set as in (2.14) of footnote 1)

$$D^H=\sum_{i=0}^{4q}c(e_i)(i^*\nabla^{\wedge^*(T^*M)\otimes E})(e_i). \tag{2.17}$$

From (2.16), (2.17), and proceeding similarly as in Sec. 2b) of footnote 1), one gets the following analogue of Prop. 2.4 of footnote 1).

Proposition 2.2. *The following identity for differential operators acting on $\Gamma(\wedge^{\frac{1-\mathrm{sgn}\det(C)}{2}}(T^*F)\otimes i^*E\otimes o_F(X))$, where we use the standard notation that* $\mathrm{sgn}\det(C)=1$ *if* $\det(C)>0$ *and* $\mathrm{sgn}\det(C)=-1$ *if* $\det(C)<0$, *holds:*

$$pc(V)D^Hp=\hat{c}(V)\hat{D}^{i^*E\otimes o_F(X)}. \tag{2.18}$$

Furthermore, the operator $\hat{c}(V)\hat{D}^{i^*E\otimes o_F(X)}$ is skew-adjoint.

Proof of Theorem 2.2. Let $c_0>0$ be such that the operator $-(\hat{c}(V)\hat{D}^{i^*E\otimes o_F(X)})^2$ acting on

1) Zhang, W., A counting formula for the Kervaire semi-characteristic, To appear in *Topology*.

$\Gamma(\Lambda^{\frac{1-\operatorname{sgn}\det(C)}{2}}(T^*F)\otimes o_F(X))$ contains no eigenvalues in $(0,2c_0)$.

By Propositions 2.1, 2.2 and Lemma 2.3 of footnote 1) which goes back to Corollary 2.2 of ref. [5], one can proceed as in Sec. 2b) and Appendix of footnote 1) to prove the following analogue of (2.17) of footnote 1). That is, there exists $T_0>0$ such that for any $T\geqslant T_0$,

$$\#\{\lambda:\lambda\in \operatorname{Sp}(-D_{V,T}^2),\lambda\leqslant c_0\} = \dim(\ker(\hat{c}(V)\hat{D}^{i^*E\otimes o_F(X)})^2). \tag{2.19}$$

From (2.13), (2.19), Theorem 2.1 and the skew-adjointness of $D_{V,T}^E$, one gets

$$k(M;E) = \operatorname{ind}_2\hat{c}(V)\hat{D}^{i^*E\otimes o_F(X)}. \tag{2.20}$$

Now by a simple application of the Atiyah-Singer index theorem (cf. ref. [2]), one sees that the following identity holds for each connected component of F,

$$\dim(\ker D^{i^*E\otimes o_F(X)}|_{\Lambda^{\text{even}}(T^*F)\otimes i^*E\otimes o_F(X)}) - \dim(\ker D^{i^*E\otimes o_F(X)}|_{\Lambda^{\text{odd}}(T^*F)\otimes i^*E\otimes o_F(X)})$$
$$= \chi(F)\dim E = 0. \tag{2.21}$$

By (2.16), (2.20), (2.21) and the definition of the twisted Signature operator $D^{i^*E\otimes o_F(X)}$, one sees directly that

$$\operatorname{ind}_2\hat{c}(V)\hat{D}^{i^*E\otimes o_F(X)} = \operatorname{ind}_2 D^{i^*E\otimes o_F(X)}. \tag{2.22}$$

Theorem 2.2 follows from (2.20) and (2.22). Q.E.D.

3 A proof of Theorem 1.1

By Theorem 2.2 and its application to the $E=\mathbb{R}$ case, which has been proved in footnote 1), one needs only to prove the following result.

Proposition 3.1. *The following identity holds*:

$$\operatorname{ind}_2 D^{i^*E\otimes o_F(X)} = (\operatorname{ind}_2 D^{o_F(X)})\dim E + \langle w_1(E)w_{4q}(M),[M]\rangle \mod 2. \tag{3.1}$$

Proof. We first assume that $\dim E$ is even. By the standard obstruction theory one knows that $[F]\in H_1(M;\mathbb{Z}_2)$ is dual to $w_{4q}(M)$. Thus, one has

$$\langle w_1(E)w_{4q}(M),[M]\rangle = \langle w_1(i^*E),[F]\rangle. \tag{3.2}$$

Thus in order to prove (3.1), one needs only to prove that

$$\operatorname{ind}_2 D^{i^*E\otimes o_F(X)} = (\operatorname{ind}_2 D^{o_F(X)})\dim E + \langle w_1(i^*E),[F]\rangle. \tag{3.3}$$

Let $F_1,\cdots,F_p$ be the connected components of F. Then each F_i is a circle. For each $1\leqslant i\leqslant p$, let v_{F_i} denote the Möbius line bundle, which is the unique nonorientable line bundle, over F_i. We now discuss the situation in three cases.

Case (i). If $i^*(E)|_{F_i}$ is orientable, then since $\dim E$ is even, one verifies that $(i^*E\otimes o_F(X))|_{F_i}$ is orientable.

Case (ii). If $(i^*E)|_{F_i}$ is nonorientable and $(o_F(X))|_{F_i}$ is orientable, then one verifies that topologically, $(i^*E\otimes o_F(X))|_{F_i} = (\mathbb{R}|_{F_i})^{\dim E-1}\oplus v_{F_i}$.

Case (iii). If both $(i^*E)|_{F_i}$ and $(o_F(X))|_{F_i}$ are nonorientable, then one verifies, in using the fact that $\dim E$ is even, that $(i^*E\otimes o_F(X))|_{F_i} = (\mathbb{R}|_{F_i}{}^{\dim E-1}\oplus v_{F_i})$.

1) Zhang, W., A counting formula for the Kervaire semi-characteristic, To appear in *Topology*.

From the discussions about these three cases, one sees easily that (3.3) holds on each F_i when $\dim E$ is even. Summing up over i, one gets Proposition 3.1 in this case.

When $\dim E$ is odd, one can apply the above discussion to $E \oplus \mathbb{R}$ to get Proposition 3.1.

Q.E.D.

The proof of Theorem 1.1 is also completed. Q.E.D.

4 An analogue in $4q+3$ dimension

We now assume that $\dim M = 4q+3$, instead of $4q+1$. In this situation, the twisted Signature operators defined as in (1.1) are no longer skew-adjoint. However, the analysis concerning the operators D_V^E and $D_{V,T}^E$ still makes sense. It turns out that in this dimension, instead of Theorem 1.1, what one gets is

Proposition 4.1. *The following identity holds*:

$$\mathrm{ind}_2\, D_V^E \equiv (\mathrm{ind}_2 D_V^{\mathbb{R}})\dim E + \langle w_1(E)\, w_{4q+2}(M), [M]\rangle \mod 2. \tag{4.1}$$

Now, by proceeding similarly as in the proof of Theorem 3.2 of footnote 1) one verifies easily that $\mathrm{ind}_2 D_V^E = 0$ for any real vector bundle over M. Thus by Proposition 4.1 one has

$$\langle w_1(E)\, w_{4q+2}(M), [M]\rangle = 0. \tag{4.2}$$

Since the map w_1: $\mathrm{Vect}_{\mathbb{R}}(M) \to H^1(M; \mathbb{Z}_2)$ is onto, from (4.2) one recovers analytically the following result of Massey[8]: on a $4q+3$ dimensional smooth orientable closed manifold M, $w_{4q+2}(M) = 0$.

References

1 Atiyah, M. F., Singer, I. M., The index of elliptic operators V, *Ann. of Math.*, 1971, 93: 139.

2 Lawson, H. B., Michelsohn, M. -L., *Spin Geometry*, Princeton: Princeton Univ. Press, 1989.

3 Zhang, W., Analytic and topological invariants associated to nowhere zero vector fields, *Pacific J. Math.*, 1999, 187: 379.

4 Steenrod, N., *The Topology of Fiber Bundles*, Princeton: Princeton Univ. Press, 1951.

5 Shubin, M., Novikov inequalities for vector fields, in *The Gelfand Mathematical Seminar*, 1993—1995 (eds. Gelfand, I. M., Lepowski, J., Smirnov, M.), Boston: Birkhäuser, 1996, 243—274.

6 Atiyah, M. F., Vector fields on manifolds, *Arbeitsgemeinschaft für Forschung des Landes Nordrhein-Westfalen*, *Dusseldorf* 1969, 1970, 200: 7.

7 Bismut, J. -M., Lebeau, G., Complex immersions and Quillen metrics, *Publ. Math. IHES*, 1991, 74: 1.

8 Massey, W., On the Stiefel-Whitney classes of a manifold, *Amer. J. Math.*, 1960, 82: 92.

1) Zhang, W., A counting formula for the Kervaire semi-characteristic, To appear in *Topology*.