Chapter 13

η-INVARIANT AND FLAT VECTOR BUNDLES II

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We first apply the method and results in the previous paper to give a new proof of a result (holds in $\mathbf{C}/\mathbf{Z}$) of Gilkey on the variation of η-invariants associated to non self-adjoint Dirac type operators. We then give an explicit local expression of certain η-invariant appearing in recent papers of Braverman-Kappeler on what they call refined analytic torsion, and propose an alternate formulation of their definition of the refined analytic torsion. A refinement in $\mathbf{C}$ of the above variation formula is also proposed.

Keywords: Flat vector bundle, η-invariant, refined analytic torsion.

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1. Introduction

In a previous paper,[14] we have given an alternate formulation of (the mod $\mathbf{Z}$ part of) the η-invariant of Atiyah-Patodi-Singer[1–3] associated to non-unitary flat vector bundles by identifying explicitly its real and imaginary parts.

On the other hand, Gilkey has studied this kind of η-invariants systematically in,[13] and in particular proved a general variation formula for them.

However, it lacks in[13] the identification of the real and imaginary parts of the η-invariants as we did in.[14]

In this article, we first show that our results in[14] lead to a direct derivation of Gilkey's variation formula Theorem 3.7.[13]

The second purpose of this paper is to apply the results in[14] to examine the η-invariants appearing in the recent papers of Braverman-Kappeler[7–9] on refined analytic torsions. We show that the imaginary part of the η-invariant appeared in these articles admits an explicit local expression which suggests an alternate formulation of the definition of the refined analytic torsion there. This reformulation provides an analytic resolution of a problem due to Burghelea[10,11] on the existence of a univalent holomorphic function on the representation space having the Ray-Singer analytic torsion as its absolute value.

Finally, using the extension (to the case of non-self-adjoint operators) given in[18] of the concept of spectral flow,[3] we propose a refinement in $\mathbf{C}$ of the above variation formula for η-invariants.

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2. η-Invariant and the Variation Formula

Let M be an odd dimensional oriented closed spin manifold carrying a Riemannian metric g^{TM}. Let $S(TM)$ be the associated Hermitian vector bundle of spinors. Let (E, g^E) be a Hermitian vector bundle over M carrying a unitary connection ∇^E. Moreover, let (F, g^F) be a Hermitian vector bundle over M carrying a flat connection ∇^F. We do not assume that ∇^F preserves the Hermitian metric g^F on F.

Let $D^{E\otimes F} : \Gamma(S(TM) \otimes E \otimes F) \longrightarrow \Gamma(S(TM) \otimes E \otimes F)$ denote the corresponding (twisted) Dirac operator.

It is pointed out in Page 93[3] that one can define the reduced η-invariant of $D^{E\otimes F}$, denoted by $\overline{\eta}(D^{E\otimes F})$, by working on (possibly) non-self-adjoint elliptic operators.

In this section, we will first recall the main result in Ref. 14 on $\overline{\eta}(D^{E\otimes F})$ and then show how it leads directly to a proof of the variation formula of Gilkey, Theorem 3.7.[13]

2.1. *Chern-Simons classes and flat vector bundles*

We fix a square root of $\sqrt{-1}$ and let $\varphi : \Lambda(T^*M) \to \Lambda(T^*M)$ be the homomorphism defined by $\varphi : \omega \in \Lambda^i(T^*M) \to (2\pi\sqrt{-1})^{-i/2}\omega$. The formulas in what follows will not depend on the choice of the square root of $\sqrt{-1}$.

If W is a complex vector bundle over M and ∇_0^W, ∇_1^W are two connections on W. Let W_t, $0 \le t \le 1$, be a smooth path of connections on W connecting ∇_0^W and ∇_1^W. We define the Chern-Simons form $CS(\nabla_0^W, \nabla_1^W)$ to be the differential form given by

$$CS\left(\nabla_0^W, \nabla_1^W\right) = -\left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{1}{2}} \varphi \int_0^1 \mathrm{Tr}\left[\frac{\partial \nabla_t^W}{\partial t} \exp\left(-\left(\nabla_t^W\right)^2\right)\right] dt. \tag{2.1}$$

Then (cf. Chapter 1[17])

$$dCS\left(\nabla_0^W, \nabla_1^W\right) = \mathrm{ch}\left(W, \nabla_1^W\right) - \mathrm{ch}\left(W, \nabla_0^W\right). \tag{2.2}$$

Moreover, it is well-known that up to exact forms, $CS(\nabla_0^W, \nabla_1^W)$ does not depend on the path of connections on W connecting ∇_0^W and ∇_1^W.

Let (F, ∇^F) be a flat vector bundle carrying the flat connection ∇^F. Let g^F be a Hermitian metric on F. We do not assume that ∇^F preserves g^F. Let $(\nabla^F)^*$ be the adjoint connection of ∇^F with respect to g^F.

From (4.1), (4.2)[6] and §1(g),[5] one has

$$\left(\nabla^F\right)^* = \nabla^F + \omega\left(F, g^F\right) \tag{2.3}$$

with

$$\omega\left(F, g^F\right) = \left(g^F\right)^{-1}\left(\nabla^F g^F\right). \tag{2.4}$$

Then

$$\nabla^{F,e} = \nabla^F + \frac{1}{2}\omega\left(F, g^F\right) \tag{2.5}$$

is a Hermitian connection on (F, g^F) (cf. (4.3)[6]).

Following (2.6)[14] and (2.47),[15] for any $r \in \mathbf{C}$, set

$$\nabla^{F,e,(r)} = \nabla^{F,e} + \frac{\sqrt{-1}r}{2}\omega\left(F, g^F\right). \tag{2.6}$$

Then for any $r \in \mathbf{R}$, $\nabla^{F,e,(r)}$ is a Hermitian connection on (F, g^F).

On the other hand, following (0.2),[5] for any integer $j \ge 0$, let $c_{2j+1}(F, g^F)$ be the Chern form defined by

$$c_{2j+1}\left(F, g^F\right) = \left(2\pi\sqrt{-1}\right)^{-j} 2^{-(2j+1)} \mathrm{Tr}\left[\omega^{2j+1}\left(F, g^F\right)\right]. \tag{2.7}$$

Then $c_{2j+1}(F, g^F)$ is a closed form on M. Let $c_{2j+1}(F)$ be the associated cohomology class in $H^{2j+1}(M, \mathbf{R})$, which does not depend on the choice of g^F.

For any $j \geq 0$ and $r \in \mathbf{R}$, let $a_j(r) \in \mathbf{R}$ be defined as

$$a_j(r) = \int_0^1 \left(1 + u^2 r^2\right)^j du \,. \tag{2.8}$$

With these notation we can now state the following result first proved in Lemma 2.12.[15]

Proposition 1: *The following identity in $H^{\mathrm{odd}}(M, \mathbf{R})$ holds for any $r \in \mathbf{R}$,*

$$CS\left(\nabla^{F,e}, \nabla^{F,e,(r)}\right) = -\frac{r}{2\pi} \sum_{j=0}^{+\infty} \frac{a_j(r)}{j!} c_{2j+1}(F) \,. \tag{2.9}$$

2.2. *η-invariant associated to flat vector bundles*

Let

$$D^{E\otimes F,e} : \Gamma(S(TM) \otimes E \otimes F) \longrightarrow \Gamma(S(TM) \otimes E \otimes F) \tag{2.10}$$

denote the Dirac operator associated to the connection $\nabla^{F,e}$ on F and ∇^E on E. Then $D^{E\otimes F,e}$ is formally self-adjoint and one can define the associated reduced η-invariant as in.[1]

In view of Proposition 1, one can restate the main result of,[14] which is Theorem 2.2,[14] as follows,

$$\overline{\eta}\left(D^{E\otimes F}\right) \equiv \overline{\eta}\left(D^{E\otimes F,e}\right) + \int_M \widehat{A}(TM)\mathrm{ch}(E) CS\left(\nabla^{F,e}, \nabla^F\right) \mod \mathbf{Z}\,, \tag{2.11}$$

where $\widehat{A}(TM)$ and $\mathrm{ch}(E)$ are the $\widehat{A}$ class of TM and the Chern character of E respectively.[17]

Now let $\widetilde{\nabla}^F$ be another flat connection on F. We use the notation with $\widetilde{\ }$ to denote the objects associated with this flat connection.

Then one has

$$\overline{\eta}\left(\widetilde{D}^{E\otimes F}\right) \equiv \overline{\eta}\left(\widetilde{D}^{E\otimes F,e}\right) + \int_M \widehat{A}(TM)\mathrm{ch}(E) CS\left(\widetilde{\nabla}^{F,e}, \widetilde{\nabla}^F\right) \mod \mathbf{Z}\,. \tag{2.12}$$

By the variation formula for η-invariants associated to self-adjoint Dirac operators,[1,4] one knows that

$$\overline{\eta}\left(\widetilde{D}^{E\otimes F,e}\right)-\overline{\eta}\left(D^{E\otimes F,e}\right)\equiv\int_M\widehat{A}(TM)\mathrm{ch}(E)CS\left(\nabla^{F,e},\widetilde{\nabla}^{F,e}\right)\mod \mathbf{Z}. \tag{2.13}$$

From (2.11)–(2.13), one deduces that

$$\overline{\eta}\left(\widetilde{D}^{E\otimes F}\right)-\overline{\eta}(D^{E\otimes F})\equiv\int_M\widehat{A}(TM)\mathrm{ch}(E)CS\left(\nabla^{F,e},\widetilde{\nabla}^{F,e}\right) \tag{2.14}$$

$$-\int_M\widehat{A}(TM)\mathrm{ch}(E)CS\left(\nabla^{F,e},\nabla^{F}\right)+\int_M\widehat{A}(TM)\mathrm{ch}(E)CS\left(\widetilde{\nabla}^{F,e},\widetilde{\nabla}^{F}\right)$$

$$=\int_M\widehat{A}(TM)\mathrm{ch}(E)CS\left(\nabla^{F},\widetilde{\nabla}^{F}\right)\mod \mathbf{Z},$$

which is exactly the Gilkey formula, Theorem 1.6[13] for the operator $P = D^E$ therein.

Remark 2: As was indicated in Remark 2.4,[14] the main result in[14] holds also for general Hermitian vector bundles equipped with a (possibly) non-Hermitian connection. Indeed, if we do not assume that ∇^F is flat, then at least (2.3)-(2.6) still holds. Thus for any $r \in \mathbf{R}$, we have well-defined (formally self-adjoint) operator $D^{E\otimes F}(r)$ which is associated to the Hermitian connection $\nabla^{F,e,(r)}$ on F. For any $r \in \mathbf{R}$, one then has the variation formula[1,4]

$$\overline{\eta}\left(D^{E\otimes F}(r)\right)-\overline{\eta}\big(D^{E\otimes F,e}\big)\equiv\int_M\widehat{A}(TM)\mathrm{ch}(E)CS\big(\nabla^{F,e},\nabla^{F,e,(r)}\big)\mod \mathbf{Z}. \tag{2.15}$$

By (2.1), one sees easily that the right hand side of (2.15) is a holomorphic function (indeed a polynomial) of r. Thus, by analytic continuity, as in,[14] one gets that for any $r \in \mathbf{C}$, (2.15) still holds. In particular, if we set $r = \sqrt{-1}$, we get

$$\overline{\eta}\left(D^{E\otimes F}\right)\equiv\overline{\eta}\left(D^{E\otimes F,e}\right)+\int_M\widehat{A}(TM)\mathrm{ch}(E)CS\left(\nabla^{F,e},\nabla^{F}\right)\mod \mathbf{Z}, \tag{2.16}$$

which generalizes (2.11). Then by proceeding as above, we see that (2.14) holds without the assumption of the flatness of connections ∇^F and $\widetilde{\nabla}^F$.

By (2.1) and (2.6),

$$CS\left(\nabla^{F,e}, \nabla^{F,e,(r)}\right) = \frac{-1}{2\pi}\int_0^1 \operatorname{Tr}\left[\frac{r}{2}\omega\left(F, g^F\right) \times \exp\left(\frac{-1}{2\pi\sqrt{-1}}\left(\nabla^{F,e,(tr)}\right)^2\right)\right] dt = \sum_{i=0}^{\dim M} a_i\left(\nabla^F, g^F\right) r^i . \tag{2.17}$$

By (2.6), one has

$$\left(\nabla^{F,e,(r)}\right)^2 = \left(\nabla^{F,e}\right)^2 + \frac{\sqrt{-1}r}{2}\left(\nabla^{F,e}\omega\left(F, g^F\right)\right) - \frac{r^2}{4}\left(\omega\left(F, g^F\right)\right)^2 . \tag{2.18}$$

Note that

$$\nabla^{F,e}\omega\left(F, g^F\right) = \left[\nabla^{F,e}, \omega\left(F, g^F\right)\right] = 0, \quad \text{if } \nabla^F \quad \text{is flat.} \tag{2.19}$$

By taking adjoint of (2.18), we see that when $r \in \mathbf{C}$ is purely imaginary, one has

$$\left(\frac{1}{2\pi\sqrt{-1}}\left(\nabla^{F,e,(r)}\right)^2\right)^* = \frac{1}{2\pi\sqrt{-1}}\left(\left(\nabla^{F,e}\right)^2 - \frac{r^2}{4}\left(\omega\left(F, g^F\right)\right)^2\right) - \left(\frac{1}{2\pi\sqrt{-1}}\right)\frac{\sqrt{-1}r}{2}\left(\nabla^{F,e}\omega\left(F, g^F\right)\right) . \tag{2.20}$$

From (2.1), (2.17) and (2.20), one sees that when $r \in \mathbf{C}$ is purely imaginary, then

$$\begin{aligned} \operatorname{Re}\left(CS\left(\nabla^{F,e}, \nabla^{F,e,(r)}\right)\right) &= \sum_{i \text{ even}} a_i\left(\nabla^F, g^F\right) r^i , \\ \operatorname{Im}\left(CS\left(\nabla^{F,e}, \nabla^{F,e,(r)}\right)\right) &= \frac{1}{\sqrt{-1}}\sum_{i \text{ odd}} a_i\left(\nabla^F, g^F\right) r^i . \end{aligned} \tag{2.21}$$

Thus when $r \in \mathbf{C}$ is purely imaginary, from (2.16) and (2.21), we have

$$\operatorname{Re}(\overline{\eta}(D^{E\otimes F}(r))) \equiv \overline{\eta}(D^{E\otimes F,e}) + \sum_{i \text{ even}} r^i \int_M \widehat{A}(TM)\operatorname{ch}(E) a_i(\nabla^F, g^F) \bmod \mathbf{Z},$$

$$\operatorname{Im}\left(\overline{\eta}\left(D^{E\otimes F}(r)\right)\right) = \frac{1}{\sqrt{-1}}\sum_{i \text{ odd}} r^i \int_M \widehat{A}(TM)\operatorname{ch}(E) a_i\left(\nabla^F, g^F\right) . \tag{2.22}$$

In particular, by setting $r = \sqrt{-1}$, we get

$$\begin{aligned}\operatorname{Re}\left(\overline{\eta}(D^{E\otimes F})\right) &\equiv \overline{\eta}\left(D^{E\otimes F,e}\right)\\ &\quad + \sum_{i \text{ even}} (-1)^{\frac{i}{2}} \int_M \widehat{A}(TM)\operatorname{ch}(E)a_i(\nabla^F, g^F) \mod \mathbf{Z},\\ \operatorname{Im}\left(\overline{\eta}\left(D^{E\otimes F}\right)\right) &= \sum_{i \text{ odd}} (-1)^{\frac{i-1}{2}} \int_M \widehat{A}(TM)\operatorname{ch}(E)a_i\left(\nabla^F, g^F\right). \qquad (2.23)\end{aligned}$$

This generalizes the main result in Ref. 14.

3. η-Invariant and the Refined Analytic Torsion of Braverman-Kappeler

Recently, in a series of preprints,[7–9] Braverman and Kappeler introduce what they call refined analytic torsion. The η-invariant associated with flat vector bundles plays a role in their definition. In this section, we first examine the imaginary part of the η-invariant appearing in,[7–9] from the point of view of the previous sections and propose an alternate definition of the refined analytic torsion. We then combine this refined analytic torsion with the η-invariant to construct analytically a univalent holomorphic function on the space of representations of $\pi_1(M)$ having the absolute value equals to the Ray-Singer torsion, thus resolving a problem posed by Burghelea.[11]

3.1. *η-invariant and the refined analytic torsion of Braverman-Kappeler*

Since there needs no spin condition in,[7–9] here we start with a closed oriented smooth odd dimensional manifold M with $\dim M = 2n+1$. Let g^{TM} be a Riemannian metric on TM. For any $X \in TM$, let $X^* \in T^*M$ denote its metric dual and $c(X) = X^* \wedge -i_X$ denote the associated Clifford action acting on $\Lambda^*(T^*M)$, where $X^*\wedge$ and i_X are the notation for the exterior and interior multiplications of X respectively.

Let $e_1, \ldots, e_{2n+1}$ be an oriented orthonormal basis of TM. Set

$$\Gamma = \left(\sqrt{-1}\right)^{n+1} c(e_1)\cdots c(e_{2n+1}). \qquad (3.1)$$

Then $\Gamma^2 = \operatorname{Id}$ on $\Lambda^*(T^*M)$.

Let (F, g^F) be a Hermitian vector bundle over M equipped with a flat connection ∇^F which need not preserve the Hermitian metric g^F on F. Then the exterior differential d on $\Omega^*(M) = \Gamma(\Lambda^*(T^*M))$ extends naturally to the twisted exterior differential d^F acting on $\Omega^*(M,F) = \Gamma(\Lambda^*(T^*M) \otimes F)$.

We define the twisted signature operator D^F_{Sig} to be

$$D^F_{\mathrm{Sig}} = \frac{1}{2}\left(\Gamma d^F + d^F \Gamma\right) : \Omega^{\mathrm{even}}(M,F) \to \Omega^{\mathrm{even}}(M,F)\,. \tag{3.2}$$

It coincides with the odd signature operator $\frac{1}{2}\mathcal{B}_{\mathrm{even}}$ in.[7–9]

Let $\nabla^{\Lambda^{\mathrm{even}}(T^*M)\otimes F}$ (resp. $\nabla^{\Lambda^{\mathrm{even}}(T^*M)\otimes F,e}$) be the tensor product connections on $\Lambda^{\mathrm{even}}(T^*M)\otimes F$ obtained from ∇^F (resp. $\nabla^{F,e}$) and the canonical connection on $\Lambda^{\mathrm{even}}(T^*M)$ induced by the Levi-Civita connection ∇^{TM} of g^{TM}.

From (3.2), it is easy to verify that

$$D^F_{\mathrm{Sig}} = \Gamma\left(\sum_{i=1}^{2n+1} c(e_i)\nabla_{e_i}^{\Lambda^{\mathrm{even}}(T^*M)\otimes F}\right). \tag{3.3}$$

Set

$$D^{F,e}_{\mathrm{Sig}} = \Gamma\left(\sum_{i=1}^{2n+1} c(e_i)\nabla_{e_i}^{\Lambda^{\mathrm{even}}(T^*M)\otimes F,e}\right). \tag{3.4}$$

Then $D^{F,e}_{\mathrm{Sig}}$ is formally self-adjoint.

Since locally one has identification $S(TM)\otimes S(TM) = \Lambda^{\mathrm{even}}(T^*M)$, one sees that one can apply the results in the previous section to the case $E = S(TM)$ to the current situation.

In particular, we get

$$\begin{aligned}
\mathrm{Re}\left(\overline{\eta}\left(D^F_{\mathrm{Sig}}\right)\right) &\equiv \overline{\eta}\left(D^{F,e}_{\mathrm{Sig}}\right) \quad \mod \mathbf{Z}\,,\\
\mathrm{Im}\left(\overline{\eta}\left(D^F_{\mathrm{Sig}}\right)\right) &= \frac{1}{\sqrt{-1}}\int_M \mathrm{L}\left(TM,\nabla^{TM}\right) CS\left(\nabla^{F,e},\nabla^F\right)\\
&= -\frac{1}{2\pi}\int_M \mathrm{L}(TM)\sum_{j=0}^{+\infty}\frac{2^{2j}j!}{(2j+1)!}c_{2j+1}(F)\,,
\end{aligned} \tag{3.5}$$

where $\mathrm{L}(TM,\nabla^TM)$ is the Hirzebruch L-form defined by

$$\mathrm{L}\left(TM,\nabla^TM\right) = \varphi\det{}^{1/2}\left(\frac{R^{TM}}{\tanh\left(R^{TM}/2\right)}\right), \tag{3.6}$$

with $R^{TM} = (\nabla^{TM})^2$ the curvature of ∇^{TM}, and $\mathrm{L}(TM)$ is the associated class.

Remark 3: By proceeding as in Section 2, we can get Theorem 3.7[13] easily by using the results in Remark 2.

Proposition 4: *The function*

$$\Psi\left(F, \nabla^F\right) = \operatorname{Im}\left(\overline{\eta}\left(D^F_{\mathrm{Sig}}\right)\right) + \frac{1}{2\pi}\int_M \mathrm{L}(TM)c_1(F) \tag{3.7}$$

is a locally constant function on the set of flat connections on F. In particular, $\Psi(F, \nabla^F) = 0$ if ∇^F can be connected to a unitary flat connection through a path of flat connections.

Proof. Let ∇^F_t, $0 \leq t \leq 1$, be a smooth pass of flat connections on F.

From (3.5), we get

$$\begin{aligned}
&\sqrt{-1}\operatorname{Im}\left(\overline{\eta}\left(D^F_{\mathrm{Sig},1}\right)\right) - \sqrt{-1}\operatorname{Im}\left(\overline{\eta}\left(D^F_{\mathrm{Sig},0}\right)\right) \\
&= \int_M \mathrm{L}\left(TM, \nabla^{TM}\right) CS\left(\nabla^{F,e}_1, \nabla^F_1\right) - \int_M \mathrm{L}\left(TM, \nabla^{TM}\right) CS\left(\nabla^{F,e}_0, \nabla^F_0\right) \\
&= \sqrt{-1}\int_M \mathrm{L}\left(TM, \nabla^{TM}\right) \operatorname{Im}\left(CS\left(\nabla^{F,e}_1, \nabla^{F,e}_0\right) - CS\left(\nabla^F_1, \nabla^F_0\right)\right) \\
&= \sqrt{-1}\int_M \mathrm{L}\left(TM, \nabla^{TM}\right) \operatorname{Im}\left(CS\left(\nabla^F_0, \nabla^F_1\right)\right).
\end{aligned} \tag{3.8}$$

Now consider the path of flat connections ∇^F_t, $0 \leq t \leq 1$. Since for any $t \in [0,1]$, $(\nabla^F_t)^2 = 0$, from (2.1), (2.5), one gets

$$\begin{aligned}
CS\left(\nabla^F_0, \nabla^F_1\right) &= \left(\frac{1}{2\pi\sqrt{-1}}\right)\operatorname{Tr}\left(\nabla^F_0 - \nabla^F_1\right) = \left(\frac{1}{2\pi\sqrt{-1}}\right)\operatorname{Tr}\left(\nabla^{F,e}_0 - \nabla^{F,e}_1\right) \\
&\quad - \left(\frac{1}{2\pi\sqrt{-1}}\right)\operatorname{Tr}\left(\frac{1}{2}\omega_0(F, g^F) - \frac{1}{2}\omega_1(F, g^F)\right).
\end{aligned} \tag{3.9}$$

Thus, one has

$$\begin{aligned}
\sqrt{-1}\operatorname{Im}\left(CS\left(\nabla^F_0, \nabla^F_1\right)\right) &= -\frac{1}{2\pi\sqrt{-1}}\operatorname{Tr}\left(\frac{1}{2}\omega_0(F, g^F) - \frac{1}{2}\omega_1(F, g^F)\right) \\
&= -\frac{1}{2\pi\sqrt{-1}}\left(c_1\left(F, \nabla^F_0\right) - c_1\left(F, \nabla^F_1\right)\right).
\end{aligned} \tag{3.10}$$

From (3.8) and (3.10), we get

$$\begin{aligned}
&\operatorname{Im}\left(\overline{\eta}\left(D^F_{\mathrm{Sig},1}\right)\right) + \frac{1}{2\pi}\int_M \mathrm{L}\left(TM, \nabla^{TM}\right) c_1\left(F, \nabla^F_1\right) \\
&\quad = \operatorname{Im}\left(\overline{\eta}\left(D^F_{\mathrm{Sig},0}\right)\right) + \frac{1}{2\pi}\int_M \mathrm{L}\left(TM, \nabla^{TM}\right) c_1\left(F, \nabla^F_0\right),
\end{aligned} \tag{3.11}$$

from which Proposition 4 follows. Q.E.D.

Remark 5: Formula (3.11) is closely related to Theorem 12.3.[7] Moreover, for any representation α of the fundamental group $\pi_1(M)$, let $(F_\alpha, \nabla^{F_\alpha})$ be the associated flat vector bundle. One has

$$\exp\left(\pi\Psi\left(F_\alpha, \nabla^{F_\alpha}\right)\right) = r(\alpha)\,, \tag{3.12}$$

where $r(\alpha)$ is the function appearing in Lemma 5.5.[9] While from (3.5) and (3.7), one has

$$\Psi\left(F, \nabla^F\right) = -\frac{1}{2\pi}\int_M \mathrm{L}(TM)\sum_{j=1}^{+\infty}\frac{2^{2j}j!}{(2j+1)!}c_{2j+1}(F)\,. \tag{3.13}$$

Combining with (3.12), this gives an explicit local expression of $r(\alpha)$ as well as the locally constant function $r_{\mathcal{C}}$ defined in Definition 5.6.[9]

Remark 6: To conclude this subsection, we recall the recent modification due to Braverman-Kappeler (Braverman mentioned this in a recent Oberwolfach conference) themselves of the original definition of the refined analytic torsion in[7–9] as follows: for any Hermitian vector bundle equipped with a flat connection ∇^F over an oriented closed smooth odd dimensional manifold M equipped with a Riemannian metric g^{TM}, let $\rho(\nabla^F, g^{TM})$ be the element defined in (2.13).[9] Then the modified definition of the refined analytic torsion is given by

$$\rho'_{\rm an}\left(\nabla^F, g^{TM}\right) = \rho\left(\nabla^F, g^{TM}\right)e^{\pi\sqrt{-1}\mathrm{rk}(F)\overline{\eta}(D_{\rm sig})}\,, \tag{3.14}$$

where $\overline{\eta}(D_{\rm sig})$ is the reduced η invariant in the sense of Atiyah-Patodi-Singer[1] of the signature operator coupled with the trivial complex line bundle over M (i.e. $D_{\rm sig} := D^{\mathbf{C}}_{\rm sig}$). There are two advantages of this reformulation. First, by multiplying the local factor $e^{-\pi\Psi(F,\nabla^F)}$ makes the comparison formula [9, (5.8)] of the refined analytic torsion has closer resemblance in comparing with the formulas of Cheeger-Müller and Bismut-Zhang (cf.[6]). The advantage of this reformulation is that since $\overline{\eta}(D_{\rm sig})$ various smoothly with respect to the metric g^{TM} (as the dimension of $\ker(D_{\rm sig})$ does not depend on the metric g^{TM}), the ambiguity of the power of $\sqrt{-1}$ disappears if one uses $e^{\pi\sqrt{-1}\mathrm{rk}(F)\overline{\eta}(D_{\rm sig})}$ to replace the factor $e^{\frac{\pi\sqrt{-1}\mathrm{rk}(F)}{2}\int_N L(p,g^M)}$ in (2.14).[9]

3.2. *Ray-Singer analytic torsion and univalent holomorphic functions on the representation space*

Let (F, ∇^F) be a complex flat vector bundle. Let g^F be an Hermitian metric on F. We fix a flat connection $\widetilde{\nabla}^F$ on F (note here that we do not assume that ∇^F and $\widetilde{\nabla}^F$ can be connected by a smooth path of flat connections).

Let g^{TM} be a Riemannian metric on TM and ∇^{TM} be the associated Levi-Civita connection.

Let $\widetilde{\eta}(\nabla^F, \widetilde{\nabla}^F) \in \mathbf{C}$ be defined by

$$\widetilde{\eta}\left(\nabla^F, \widetilde{\nabla}^F\right) = \int_M \mathrm{L}\left(TM, \nabla^{TM}\right) CS\left(\widetilde{\nabla}^{F,e}, \nabla^F\right). \tag{3.15}$$

One verifies easily that $\widetilde{\eta}(\nabla^F, \widetilde{\nabla}^F) \in \mathbf{C}$ does not depend on g^{TM}, and is a holomorphic function of ∇^F. Moreover, by (3.5) one has

$$\operatorname{Im}\left(\widetilde{\eta}\left(\nabla^F, \widetilde{\nabla}^F\right)\right) = \operatorname{Im}\left(\overline{\eta}\left(D^F_{\mathrm{Sig}}\right)\right). \tag{3.16}$$

Recall that the refined analytic torsion of[7–9] has been modified in (3.14).

Set

$$\mathcal{T}_{\mathrm{an}}\left(\nabla^F, g^{TM}\right) = \rho'_{\mathrm{an}}\left(\nabla^F, g^{TM}\right) \exp\left(\sqrt{-1}\pi\widetilde{\eta}\left(\nabla^F, \widetilde{\nabla}^F\right)\right). \tag{3.17}$$

Then $\mathcal{T}_{\mathrm{an}}(\nabla^F, g^{TM})$ is a holomorphic section in the sense of Definition 3.4.[9]

By Theorem 11.3[8] (cf. (5.13)[9]), (3.14), (3.16) and (3.17), one gets the following formula for the Ray-Singer norm of $\mathcal{T}_{\mathrm{an}}(\nabla^F, g^{TM})$,

$$\left\|\mathcal{T}_{\mathrm{an}}\left(\nabla^F, g^{TM}\right)\right\|^{\mathrm{RS}} = 1. \tag{3.18}$$

In particular, when restricted to the space of acyclic representations, $\mathcal{T}_{\mathrm{an}}(\nabla^F, g^{TM})$ becomes a (univalent) holomorphic function such that

$$\left|\mathcal{T}_{\mathrm{an}}\left(\nabla^F, g^{TM}\right)\right| = T^{RS}(\nabla^F), \tag{3.19}$$

the usual Ray-Singer analytic torsion. This provides an analytic resolution of a question of Burghelea.[11]

Remark 7: If one considers $\mathcal{T}^2_{\mathrm{an}}$, then one can further modify it to

$$\begin{aligned}\mathcal{T}^2_{\mathrm{an}}\left(\nabla^F, g^{TM}\right)' &= \mathcal{T}^2_{\mathrm{an}}\left(\nabla^F, g^{TM}\right)\\ &\quad\times \exp\left(2\pi\sqrt{-1}\left(\overline{\eta}\left(\widetilde{D}^{F,e}_{\mathrm{Sig}}\right) - \mathrm{rk}(F)\overline{\eta}\left(D_{\mathrm{Sig}}\right)\right)\right),\end{aligned} \tag{3.20}$$

which does not depend on the choice of $\widetilde{\nabla}^F$, and thus gives an intrinsic definition of a holomorphic section of the square of the determinant line bundle, having the same norm as that of $\mathcal{T}^2_{\mathrm{an}}(\nabla^F, g^{TM})$. The dependence of $\mathcal{T}_{\mathrm{an}}$ on α indicates in part the subtleness of the analytic meaning of the phase of the Turaev torsion (cf.[12,16]).

Next, we show how to modify the Turaev torsion[12,16] to get a holomorphic section with Ray-Singer norm equal to one.

Let ε be an Euler structure on M and $\mathbf{o}$ a cohomological orientation. We use the notation as in[9] to denote the associated Turaev torsion by $\rho_{\varepsilon,\mathbf{o}}$.

Let $c(\varepsilon) \in H_1(M, \mathbf{Z})$ be the canonical class associated to the Euler structure ε (cf.[16] or Section 5.2[12]). Then for any representation α_F corresponding to a flat vector bundle (F, ∇^F), by Theorem 10.2[12] one has

$$\|\rho_{\varepsilon,\mathbf{o}}(\alpha_F)\|^{\mathrm{RS}} = |\det \alpha_F(c(\varepsilon))|^{1/2}. \tag{3.21}$$

Let $\mathrm{L}_{\dim M-1}(TM) \in H^{\dim M-1}(M, \mathbf{Z})$ be the degree $\dim M - 1$ component of the characteristic class $\mathrm{L}(TM)$. Let $\widehat{\mathrm{L}}_1(TM) \in H_1(M, \mathbf{Z})$ denote its Poincaré dual. Then one verifies easily that

$$\left|\det \alpha_F\left(\widehat{\mathrm{L}}_1(TM)\right)\right| = \exp\left(\int_M \mathrm{L}\left(TM, \nabla^{TM}\right) c_1\left(F, \nabla^F\right)\right). \tag{3.22}$$

On the other hand, by Corollary 5.9,[9] $\widehat{\mathrm{L}}_1(TM) + c(\varepsilon) \in H_1(M, \mathbf{Z})$ is divisible by two, and one can define a class $\beta_\varepsilon \in H_1(M, \mathbf{Z})$ such that

$$-2\beta_\varepsilon = \widehat{\mathrm{L}}_1(TM) + c(\varepsilon). \tag{3.23}$$

From Proposition 4, (3.22) and (3.23), one finds

$$|\det \alpha_F(c(\varepsilon))|^{1/2} = |\det \alpha_F(\beta_\varepsilon)|^{-1} \exp\left(-\pi\Phi\left(F, \nabla^F\right) + \pi \mathrm{Im}\left(\overline{\eta}\left(D^F_{\mathrm{Sig}}\right)\right)\right), \tag{3.24}$$

where $\Phi(F, \nabla^F)$ is the locally constant function given by (3.13).

We now define a modified Turaev torsion as follows:

$$\mathcal{T}_{\varepsilon,\mathbf{o}}\left(F, \nabla^F\right) = \rho_{\varepsilon,\mathbf{o}}(\alpha_F)\, e^{\pi\Phi(F,\nabla^F) + \sqrt{-1}\pi\tilde{\eta}(\nabla^F, \tilde{\nabla}^F)} \left(\det \alpha_F(\beta_\varepsilon)\right). \tag{3.25}$$

Clearly, $\mathcal{T}_{\varepsilon,\mathbf{o}}(F, \nabla^F)$ is a holomorphic section in the sense of Definition 3.4.[9] Moreover, by (3.21), (3.24) and (3.25), its Ray-Singer norm equals to one. Thus it provides another resolution of Burghelea's problem mentioned above which should be closely related to what in.[10]

Combining with (3.18) we get

$$\left|\frac{\mathcal{T}_{\mathrm{an}}\left(\nabla^F, g^{TM}\right)}{\mathcal{T}_{\varepsilon,\mathbf{o}}\left(F, \nabla^F\right)}\right| = 1, \tag{3.26}$$

which, in view of (3.12), is equivalent to (5.10).[9]

On the other hand, since now $\mathcal{T}_{\mathrm{an}}(\nabla^F, g^{TM})/\mathcal{T}_{\varepsilon,\mathbf{o}}(F, \nabla^F)$ is a holomorphic function with absolute value identically equals to one, one sees that there is a real locally constant function $\theta_{\varepsilon,\mathbf{o}}(F, \nabla^F)$ such that

$$\frac{\mathcal{T}_{\mathrm{an}}\left(\nabla^F, g^{TM}\right)}{\mathcal{T}_{\varepsilon,\mathbf{o}}\left(F, \nabla^F\right)} = e^{\sqrt{-1}\theta_{\varepsilon,\mathbf{o}}(F,\nabla^F)}, \tag{3.27}$$

which is equivalent to (5.8).[9]

Remark 8: While the univalent holomorphic sections $\mathcal{T}_{\mathrm{an}}$ and $\mathcal{T}_{\varepsilon,\mathbf{o}}$ depend on the choice of an "initial" flat connection $\widetilde{\nabla}^F$, the quotients in the left hand sides of (3.26) and (3.27) do not involve it.

Remark 9: One of the advantages of (3.26) and (3.27) is that they look in closer resemblance to the theorems of Cheeger, Müller and Bismut-Zhang[6] concerning the Ray-Singer and Reidemeister torsions.

Now let ∇_1^F and ∇_2^F be two acyclic unitary flat connections on F. We do not assume that they can be connected by a smooth path of flat connections.

By (14.11)[7] (cf. (6.2)[9]), (3.15), (3.17) and the variation formula for η-invariants,[1,3,4] one finds

$$\begin{aligned}
\frac{\mathcal{T}_{\mathrm{an}}(\nabla_1^F, g^{TM})}{\mathcal{T}_{\mathrm{an}}(\nabla_2^F, g^{TM})} &= \frac{T^{\mathrm{RS}}(\nabla_1^F)}{T^{\mathrm{RS}}(\nabla_2^F)} \cdot \frac{\exp(-\sqrt{-1}\pi\overline{\eta}(D^F_{\mathrm{Sig},1}) + \sqrt{-1}\pi\widetilde{\eta}(\nabla_1^F, \widetilde{\nabla}^F))}{\exp(-\sqrt{-1}\pi\overline{\eta}(D^F_{\mathrm{Sig},2}) + \sqrt{-1}\pi\widetilde{\eta}(\nabla_2^F, \widetilde{\nabla}^F))} \\
&= \frac{T^{\mathrm{RS}}(\nabla_1^F)}{T^{\mathrm{RS}}(\nabla_2^F)} \cdot \frac{\exp(-\sqrt{-1}\pi\overline{\eta}(D^F_{\mathrm{Sig},1}) + \sqrt{-1}\pi\overline{\eta}(D^F_{\mathrm{Sig},2}))}{\exp(-\sqrt{-1}\pi\int_M \mathrm{L}(TM, \nabla^{TM})CS(\nabla_2^F, \nabla_1^F))} \\
&= \frac{T^{\mathrm{RS}}(\nabla_1^F)}{T^{\mathrm{RS}}(\nabla_2^F)} \cdot \exp(\sqrt{-1}\pi \cdot \mathrm{sf}(D^F_{\mathrm{Sig},1}, D^F_{\mathrm{Sig},2})),
\end{aligned} \tag{3.28}$$

where $D^F_{\mathrm{Sig},1}$ and $D^F_{\mathrm{Sig},2}$ are the signature operators associated to ∇_1^F and ∇_2^F respectively, while $\mathrm{sf}(D^F_{\mathrm{Sig},1}, D^F_{\mathrm{Sig},2})$ is the spectral flow of the linear path connecting $D^F_{\mathrm{Sig},1}$ and $D^F_{\mathrm{Sig},2}$, in the sense of Atiyah-Patodi-Singer.[3]

Remark 10: Since we do not assume that ∇_1^F and ∇_2^F can be connected by a path of flat connections, our formula extends the corresponding formula in Proposition 6.2.[9]

Corollary 11: *The ratio $\mathcal{T}_{\mathrm{an}}(\nabla^F, g^{TM})/T^{\mathrm{RS}}(\nabla^F)$ is a locally constant function on the set of acyclic unitary flat connections on F.*

Example 12: Let ∇^F be an acyclic unitary flat connection on F. Let $g \in \Gamma(U(F))$ be a smooth section of unitary automorphisms of F. Then

$g^{-1}\nabla^F g$ is another acyclic unitary flat connection on F. A standard calculation shows that

$$\mathrm{sf}\left(D_{\mathrm{Sig}}^{F,\nabla^F}, D_{\mathrm{Sig}}^{F,g^{-1}\nabla^F g}\right) = \int_M \mathrm{L}(TM)\mathrm{ch}(g)\,, \tag{3.29}$$

where $\mathrm{ch}(g) \in H^{\mathrm{odd}}(M,\mathbf{R})$ is the odd Chern character associated to g (cf.[17]). From (3.29), one sees that if $\int_M \mathrm{L}(TM)\mathrm{ch}(g)$ is nonzero, then ∇^F and $g^{-1}\nabla^F g$ do not lie in the same connected component in the set of acyclic unitary flat connections on F.

3.3. *More on η-invariants, spectral flow and the phase of the refined analytic torsion*

We would like to point out that the (reduced) η-invariant for non-self-adjoint operators we used above, when considered as a $\mathbf{C}$-valued function, is the original η invariant appeared in[3] (see also[13]). In this section, we show that the $\mathbf{R}$-valued variation formula for η-invariants (which has been used in (3.28)) admits an extension to a $\mathbf{C}$-valued variation formula valid also for the non-self-adjoint operators discussed in the present paper.

First, the concept of spectral flow can be extended to non-self-adjoint operators, and this has been done in[18] in a general context.

For our specific situation, if $D_{\mathrm{Sig},t}^F$, $0 \le t \le 1$, is a smooth path of (possibly) non-self-adjoint signature operators, following,[18] we define the spectral flow of this path to be, tautologically,

$$\begin{aligned}\mathrm{sf}\left(D_{\mathrm{Sig},0}^F, D_{\mathrm{Sig},1}^F\right) = \\ &\#\left\{\mathrm{spec}\left(D_{\mathrm{Sig},0}^F\right) \cap \{\mathrm{Re}(\lambda) \ge 0\} \to \mathrm{spec}\left(D_{\mathrm{Sig},1}^F\right) \cap \{\mathrm{Re}(\mu) < 0\}\right\} \\ &- \#\left\{\mathrm{spec}\left(D_{\mathrm{Sig},0}^F\right) \cap \{\mathrm{Re}(\lambda) < 0\} \to \mathrm{spec}\left(D_{\mathrm{Sig},1}^F\right) \cap \{\mathrm{Re}(\mu) \ge 0\}\right\},\end{aligned} \tag{3.30}$$

which simply replaces the number zero in the original definition for self-adjoint operators[3] by the axis of purely imaginary numbers.

Now let ∇_t^F, $0 \le t \le 1$, be a smooth path of (not necessary unitary and/or flat) connections on F. Let $D_{\mathrm{Sig},t}^F$, $0 \le t \le 1$, be the corresponding path of signature operators. With the definition of spectral flow, one then sees easily that the following variation formula holds in $\mathbf{C}$,

$$\overline{\eta}(D_{\mathrm{Sig},1}^F) - \overline{\eta}(D_{\mathrm{Sig},0}^F) = \mathrm{sf}(D_{\mathrm{Sig},0}^F, D_{\mathrm{Sig},1}^F) + \int_M \mathrm{L}(TM,\nabla^{TM})CS(\nabla_0^F,\nabla_1^F). \tag{3.31}$$

Now we observe that in,[7–9] Braverman and Kappeler propose an alternate definition of (reduced) η invariant, which if we denote by η_{BK}, then (cf. Definition 4.3[7] and Definition 5.2[9])

$$\eta_{BK}\left(D^F_{\mathrm{Sig}}\right) = \overline{\eta}\left(D^F_{\mathrm{Sig}}\right) - m_-\left(D^F_{\mathrm{Sig}}\right), \tag{3.32}$$

where $m_-(D^F_{\mathrm{Sig}})$ is the number of purely imaginary eigenvalues of D^F_{Sig} of form $\lambda\sqrt{-1}$ with $\lambda < 0$.

Formulas (3.31) and (3.32) together give a variation formula for η_{BK}, which can be used to extend (3.28) to non-unitary acyclic representations.

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