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A GENERALIZATION OF THE ATIYAH–DUPONT VECTOR FIELDS THEORY

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Dedicated to Professor Shiing-shen Chern on his 91st birthday

To generalize the Hopf index theorem and the Atiyah–Dupont vector fields theory, one is interested in the following problem: for a real vector bundle E over a closed manifold M with $\operatorname{rank} E = \dim M$, whether there exist two linearly independent cross sections of E? We provide, among others, a complete answer to this problem when both E and M are orientable. It extends the corresponding results for $E = TM$ of Thomas, Atiyah, and Atiyah–Dupont. Moreover we prove a vanishing result of a certain mod 2 index when the bundle E admits a complex structure. This vanishing result implies many known famous results as consequences. Ideas and methods from obstruction theory, K-theory and index theory are used in getting our results.

Keywords: Vector bundles; cross sections; obstruction theory; index theory.

1. Introduction

On an n-dimensional closed differentiable manifold M, there is a well known theorem of H. Hopf which asserts that the number of zeros of a smooth tangent vector field v depends only on the manifold M and is equal to the Euler characteristic $\chi(M)$. Moreover, the vanishing of the Euler characteristic $\chi(M)$ is the necessary and sufficient condition for the existence of a smooth tangent vector field without zeros. Here one assumes of course that the number of zeros of v is finite and that each is counted with an appropriate multiplicity which is called the index of v at that point. Instead of a single vector field one considers r vector fields $V = \{v_1, \ldots, v_r\}$ and is interested in their singular set A, that is, the set of points on the manifold where they become linearly dependent. In general this singular set A has dimension $r-1$. The standard primary obstruction theory provides one way of generalizing the classical Hopf Theorem. However the general theory of characteristic classes does not give complete topological information about the singular set. For instance, this theory tells nothing about the singularities if the dimension of A is less than $r-1$.

Atiyah and Dupont [1, 3] generalized the Hopf theorem by considering the opposite extreme case in which the singular set A is assumed to be finite. At each singular point p, one has a local obstruction

$$\mathrm{Ind}_V(p) \in \pi_{n-1}V_{n,r}\,,$$

where $V_{n,r}$ is the Stiefel manifold of orthonormal r-frames in $\mathbf{R}^n$. For $r=1$, one has $V_{n,r}=S^{n-1}$, $\pi_{n-1}V_{n,1}\cong\mathbf{Z}$ which give the multiplicity used in the Hopf theory. In the general case $r>1$, one can form the global obstruction $\mathrm{Ind}(V)$ and ask how far it is independent of M. Clearly the vanishing of $\mathrm{Ind}(V)$ is the necessary and sufficient condition that we can deform $v_1,\ldots,v_r$ near A so that these singularities disappear. In general, $\mathrm{Ind}(V)$ depends not only on the special choice of V, but also on the orientation of the manifold. One fundamental task is to identify this global obstruction (when it is independent of the choice of V) with certain global invariants of the manifold.

There are many results in this direction and the invariants of M which appear in these results are the Euler characteristic, the Signature and so on. For $r=2$, note that $\pi_{n-1}V_{n,2}\cong\mathbf{Z}_2$ for odd n and $\pi_{n-1}V_{n,2}\cong\mathbf{Z}\oplus\mathbf{Z}_2$ for even n. Briefly speaking, Atiyah [1] used the index of an elliptic operator and also used the mod 2 index of a real skew-adjoint elliptic operator to represent the obstruction defined above by index theory. For instance, he gave an interesting result that if M is oriented and of dimension $4q+1$, then this $\mathrm{Ind}(V)$ coincides with the Kervaire semi-characteristic $k(M)$ which is defined to be

$$k(M)=\sum_j \dim_{\mathbf{R}} H^{2j}(M;\mathbf{R}) \mod 2\,.$$

For other dimensions, see [1, Theorem 5.1] and [3, Theorem 1.1 or Theorem 2.20] for more details.

Throughout this paper, for a real vector bundle E, we denote by $W_i(E)$ and $w_i(E)$ the ith unreduced and the ith reduced Stiefel–Whitney classes, respectively. For the sake of simplicity, let $W_i(M)=W_i(TM)$, $w_i(M)=w_i(TM)$.

In this paper we present some generalizations of the Atiyah–Dupont vector fields theory. Let E be a real vector bundle with $\mathrm{rank}\,E=\dim M$. We shall be interested in the existence of $r=2$ linearly independent cross sections of E over M. What Atiyah and Dupont considered is the special case where E is the tangent bundle of M. Similar to the situation of tangent bundle, for the vector bundle E, if the Stiefel–Whitney class $W_{n-1}(E)$ vanishes then we can get a continuous field of two-frames of E with finite singularities which will be denoted by V. Again similar to the situation of tangent bundle, we can define the local obstruction around each singular point and then $\mathrm{Ind}(V)$, the global obstruction. Clearly the vanishing of $\mathrm{Ind}(V)$ is the necessary and sufficient condition that we can remove the singularities of V.

In Sec. 2, we investigate the condition under which $\mathrm{Ind}(V)$ is independent of the choice of V. We treat it separately for n is odd and n is even. In both cases, we employ the formula of Boltyanski–Liao on second obstructions for cross sections [6, 16]. The main results can be stated as follows.

Corollary 2.2. *Suppose that $n > 4$ is an odd number and E is a vector bundle of rank n over an n-dimensional, closed, connected manifold M with $w_{n-1}(E) = 0$. Then*

(A) *If $w_2(E)+w_1^2(M)+w_2(M) = 0$, then $\mathrm{Ind}(V) \in \mathbf{Z}_2$ is independent of the choice of V. In this case, this mod 2 integer will be called the generalized Kervaire semi-characteristic of the bundle E, denoted by $k(E)$; Hence E admits two linearly independent cross sections if and only if $k(E)$ vanishes.*
(B) *If $w_2(E) + w_1^2(M) + w_2(M) \neq 0$, then both 0 and 1 can occur as $\mathrm{Ind}(V)$. In particular, E admits always two linearly independent cross sections.*

Corollary 2.11. *Suppose that $n > 4$ is an even number and E is an oriented vector bundle of rank n over an n-dimensional, closed, connected, oriented manifold M with $W_{n-1}(E) = 0$. Then*

(A) *If $w_2(E) = w_2(M)$, then $\mathrm{Ind}(V)$ is independent of the choice of V. In this case, E admits two linearly independent cross sections if and only if both this invariant and $\chi(E)$ vanish.*
(B) *If $w_2(E) \neq w_2(M)$, then both $(\chi(E), 0)$ and $(\chi(E), 1)$ can occur as $\mathrm{Ind}(V)$, the index of a two-frames V of E with finite singularities. In particular, E admits always two linearly independent cross sections if and only if $\chi(E) = 0$.*

For an embedding of M into $\mathbf{R}^{2n}$ with normal bundle N_f, we try to compute $k(N_f)$. If n is odd and M is orientable, we show in Proposition 2.5 that this invariant can be expressed by a dual Stiefel–Whitney characteristic number of the manifold. We also indicate, by a concrete example, that the relevant statement in [25, p. 679] is incorrect.

In Sec. 3, we try to give $\mathrm{Ind}(V)$ a K-theoretic and an analytic interpretation. We generalize the result of Atiyah–Dupont [3, Theorem 2.20] as follows.

Theorem 3.1. *Let M be a closed oriented connected manifold of dimension $n = 4k - s$ with $0 \leq s \leq 3$, and let E be an oriented vector bundle over M of rank n, verifying the conditions in Corollaries 2.2(A) and 2.11(A) above. We fix a spin structure on $TM \oplus E$. Let $\{v_1, \ldots, v_r\}$ be a set of r-cross sections of E, linearly independent except at the finite set of points $\{A_1, \ldots, A_l\}$. Then in the group $KR^s(\mathbf{R}P^{r+s-1}, \mathbf{R}P^{s-1})$, we have the formula for the global invariant*

$$\mathrm{ind}\, \alpha^s_{E,r} = \sum_{i=1}^{l} \theta^s \mathcal{O}_{A_i}(v_1, \ldots, v_r)\,,$$

where $\mathcal{O}_A(v_1, \ldots, v_r) \in \pi_{n-1}(V_{n,r})$ *is the local obstruction to extending the cross sections at* A. *In particular,* $\operatorname{ind} \alpha^s_{E,r}$ *does not depend on the spin structure on* $TM \oplus E$.

To prove this theorem, one would have to excise small balls around the singularities and set up a suitable boundary value problem. The basic idea is to pass from elliptic operators to their symbols which lie in certain K-groups. The index theorem in its various forms implies that some analytical index (Euler characteristic number, generalized Kervaire semi-characteristic, etc.) can be computed purely in terms of K-theory from these symbols. We then want to relate these elements with $\operatorname{Ind}(V)$. For $r = 2$, we make the corresponding table:

$n > 4$	$\pi_{n-1}V_{n,2}$	$\operatorname{Ind}(V)$
$4q$	$\mathbf{Z} \oplus \mathbf{Z}_2$	$\left(\chi(E), \frac{1}{2}(\chi(E) - (-1)^q \langle \hat{A}(TM)\hat{S}(E), [M] \rangle)\right)$
$4q+1$	$\mathbf{Z}_2$	$k(E) = \operatorname{Ind}_2 P_E$
$4q+2$	$\mathbf{Z} \oplus \mathbf{Z}_2$	$(\chi(E), 0)$
$4q+3$	$\mathbf{Z}_2$	0

where $\hat{S}(E)$ is the multiplicative characteristic class of E associated to the function $2\cosh(x/2)$, and $\operatorname{Ind}_2 P_E$ is the mod 2 index in the sense of Atiyah–Singer [5] of some skew-adjoint Dirac type operator associated to the $8q+2$ dimensional spin vector bundle $TM \oplus E$.

With these computations, we can get the corresponding sufficient and necessary conditions for which E has two linearly independent cross sections. See Corollaries 3.2, 3.3, 3.5, and 3.7 for details.

The case $s = 0$ is particularly interesting. A byproduct is the vanishing of $\mathbf{Z}_2$-$\operatorname{Ind}(V)$ when E admits a complex structure, which implies many important conclusions such as the well known results of Hopf, Ehresmann, Borel–Serre, Kahn, Borel–Hirzebruch and Milnor on the non-existences of almost complex structures on even dimensional oriented manifolds. Moreover, we prove an unexpected result that every "negative" complex projective space of even complex dimension admits no almost complex structures.

2. Obstruction Theory Method

Let M be a closed, connected, n-dimensional, differentiable manifold, and let E be a real vector bundle of rank n over M. We shall be concerned with the problem of the existence of two linearly independent cross sections of E.

Let $V_{n,2}(E)$ denote the associated bundle of E with the fibre $V_{n,2}$, the Stiefel manifold of orthonormal two-frames in $\mathbf{R}^n$. As is well-known, $W_{n,2} = GL(n, \mathbf{R})/GL(n-2, \mathbf{R})$ has the same homotopy type with $V_{n,2}$ since every two-frame can be naturally orthogonalized. In such way, there is a correspondence between the cross sections of the fibre bundle $V_{n,2}(E)$ and the continuous fields of two-frames of E.

In the first part of this section we will concentrate on the case that $n > 4$ is an odd integer. Using the result that M is triangulable, the classical obstruction theory asserts that a continuous field of two-frames of E can be defined over the $(n-2)$-dimensional skeleton $M^{(n-2)}$ of M. Moreover, there exists such a field over $M^{(n-1)}$ if and only if the Stiefel–Whitney class (the first obstruction) $w_{n-1}(E) \in H^{n-1}(M; \pi_{n-2}V_{n,2})$ vanishes. Since $\pi_{n-2}V_{n,2} \cong \mathbf{Z}_2$ for odd n, $w_{n-1}(E)$ is the reduced Stiefel–Whitney class in $H^{n-1}(M; \mathbf{Z}_2)$.

Let $V = \{v_1, v_2\}$ be such a field over $M^{(n-2)}$, and assume that the first obstruction $w_{n-1}(E)$ (independent of the choice of V) vanishes so that V can be defined over $M^{(n-1)}$. Clearly the sufficient and necessary condition for that V can be extended to $M^{(n)} = M$ is the vanishing of the second and the final obstruction which we denote by $\mathrm{Ind}(V)$.

Suppose now we are given a field $V = \{v_1, v_2\}$ over $M^{(n-1)}$. The second obstruction $\mathrm{Ind}(V)$ belongs to $H^n(M; \pi_{n-1}V_{n,2})$. Since $\pi_{n-1}V_{n,2} \cong \mathbf{Z}_2$ for odd $n > 4$, the second obstruction is a mod 2 integer:

$$\mathrm{Ind}(V) \in \mathbf{Z}_2\,. \tag{2.0}$$

In fact $\mathrm{Ind}(V)$ has a geometric significance. Observe that M is compact and $\dim M = n$, the $(n-1)$-skeleton of M has the homotopy type of the space obtained by removing one point from the interior of each n-simplex of M. Thus a field $V = \{v_1, v_2\}$ over $M^{(n-1)}$ corresponds to a field $V = \{v_1, v_2\}$ with finite singularities on M. Let p be one of the singularities, say, in the interior of a simplex σ. The bundle E restricted to σ is isomorphic to the product bundle $\sigma \times \mathbf{R}^n$. For each point q in $\sigma - \{p\}$, we regard $(v_1(q), v_2(q))$ as an ordered set of two linearly independent vectors in $\mathbf{R}^n$, that is, a point in the Stiefel manifold $V_{n,2}$ ($W_{n,2} \simeq V_{n,2}$). Since the field V is defined on the boundary of σ, we obtain in this way a map $\mathrm{Bd}(\sigma) \to V_{n,2}$. But $\mathrm{Bd}(\sigma)$ is an $(n-1)$-sphere, and so the homotopy class of this map gives rise to an element of the homotopy group $\pi_{n-1}V_{n,2}$. This class is defined to be the index of V at the point p and is denoted by $\mathrm{Ind}_V(p)$. Finally, we define

$$\mathrm{Ind}(V) = \sum \mathrm{Ind}_V(p)\,, \tag{2.1}$$

where the sum is taken over all singularities of V. Thus

$$\mathrm{Ind}(V) \in \pi_{n-1}V_{n,2} \cong \mathbf{Z}_2\,,$$

as n is assumed to be odd and $n > 4$.

Thus the mod 2 integer $\mathrm{Ind}(V)$ measures whether or not one can alter a field so as to remove its singularities.

The following result shows that $\mathrm{Ind}(V)$, the (total) index of a field V, is not necessarily independent of the choice of V.

Theorem 2.1. *Let $n > 4$ be an odd number and $V = \{v_1, v_2\}$, $U = \{u_1, u_2\}$ be two cross sections of $V_{n,2}(E)$ over $M^{(n-1)}$. Then their second obstructions can be related by*

$$\mathrm{Ind}(V) - \mathrm{Ind}(U) = (w_2(E) + w_1^2(M) + w_2(M))\Omega(V, U)\,, \tag{2.2}$$

where $w_2(E) \in H^2(M;\mathbf{Z}_2)$ *and* $w_j(M) = w_j(TM) \in H^j(M;\mathbf{Z}_2)$, $j = 1$ *or* 2, *are the mod* 2 *Stiefel–Whitney classes of the vector bundle* E *and the tangent bundle* TM, *respectively, and* $\Omega(V,U)$ *is the first difference of* V *and* U,

$$\Omega(V,U) \in H^{n-2}(M;\pi_{n-2}V_{n,2}) \cong H^{n-2}(M;\mathbf{Z}_2)\,.$$

Proof. Observe first that

$$\begin{aligned}&w_2(E) \in H^2(M;\pi_1 V_{n,n-1})\,,\\ &\Omega(V,U) \in H^{n-2}(M;\pi_{n-2}V_{n,2})\,,\\ &\mathrm{Ind}(V), \mathrm{Ind}(U) \in H^n(M;\pi_{n-1}V_{n,2})\,.\end{aligned}$$

Since $n > 4$ is odd, the homomorphism $\pi_{n-1}S^{n-2} \to \pi_{n-1}V_{n,2}$ induced by the inclusion $S^{n-2} = V_{n-1,1} \to V_{n,2}$ is an isomorphism. The formula of Boltyanski–Liao ([6, p.66], [16]) then gives for $n > 4$ that

$$\mathrm{Ind}(V) = \mathrm{Ind}(U) + \mathrm{Sq}^2\Omega(V,U) + w_2(E)\cdot\Omega(V,U)\,, \tag{2.3}$$

in which the cup product is determined by the pairing of groups $G_1 \cdot G_2 \subset G_3$ with $\alpha_1\cdot\alpha_2 = \alpha_3$ where $G_1 = \pi_1 V_{n,n-1} \cong \mathbf{Z}_2$ ($n \geq 3$), $G_2 = \pi_{n-2}V_{n,2} \cong \mathbf{Z}_2$ ($n > 4$ is odd), $G_3 = \pi_{n-1}V_{n,2} \cong \pi_{n-1}S^{n-2} \cong \mathbf{Z}_2$($n > 4$ is odd) with generators α_1, α_2 and α_3, respectively.

Note that $\mathrm{Sq}^2\Omega(V,U) = u_2(M)\cdot\Omega(V,U)$, the cup product of $\Omega(V,U)$ with the second Wu class $u_2(M)$ [21]. By using the Wu formula [21] that

$$u_2(M) = w_1^2(M) + w_2(M)\,,$$

one completes the proof of the theorem. □

Corollary 2.2. *Suppose that* $n > 4$ *is an odd number and* $w_{n-1}(E) = 0$. *Then*

(A) *If* $w_2(E) + w_1^2(M) + w_2(M) = 0$, *then* $\mathrm{Ind}(V)$ *is independent of the choice of* V. *In this case, this mod* 2 *integer will be called the generalized Kervaire semi-characteristic of the bundle* E, *denoted by* $k(E)$; *Hence* E *admits two linearly independent cross sections if and only if* $k(E)$ *vanishes.*

(B) *If* $w_2(E) + w_1^2(M) + w_2(M) \neq 0$, *then both* 0 *and* 1 *can occur as* $\mathrm{Ind}(V)$. *In particular,* E *admits always two linearly independent cross sections.*

Proof. Claim (A) is clear. For claim (B), since $w_2(E) + w_1^2(M) + w_2(M) \neq 0$, Poincaré duality implies that

$$(w_2(E) + w_1^2(M) + w_2(M))\cdot y = 1 \in H^n(M;\mathbf{Z}_2) \cong \mathbf{Z}_2$$

for some cohomology class $y \in H^{n-2}(M;\mathbf{Z}_2)$. Furthermore, with a fixed field V, the class $y \in H^{n-2}(M;\mathbf{Z}_2)$ can be realized as the first difference $\Omega(V,U)$ for some field U (cf. [24, 36.7 and 37.5]). □

It is interesting to apply this corollary to several important cases. We first consider the case of $E = TM$, the tangent bundle of the closed connected

n-dimensional differentiable manifold M. Corollary 2.2 implies immediately the following result of Thomas [25].

Corollary 2.3. *Let M be a closed connected n-dimensional differentiable manifold with $w_{n-1}(M) = 0$. Assume that $n > 4$ is an odd number. Then*

(A) *If $w_1^2(M) = 0$, then* $\mathrm{Ind}(V)$ *is independent of the choice of V, and hence M admits two linearly independent vector fields if and only if $k(TM)$ vanishes*;

(B) *If $w_1^2(M) \neq 0$, then both* 0 *and* 1 *occur as* $\mathrm{Ind}(V)$. *In particular, M admits always two linearly independent vector fields.*

Remark 2.4. It should be remarked that, the preceding corollary was stated by E. Thomas in [25, Theorem 6] without proof. When $n \equiv 1 \mod 4$ and M is orientable, $k(TM)$ was shown to be the original Kervaire semi-characteristic defined by (cf. [1, Theorem 5.1])

$$k(TM) = \sum_j \dim_{\mathbf{R}} H^{2j}(M; \mathbf{R}) \mod 2\,. \tag{2.4}$$

If M is a nonorientable manifold of dimension $4q + 1$ with $w_1^2(M) = 0$, Atiyah and Dupont (cf. [3, Theorem 7.6]) established an analogue of (2.4) involving a semi-characteristic based on cohomology with coefficients in a local system.

We next turn to the normal bundle in the theory of embedding. As before, let M be a closed connected n-dimensional differentiable manifold ($n > 4$ is odd). The famous Whitney theorem asserts that M can be immersed (even embedded) into $\mathbf{R}^{2n}$, the $2n$-dimensional Euclidean space. Let $f : M \to \mathbf{R}^{2n}$ be an immersion and N_f be the normal bundle of rank n over M. Since $w_2(N_f) = \bar{w}_2(M) = w_1^2(M) + w_2(M)$, $w_{n-1}(N_f) = \bar{w}_{n-1}(M) = 0$ (cf. [19, Corollary 1]), one has

$$w_2(N_f) + w_1^2(M) + w_2(M) = 0\,.$$

Therefore in virtue of Corollary 2.2 (A) we have a well-defined generalized Kervaire semi-characteristic, $k(N_f) \in \mathbf{Z}_2$.

The computation of $k(N_f)$ seems to be very difficult. However we can do it if the immersion is assumed to be an embedding.

Proposition 2.5. *Let $n > 4$ be odd, and let $f : M \to \mathbf{R}^{2n}$ be an embedding of an orientable manifold M. Then*

$$k(N_f) = \bar{w}_2(M) \cdot \bar{w}_{n-2}(M)\,. \tag{2.5}$$

In particular $k(N_f)$ is independent of the choice of the embedding f.

Proof. The idea is to combine Theorem 3.5.1 with 4.1.1 of Mahowald and Peterson [18], as well as results in [19]. The details will be omitted. □

We conjecture that (2.5) still holds even when M is non-orientable.

Remark 2.6. It is worth emphasizing that in the preceding proposition the assumption that f is an embedding is necessary. The relevant statement in [25, p. 679] is incorrect. In other words, the assumption that f is an embedding cannot be weakened to that f is an immersion. We give an example to illustrate the difference. Following Ucci [26], let $S^m \subset \mathbf{R}^{m+1}$ be the usual m-sphere and $\mathbf{C}P^t$ the usual complex projective t-space. Let $P(m,t)$ be the Dold manifold of dimension $m+2t$ obtained from $S^m \times \mathbf{C}P^t$ by identifying (x,z) with $(-x,\bar{z})$ for $(x,z) \in S^m \times \mathbf{C}P^t$. Clearly $P(m,0)$ and $P(0,t)$ are $\mathbf{R}P^m$ and $\mathbf{C}P^t$, respectively. The ring structure of $H^*(P(m,t);\mathbf{Z}_2)$ is given by

$$H^*(P(m,t);\mathbf{Z}_2) = \mathbf{Z}_2[c]/c^{m+1} \otimes \mathbf{Z}_2[d]/d^{t+1}\,.$$

Moreover there is a bundle equivalence

$$TP(m,t) \oplus \xi \oplus 2 \cong (m+1)\xi \oplus (t+1)\eta$$

where ξ and η are vector bundles of rank one and rank two over $P(m,t)$ with the total Stiefel–Whitney classes $w(\xi) = 1+c$ and $w(\eta) = 1+c+d$, respectively.

We take $m=1,\ t=2$. Then $P(1,2)$ is an orientable manifold with the total dual Stiefel–Whitney class $\bar{w}(P(1,2)) = 1+c+d+cd$, thus $\bar{w}_2 \cdot \bar{w}_3 = cd^2 \neq 0$. This observation is due to [27].

Let $h : P(1,2) \to \mathbf{R}^{10}$ be an embedding with normal bundle N_h. It follows from Proposition 2.5 that $k(N_h)$, the generalized Kervaire semi-characteristic of N_h, is equal to $\bar{w}_2 \cdot \bar{w}_3 = 1$. On the other hand, $P(1,2)$ is readily seen to be immersed into $\mathbf{R}^8$. Composing with the obvious inclusion $\mathbf{R}^8 \to \mathbf{R}^{10}$, we get an immersion $g : P(1,2) \to \mathbf{R}^{10}$ which has two linearly independent normal vector fields, thus $k(N_g) = 0$.

The above arguments tell us that $k(N_f)$ for an immersion $f : M \to \mathbf{R}^{2n}$ depends not only on the manifold M, but also on the map f. Here we wish to pose a problem:

> *Compute the mod* 2 *integer which is the index for any normal two-frame on* M *with finite singularities for an immersion* $M^n \to \mathbf{R}^{2n}$ $(n > 4$ *is odd*$)$.

Remark 2.7. When $n \neq 1 \bmod 4$, M is orientable, Li [27] has obtained some results. In fact, to generalize the work of Mahowald [17], Li considered the problem when is a map $M^n \to N^{2n-2}$ between two manifolds homotopic to an immersion and established some interesting explicit results.

To understand well the generalized Kervaire semi-characteristic, we give now an interesting computation of $k(E)$ for vector bundles over spheres. Let $M = S^n$ be the n-sphere with $n = 4q+1,\ q \geq 2$, and let E be a vector bundle of rank n over S^n. In virtue of [21, (8-B)], $w(E) = w(TS^n) = 1$.

By Corollary 2.2 (A), we have the well-defined generalized Kervaire semi-characteristic $k(E) \in \mathbf{Z}_2$. Notice that since n is odd and E is orientable, E has always a nowhere zero cross section. Furthermore, E has two linearly independent cross sections if and only if $k(E) = 0$.

We shall use $\mathrm{Vect}_n(S^n)$ to denote the set of isomorphism classes of rank n vector bundles over S^n. According to [24, 18.5], there is a one-to-one correspondence between the elements of $\mathrm{Vect}_n(S^n)$ and that of $\pi_{n-1}SO(n)$ which is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ for $n = 4q + 1$ with even q, and isomorphic to $\mathbf{Z}_2$ for $n = 4q + 1$ with odd q (cf. [15]).

Example 2.8. Let $f_E : S^{n-1} \to SO(n)$ be the characteristic map of the bundle E. Consider the exact homotopy sequence

$$\pi_{n-1}SO(n-2) \xrightarrow{j_*} \pi_{n-1}SO(n) \xrightarrow{\pi_*} \pi_{n-1}V_{n,2} \to \pi_{n-2}SO(n-2)$$

induced by the fibration $SO(n-2) \xrightarrow{j} SO(n) \xrightarrow{\pi} V_{n,2}$. Clearly we see $\pi_*(f_E) = k(E)$. Since $\pi_{n-2}SO(n-2) \cong \mathbf{Z}$ for $n \equiv 1 \bmod 4$ [15] and $\pi_{n-1}V_{n,2}$ is finite, it follows that π_* is surjective. By the exactness, $k(E) = 0$ if and only if f_E belongs to the image of the homomorphism j_*. We consider separately the two cases. Case (1), q is odd. In this case, $\pi_{n-1}SO(n) \cong \mathbf{Z}_2$ is generated by the tangent bundle TS^n of S^n. As is well known, $k(TS^n) = 1$. Case (2), q is even. In this case, $\pi_{n-1}SO(n) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and TS^n is a generator whose Kervaire semi-characteristic is equal to 1. Another generator η can be chosen as the generator of $\widetilde{KO}(S^n) \cong \mathbf{Z}_2$ ($n \equiv 1 \bmod 8$). In other words, f_η is mapped to the generator of $\pi_{n-1}SO(N) \cong \mathbf{Z}_2$ for $N \geq n+1$, by the homomorphism $\pi_{n-1}SO(n) \to \pi_{n-1}SO(N)$. Next consider the exact homotopy sequence

$$\pi_n V_{N,N-(n-2)} \to \pi_{n-1}SO(n-2) \xrightarrow{i_*} \pi_{n-1}SO(N) \to \pi_{n-1}V_{N,N-(n-2)}$$

induced by the fibration $SO(n-2) \xrightarrow{i} SO(N) \to V_{N,N-(n-2)}$.

Fortunately we have $\pi_{n-1}V_{N,N-(n-2)} \cong 0$ [22], thus $i_* : \pi_{n-1}SO(n-2) \to \pi_{n-1}SO(N)$ is surjective. Therefore we deduce that $k(\eta) = \pi_*(f_\eta) = 0$.

The second part of this section will be devoted to the case of even dimensions. From now on we assume always that $n > 4$ is an even number. As mentioned before, the classical obstruction theory asserts that a continuous field of two-frames of E can be defined over the $(n-2)$-dimensional skeleton $M^{(n-2)}$ of M, moreover there exists such a field over $M^{(n-1)}$ if and only if the Stiefel–Whitney class (the first obstruction) $W_{n-1}(E)$ in $H^{n-1}(M; (\pi_{n-2}V_{n,2})_E)$ vanishes. Since $\pi_{n-2}V_{n,2} \cong \mathbf{Z}$ for even n, $W_{n-1}(E)$ is the unreduced Stiefel–Whitney class in $H^{n-1}(M; \mathbf{Z}_E)$ where $\mathbf{Z}_E$ denotes the local coefficient system determined by E.

Let $V = \{v_1, v_2\}$ be such a field over $M^{(n-2)}$, and assume that the first obstruction $W_{n-1}(E)$ (independent of the choice of V) vanishes so that V can be defined over $M^{(n-1)}$. Clearly the sufficient and necessary condition for that V can be extended to $M^{(n)} = M$ is the vanishing of the second and the final obstruction, say again $\mathrm{Ind}(V)$.

Suppose now we are given a field $V = \{v_1, v_2\}$ over $M^{(n-1)}$. The second obstruction $\mathrm{Ind}(V)$ belongs to $H^n(M; (\pi_{n-1}V_{n,2})_E)$. If M is nonorientable, it was proved (cf. [23, Theorem 3.1]) that E adimits two linearly independent cross sections if and only if $W_{n-1}(E) = 0$ and $e(E) = 0$, where $e(E)$ is the Euler class of the bundle E. It remains to deal with the case that M is orientable. For simplicity, we assume that both E and TM are oriented. Since $\pi_{n-1}V_{n,2} \cong \mathbf{Z} \oplus \mathbf{Z}_2$ for even $n > 4$, the second obstruction

$$\mathrm{Ind}(V) = (\mathbf{Z}\text{-}\mathrm{Ind}(V), \mathbf{Z}_2\text{-}\mathrm{Ind}(V)) \in \mathbf{Z} \oplus \mathbf{Z}_2 \,.$$

Similar to the case of odd dimension, $\mathrm{Ind}(V)$ has a geometric significance. As we shall see, $\mathrm{Ind}(V)$, the index of a field V, is not necessarily independent of the choice of a particular field. Furthermore, in contrast with the odd dimension case, this index depends on the choice of the orientations of E and M.

We first determine $\mathbf{Z}$-$\mathrm{Ind}(V)$. Let $e(E) \in H^n(M;\mathbf{Z})$ be the Euler class of the oriented bundle E. Define $\chi(E) = \langle e(E), [M]\rangle \in \mathbf{Z}$ to be the Euler characteristic number corresponding to $[M]$, the fundamental class of M. Note that $\chi(E)$ changes sign if the orientation of one of either E or M is reversed.

Lemma 2.9. *Let $n > 4$ be an even number and $V = \{v_1, v_2\}$ be a cross section of $V_{n,2}(E)$ over $M^{(n-1)}$. Then $p_*(\mathbf{Z}\text{-}\mathrm{Ind}(V)) = \chi(E)$, where p_* is induced by the bundle projection $p : V_{n,2} \to V_{n,1} = S^{n-1}$. In particular, $\mathbf{Z}$-$\mathrm{Ind}(V)$ is independent of the choice of a particular V.*

Proof. The fiber bundle

$$S^{n-2} \to V_{n,2} \xrightarrow{p} S^{n-1}$$

induces an exact homotopy sequence:

$$\pi_{n-1}S^{n-2} \to \pi_{n-1}V_{n,2} \to \pi_{n-1}S^{n-1} \to \pi_{n-2}S^{n-2} \to \pi_{n-2}V_{n,2} \to \pi_{n-2}S^{n-1} \,.$$

Note that $\pi_{n-2}V_{n,2} \cong \mathbf{Z}$ ($n > 4$ is even) and $\pi_{n-2}S^{n-1} \cong 0$, the above exact sequence gives

$$\pi_{n-1}S^{n-2} \cong \mathbf{Z}_2 \to \pi_{n-1}V_{n,2} \cong \mathbf{Z} \oplus \mathbf{Z}_2 \to \pi_{n-1}S^{n-1} \cong \mathbf{Z} \to 0 \,,$$

from which the proof is complete. □

The next theorem is an analogue of Theorem 2.1.

Theorem 2.10. *Let $n > 4$ be an even number and let E be an oriented vector bundle of rank n over an n-dimensional, closed, connected, oriented manidold M. Suppose that $V = \{v_1, v_2\}$, $U = \{u_1, u_2\}$ are two cross sections of $V_{n,2}(E)$ over $M^{(n-1)}$. Then their second obstructions can be related by*

$$\mathrm{Ind}(V) - \mathrm{Ind}(U) = (w_2(E) + w_2(M))\rho(\Omega(V, U))$$

where $w_2(E) \in H^2(M;\mathbf{Z}_2)$ and $w_2(M) = w_2(TM) \in H^2(M;\mathbf{Z}_2)$ are the mod 2 Stiefel–Whitney classes of the vector bundle E and the tangent bundle TM, respectively; $\Omega(V, U)$ is the first difference of V and U,

$$\Omega(V, U) \in H^{n-2}(M; \pi_{n-2}V_{n,2}) \cong H^{n-2}(M;\mathbf{Z}) \,,$$

and $\rho : H^{n-2}(M;\mathbf{Z}) \to H^{n-2}(M;\mathbf{Z}_2)$ is the mod 2 reduction.

Proof. The proof is similar to that of Theorem 2.1. Observe first that

$$w_2(E) \in H^2(M; \pi_1 V_{n,n-1}),$$
$$\Omega(V,U) \in H^{n-2}(M; \pi_{n-2} V_{n,2}),$$
$$\mathrm{Ind}(V),\ \mathrm{Ind}(U) \in H^n(M; \pi_{n-1} V_{n,2}).$$

Since $n > 4$ is even, the homomorphism $\pi_{n-1} S^{n-2} \to \pi_{n-1} V_{n,2}$ induced by the inclusion $S^{n-2} = V_{n-1,1} \to V_{n,2}$ is injective. In other words, the generator of $\pi_{n-1} S^{n-2}$ is mapped to the generator of the cyclic group of order two of $\pi_{n-1} V_{n,2}$. The formula of Boltyanski–Liao gives for $n > 4$,

$$\mathrm{Ind}(V) = \mathrm{Ind}(U) + \mathrm{Sq}^2(\rho(\Omega(V,U)) + w_2(E) \cdot \rho\Omega(V,U),$$

in which the cup product is determined by the pairing of groups $G_1 \cdot G_2 \subset G_3$ with $\alpha_1 \cdot \alpha_2 = \alpha_3$ where $G_1 = \pi_1 V_{n,n-1} \cong \mathbf{Z}_2$ $(n \geq 3)$, $G_2 = \pi_{n-2} V_{n,2} \cong \mathbf{Z}$ ($n > 4$ is even), $G_3 = \pi_{n-1} V_{n,2} \cong \mathbf{Z} \oplus \mathbf{Z}_2$ ($n > 4$ is even) with generators α_1, α_2 and (β_3, α_3), respectively. The proof is now complete. □

Corollary 2.11. *Let* n, E *and* M *be as in Theorem* 2.10, *and suppose that* $W_{n-1}(E) = 0$. *Then*

(A) *If* $w_2(E) = w_2(M)$, *then* $\mathrm{Ind}(V) = (\chi(E), \mathbf{Z}_2\text{-}\mathrm{Ind}(V))$ *is independent of the choice of* V. *In this case*, E *admits two linearly independent cross sections if and only if both* $\chi(E)$ *and* $\mathbf{Z}_2$-$\mathrm{Ind}(V)$ *vanish.*

(B) *If* $w_2(E) \neq w_2(M)$, *then both* $(\chi(E), 0)$ *and* $(\chi(E), 1)$ *can occur as* $\mathrm{Ind}(V)$. *In particular*, E *admits two linearly independent cross sections if and only if* $\chi(E) = 0$.

Proof. Using Lemma 2.9 and the similar arguments in the proof of Corollary 2.2. We omit the details. □

It should be remarked that in the next section, by making use of K-theory and index theory, we will be able to show that in case (A) of the preceding corollary, the invariant $\mathrm{Ind}(V)$ can be represented by some well-known characteristic classes. These generalize the theory of Atiyah–Dupont on the existence of two linearly independent tangent fields on M.

3. *K*-Theory and Index Theory Methods

We still assume that M is a closed, connected, n-dimensional, differentiable manifold with $n > 4$. Let E be a vector bundle of rank n over M verifying that

$$\begin{aligned} W_{n-1}(E) &= 0 \quad \text{if } n \text{ is even}, \\ w_{n-1}(E) &= 0 \quad \text{if } n \text{ is odd}. \end{aligned} \tag{3.0}$$

In this section, for simplicity, we further assume that both M and E are oriented. We also make the assumption that

$$w_2(E) = w_2(M)\,. \tag{3.1}$$

Then by Corollary 2.2 and Corollary 2.11, the mod 2 invariants $k(E)$ (for odd n) and $\mathbf{Z}_2$-Ind(V) (for even n) are well-defined.

This section has two purposes. The first is to give $k(E)$ (for odd n) and $\mathbf{Z}_2$-Ind(V) (for even n) global K-theoretic, as well as analytic interpretations, respectively. The second is to establish two vanishing results: $k(E) = 0$ for $n \equiv 3 \bmod 4$ and $\mathbf{Z}_2$-Ind$(V) = 0$ for $n \equiv 2 \bmod 4$.

We will apply the method developed by Atiyah and Dupont in [3] to our situation with suitable modifications. The key observation is that under the orientibility assumption and also the assumption (3.1), the vector bundle $TM \oplus E$ is oriented and spin over M.

Following [3], let s be the integer such that $0 \le s < 4$ and that $n + s \equiv 0 \bmod 4$. We equip TM and E with Euclidean metrics respectively. Let $\mathbf{R}^s \to M$ be the trivial Euclidean vector bundle of rank s over M.

Set $\widetilde{TM} = TM \oplus \mathbf{R}^s$, $\tilde{E} = E \oplus \mathbf{R}^s$ Then $\widetilde{TM} \oplus \tilde{E}$ is an oriented Euclidean spin vector bundle over M. We equip it with a spin structure and denote by $S(\widetilde{TM} \oplus \tilde{E})$ the corresponding bundle of spinors.

Let $e_1, \dots, e_{n+s}$ be an oriented orthonormal basis of $\widetilde{TM}$ such that $e_{n+1}, \dots, e_{n+s}$ is an oriented orthonormal basis of $\mathbf{R}^s \subset \widetilde{TM}$, and let $f_1, \dots, f_{n+s}$ be an oriented orthonormal basis of $\tilde{E}$ such that $f_{n+1}, \dots, f_{n+s}$ is an oriented orthonormal basis of $\mathbf{R}^s \subset \tilde{E}$.

For any $X \in \widetilde{TM} \oplus \tilde{E}$, let $c(X)$ denote the Clifford action of X on $S(\widetilde{TM} \oplus \tilde{E})$. Set

$$\tau_1 = c(e_1) \cdots c(e_{n+s}) c(f_1) \cdots c(f_{n+s})\,.$$

Since $n + s \equiv 0 \bmod 4$, one verifies that $\tau_1^2 = \mathrm{Id}$. Define

$$S^{\mathrm{even/odd}}(\widetilde{TM} \oplus \tilde{E}) = \{ s \in S(\widetilde{TM} \oplus \tilde{E}) : \tau_1 s = \pm s \}\,.$$

Then one has the $\mathbf{Z}_2$-splitting

$$S(\widetilde{TM} \oplus \tilde{E}) = S^{\mathrm{even}}(\widetilde{TM} \oplus \tilde{E}) \oplus S^{\mathrm{odd}}(\widetilde{TM} \oplus \tilde{E})\,.$$

For any $X \in \widetilde{TM} \oplus \tilde{E}$, $c(X)$ exchanges $S^{\mathrm{even}}(\widetilde{TM} \oplus \tilde{E})$ and $S^{\mathrm{odd}}(\widetilde{TM} \oplus \tilde{E})$. Thus,

$$\tau_2 = c(e_1) \cdots c(e_{n+s})$$

preserves each of $S^{\mathrm{even/odd}}(\widetilde{TM} \oplus \tilde{E})$.

Now again, since $n + s \equiv 0 \bmod 4$, it is easy to check that $\tau_2^2 = \mathrm{Id}$. Define

$$S_{\pm}^{\mathrm{even}}(\widetilde{TM} \oplus \tilde{E}) = \{ s \in S^{\mathrm{even}}(\widetilde{TM} \oplus \tilde{E}) : \tau_2 s = \pm s \}\,,$$

$$S_{\pm}^{\mathrm{odd}}(\widetilde{TM} \oplus \tilde{E}) = \{ s \in S^{\mathrm{odd}}(\widetilde{TM} \oplus \tilde{E}) : \tau_2 s = \pm s \}\,.$$

Let $i\widetilde{TM}$ denote the *Real* vector bundle $\widetilde{TM}$ in the sense of [2], with the canonical involution given by $X \in \widetilde{TM} \to -X \in \widetilde{TM}$. One can verify easily that the lifts of $S_{\pm}^{\text{even}}(\widetilde{TM} \oplus \tilde{E})$ and $S_{\pm}^{\text{odd}}(\widetilde{TM} \oplus \tilde{E})$ are *Real* vector bundles over $i\widetilde{TM}$ in the sense of [2] and [3].

We will use the same notation $S_{\pm}^{\text{even/odd}}(\widetilde{TM} \oplus \tilde{E})$ for their lifts over $\widetilde{TM}$, etc., when there will be no confusion.

Now let $v_1, \ldots, v_r$ be r-cross sections of E over M which are linearly independent over a closed subset $Y \subset M$. Clearly, we can and we will assume that $v_1, \ldots, v_r$ are orthogonal to each other over Y.

Following [3], consider a point $(v, x) \in \widetilde{TM} \times S^{r+s-1}$, where v is in the fiber over $y \in M$ and put

$$x(y) = \sum_{i=1}^{r} x_i v_i(y) + \sum_{j=1}^{s} x_{n+j} f_{n+j}\,.$$

Then consider the following square over $\widetilde{TM} \times S^{r+s-1}$ which is analogous to [3, (2.7)],

$$\begin{array}{ccc} S_{+}^{\text{even}}(\widetilde{TM} \oplus \tilde{E}) & \xrightarrow{ic(v)} & S_{-}^{\text{odd}}(\widetilde{TM} \oplus \tilde{E}) \\ \downarrow c(x(y)) & & \downarrow -c(x(y)) \\ S_{+}^{\text{odd}}(\widetilde{TM} \oplus \tilde{E}) & \xrightarrow{ic(v)} & S_{-}^{\text{even}}(\widetilde{TM} \oplus \tilde{E})\,. \end{array}$$

This defines a square of *Real* vector bundles and homomorphisms over $\widetilde{TM} \times S^{r+s-1}$. Clearly, the maps are $\mathbf{Z}_2$-equivariant with respect to the anti-podal involution on S^{r+s-1} and the action on the bundles defined by the trivial action on the upper row and multiplication by -1 on the lower row.

Hence over $\widetilde{TM} \times \mathbf{R}P^{r+s-1}$ we have the square

$$\begin{array}{ccc} S_{+}^{\text{even}}(\widetilde{TM} \oplus \tilde{E}) & \xrightarrow{ic(v)} & S_{-}^{\text{odd}}(\widetilde{TM} \oplus \tilde{E}) \\ \downarrow c(x(y)) & & \downarrow -c(x(y)) \\ S_{+}^{\text{odd}}(\widetilde{TM} \oplus \tilde{E}) \otimes H & \xrightarrow{ic(v)} & S_{-}^{\text{even}}(\widetilde{TM} \oplus \tilde{E}) \otimes H\,, \end{array}$$

where H denotes the Hopf line bundle over $\mathbf{R}P^{r+s-1}$.

Now since $f_{n+1}, \ldots, f_{n+s}$ are linearly independent over the whole of M, it follows easily that the vertical maps in the above square are isomorphisms over $(\widetilde{TM} \times \mathbf{R}P^{s-1}) \cup (Y \times \mathbf{R}P^{r+s-1})$, and thus the square defines an element $\alpha_E^s(v_1, \ldots, v_r)$ in

$$KR((i\widetilde{TM}|_{M\setminus Y} \times (\mathbf{R}P^{r+s-1} \setminus \mathbf{R}P^{s-1})) = KR^s((iTM|_{M\setminus Y} \times (\mathbf{R}P^{r+s-1} \setminus \mathbf{R}P^{s-1}))\,,$$

where KR is the KR-group in the sense of [2].

By composing with the index map defined in [3, (2.12)], one then gets at last an element

$$\operatorname{Ind} \alpha_E^s(v_1, \ldots, v_r) \in KR^s(\mathbf{R}P^{r+s-1} \setminus \mathbf{R}P^{s-1})\,. \tag{3.2}$$

In particular, if $Y = \emptyset$, we get a **global invariant** for E:

$$\operatorname{Ind} \alpha^s_{E,r} \in KR^s(\mathbf{R}P^{r+s-1}, \mathbf{R}P^{s-1}) . \tag{3.3}$$

We pass now to the case where $M = B^n$, the unit ball in $\mathbf{R}^n$, and $Y = S^{n-1}$ the unit sphere. Let E be the trivial vector bundle of rank n over B^n. Then the construction in (3.2) induces a map

$$\theta^s : \pi_{n-1}(V_{n,r}) \longrightarrow KR^s(\mathbf{R}P^{r+s-1}, \mathbf{R}P^{s-1}) . \tag{3.4}$$

Clearly, on $M = B^n$, TM is also a trivial vector bundle. Moreover, as $n + s \equiv 0 \bmod 4$, one has the canonical identification that

$$S(\mathbf{R}^{n+s} \oplus \mathbf{R}^{n+s}) \simeq \Lambda^*(\mathbf{R}^{n+s}) ,$$

from which one sees easily that the homomorphism θ^s in (2.4) is exactly the same as the homomorphism defined in [3, (2.15)].

By proceeding in exactly the same way as in proof of [3, Theorem 2.20], one obtains the following generalization of [3, Theorem 2.20].

Theorem 3.1. *Let M be a closed oriented connected manifold of dimension $n = 4k - s$ with $0 \leq s \leq 3$, and let E be an oriented vector bundle over M of rank n, verifying* (3.0) *and* (3.1). *We fix a spin structure on $TM \oplus E$. Let $\{v_1, \ldots, v_r\}$ be a set of r-cross sections of E, linearly independent except at the finite set of points $\{A_1, \ldots, A_l\}$. Then in the group $KR^s(\mathbf{R}P^{r+s-1}, \mathbf{R}P^{s-1})$, we have the formula for the global invariant*

$$\operatorname{ind} \alpha^s_{E,r} = \sum_{i=1}^{l} \theta^s \mathcal{O}_{A_i}(v_1, \ldots, v_r) , \tag{3.5}$$

where $\mathcal{O}_A(v_1, \ldots, v_r) \in \pi_{n-1}(V_{n,r})$ is the local obstruction to extending the cross sections at A. In particular, $\operatorname{ind} \alpha^s_{E,r}$ *does not depend on the spin structure on $TM \oplus E$.*

In the rest of this section, we restrict to the case $r = 2$.

We first recall from [3, Sec. 3] that $KR^1(\mathbf{R}P^\infty, \mathbf{R}P^{s-1}) = 0$. Thus, when $s = 1$, the left hand side of (3.5) vanishes as we can set $Y = \emptyset$ and $r = \infty$. As a consequence, the right hand side of (3.5) vanishes identically when $s = 1$.

Also recall from [3, Sec. 5] that the homomophism θ^s in (3.4) is an isomorphism when $r \leq 3$ and $n \geq r + 3$.

The following corollary is now clear.

Corollary 3.2. *Let M, E be as in Theorem* 3.1 *and assume $n \equiv 3 \bmod 4$, $n > 4$. Then the invariant $k(E)$ defined in Corollary* 2.2(A) *vanishes. In particular, there exist two linearly independent cross sections of E.*

We turn next to the remaining cases of $s = 0,\ 2,\ 3$, separately. Clearly, it suffices to discuss the global invariant $\operatorname{Ind} \alpha^s_{E,\infty}$.

(1). The case $s = 0$.

Let ∇^{TM} be the Levi–Civita connection on TM associated to g^{TM}, and let ∇^E be a Euclidean connection on E. The induced Hermitian connection on $S(TM \oplus E)$ will be denoted by $\nabla^{S(TM\oplus E)}$. Then $\nabla^{S(TM\oplus E)}$ preserves each $S_{\pm}^{\mathrm{even/odd}}(TM \oplus E)$.

Define a Dirac type operator $D : \Gamma(S(TM \oplus E)) \to \Gamma(S(TM \oplus E))$ by

$$D = \sum_{i=1}^{n} c(e_i)\nabla_{e_i}^{S(TM\oplus E)} .$$

Let $D_{\pm} : \Gamma(S_{+}^{\mathrm{even/odd}}(TM \oplus E)) \to \Gamma(S_{-}^{\mathrm{odd/even}}(TM \oplus E))$ be the restriction of D on $S_{+}^{\mathrm{even/odd}}(TM \oplus E)$ respectively. By proceeding as in [3], one gets in an obvious way the following analogue of [3, (4.2)]:

$$\mathrm{Ind}\, \alpha_{E,\infty}^{0} = (\mathrm{Ind}\, D_+ - \mathrm{Ind}\, D_-) - (\mathrm{Ind}\, D_-)(H-1) . \tag{3.6}$$

There are two ways to identify $\mathrm{Ind}\, D_{\pm}$. The first is to apply the Atiyah–Singer index theorem [4]. The second is that in view of local index theory, to compute $\mathrm{Ind}\, D_{\pm}$, one may assume that both TM and E are spin. Then one has the splitting of bundles of spinors $S(TM) = S_+(TM) \oplus S_-(TM)$ and $S(E) = S_+(E) \oplus S_-(E)$. Moreover, $D_{\pm}$ are the canonical Dirac operators

$$D_{\pm} : \Gamma(S_+(TM) \otimes S_{\pm}(E)) \longrightarrow \Gamma(S_-(TM) \otimes S_{\pm}(E))$$

respectively. Thus, one deduces directly that, as $\mathrm{rank}\, E = \dim M$,

$$\mathrm{Ind}\, D_+ - \mathrm{Ind}\, D_- = \langle \hat{A}(TM)\mathrm{ch}(S_+(E) - S_-(E)), [M]\rangle = \chi(E) \tag{3.7}$$

and

$$\begin{aligned}\mathrm{Ind}\, D_+ + \mathrm{Ind}\, D_- &= (-1)^{n/4}\langle \hat{A}(TM)\mathrm{ch}(S(E)), [M]\rangle \\ &= (-1)^{n/4}\langle \hat{A}(TM)\hat{S}(E), [M]\rangle ,\end{aligned} \tag{3.8}$$

where $\hat{S}(E)$ is the multiplicative characteristic class of E associated to the function $2\cosh(x/2)$.

From (3.6), (3.7) and (3.8), we obtain

$$\mathrm{Ind}\, \alpha_{E,\infty}^{0} = \chi(E) + \frac{1}{2}(\chi(E) - (-1)^{n/4}\langle \hat{A}(TM)\hat{S}(E), [M]\rangle)(H-1) . \tag{3.9}$$

Combining with Theorem 3.1, we deduce finally

Corollary 3.3. *Let M, E be as in Theorem* 3.1 *and assume* $n \equiv 0 \bmod 4$, $n > 4$. *Then E admits two linearly independent cross sections if and only if $\chi(E) = 0$ and* $\chi(E) \equiv (-1)^{n/4}\langle \hat{A}(TM)\hat{S}(E), [M]\rangle \bmod 4$.

Remark 3.4. Note that the Dirac type operators $D_{\pm}$ and the characteristic number $(-1)^{n/4}\langle \hat{A}(TM)\hat{S}(E), [M]\rangle$ have appeared before in [20]. It is remarkable that they also appear in the current context (compare also with [9]).

(2). The case $s = 2$.

By proceeding as in [3], one verifies that in this case

$$\mathrm{Ind}\,\alpha^2_{E,\infty} = \chi(E)\,. \tag{3.10}$$

We leave the details to the interested reader.

Corollary 3.5. *Let M, E be as in Theorem 3.1 and assume $n \equiv 2 \bmod 4$, $n > 4$. Then E admits two linearly independent cross sections if and only if $\chi(E) = 0$.*

(3). The case $s = 3$.

By Corollary 2.2, for M, E as in Theorem 3.1, the mod 2 invariant $k(E)$ is well-defined. By Theorem 3.1 one has a K-theoretic interpretation of $k(E)$. Here we give an analytic interpretation of $k(E)$ which generalizes the analytic interpretation (3.4) for the case of $E = TM$.

We write $n = 4q+1$. Then $TM \oplus E$ is an oriented spin vector bundle of rank $8q+2$ over M. Set

$$I = c(e_1)\cdots c(e_{4q+1})\,,$$

$$J = c(e_1)\cdots c(e_{4q+1})c(f_1)\cdots c(f_{4q+1})\,,$$

$$K = IJ\,.$$

Then one verifies directly that $S(TM \oplus E)$ carries a natural quaternionic structure associated to $1,\ I,\ J,\ K$.

Observe that for any $X \in TM$, $c(X)$ commutes with I and anti-commutes with $J,\ K$.

Setting $S^0(TM \oplus E) = (1-J)S(TM \oplus E)$, then one has the natural splitting

$$S(TM \oplus E) = S^0(TM \oplus E) \oplus JS^0(TM \oplus E)\,.$$

Now since for any $1 \le i \le n$, $c(e_i)$ anti-commutes with J, the Dirac operator D maps $\Gamma(S^0(TM \oplus E))$ to $\Gamma(JS^0(TM \oplus E))$. Thus,

$$P_E = ID \tag{3.11}$$

maps $\Gamma(S^0(TM \oplus E))$ to itself. Moreover, one verifies clearly that P_E is Hermitian skew-adjoint: $P_E^* = -P_E$. Let

$$\mathrm{Ind}_2 P_E = \dim_{\mathbf{C}} \ker P_E \mod 2$$

be the corresponding mod 2 index in the sense of Atiyah and Singer [5].

Recalling from [3, Sec. 3] that $KR^3(\mathbf{R}P^\infty, \mathbf{R}P^2) = \mathbf{Z}_2$, we may view $\mathrm{Ind}\,\alpha^3_{E,\infty}$ as an element in $\mathbf{Z}_2$. Then by proceeding as in [3], one gets easily the following analogue of [3, Lemma 4.3],

$$\mathrm{Ind}_2 P_E = \mathrm{Ind}\,\alpha^3_{E,\infty}\,, \tag{3.12}$$

from which one gets the following analytic interpretation of $k(E)$, when $\dim M = 4q+1$ and E verifies the conditions in Theorem 3.1,

$$k(E) = \mathrm{Ind}_2 P_E\,. \tag{3.13}$$

Remark 3.6. If $E = TM$, then $S^0(TM \oplus E) \simeq \Lambda^{\mathrm{even}}(T^*M)$, and one recovers (2.4).

As a consequence, one deduces at once the following result.

Corollary 3.7. *If E is an oriented vector bundle of rank $4q+1$ over a $4q+1$ dimensional closed oriented connected manifold M verifying that $w_2(E) = w_2(M)$ and $w_{4q}(E) = 0$, then E admits two linearly independent cross sections if and only if* $\mathrm{Ind}_2 P_E = 0$.

Now we go back to the case of $s = 0$.

Here we are going to prove an interesting and strong result that $\mathbf{Z}_2$-$\mathrm{Ind}(V)$ vanishes when the bundle E admits a complex structure. More precisely we have

Corollary 3.8. *Let E and M be given as in Theorem* 3.1, *and assume that $n = 4q$ $(q > 1)$. If E admits a complex structure, then*

$$\chi(E) \equiv (-1)^q \langle \hat{A}(TM)\hat{S}(E), [M]\rangle \bmod 4\,.$$

Proof. Let $v : M \to E$ be a cross section of E over M with finite singularities. Using the complex structure J of E, we get two (real) linearly independent cross sections v and Jv with finite singularities. We hope to show that the $\mathbf{Z}_2$-$\mathrm{Ind}(V)$ of $V = \{v, Jv\}$ vanishes. Observe that the fiber bundle

$$S^{4q-2} \xrightarrow{i} V_{4q,2} \xrightarrow{\pi} V_{4q,1} = S^{4q-1}$$

has a canonical section $f_0 : S^{4q-1} \to V_{4q,2}$ defined by $f_0(v) = (v, J_0 v)$, where $J_0 v = J_0(v_1, v_2, \ldots, v_{4q}) = (-v_2, v_1, \ldots, -v_{4q}, v_{4q-1})$. It follows that in the homotopy sequence

$$\mathbf{Z}_2 \cong \pi_{4q-1}S^{4q-2} \xrightarrow{i_*} \pi_{4q-1}V_{4q,2} \xrightarrow{\pi_*} \pi_{4q-1}S^{4q-1} \cong \mathbf{Z}\,,$$

$\pi_*(f_0)_*(a) = a$, where a is a generator of $\pi_{4q-1}S^{4q-1}$. Recall that $\pi_{4q-1}V_{4q,2} \cong \mathbf{Z} \oplus \mathbf{Z}_2$ has a natural decomposition, in other words, the generators of $\pi_{4q-1}V_{4q,2} \cong \mathbf{Z} \oplus \mathbf{Z}_2$ have been selected so that $(1,0) = (f_0)_*(a)$ and $(0,1) = i_*(1)$.

Let $f : S^{4q-1} \to V_{4q,2}$ be given by $f(v) = (v, Jv)$. Clearly there exists an element S of $GL(4q, \mathbf{R})$ such that $J = SJ_0S^{-1}$. Moreover, one can assume that the determinant of S is positive. It follows that $f(v) = (v, Jv) = (v, SJ_0S^{-1}v) \simeq (v, J_0v) = f_0 : S^{4q-1} \to V_{4q,2}$.

Therefore the mapping

$$H^{4q}(M; \pi_{4q-1}S^{4q-1}) \to H^{4q}(M; \pi_{4q-1}V_{4q,2})$$

induced by $(f_0)_* = f_* : \pi_{4q-1}S^{4q-1} \to \pi_{4q-1}V_{4q,2}$ sends $\chi(E)$, the Euler number of E, to $(\chi(E), 0) \in \mathbf{Z} \oplus \mathbf{Z}_2$. Thus it follows from (3.9) that

$$\frac{1}{2}(\chi(E) - (-1)^q \langle \hat{A}(TM)\hat{S}(E), [M]\rangle) \equiv 0 \bmod 2\,,$$

as desired. □

Recall that an even dimensional differentiable manifold is said to admit an almost complex structure if its tangent bundle admits a complex structure. As we noted before, an almost complex structure induces a canonical orientation on the manifold. As pointed out by Hopf [Ho], the study and construction of almost complex manifolds have revealed some interesting phenomena. For instance, the property of admitting an almost complex structure in not a homotopy invariant. This property depends on the given orientation of the manifold. There are many applications of Corollary 3.8, for instance, we have immediately

Corollary 3.9. *The following manifolds admit no almost complex structures*

(1) *The spheres* S^{4k} $(k > 1)$;
(2) *The "negative" complex projective space* $\overline{\mathbf{C}P}^{2k}$ *which has opposite orientation with the standard one of* $\mathbf{C}P^{2k}$ $(k > 1)$;
(3) *The quaternionic projective spaces* $\mathbf{H}P^{4k+1}$ $(k > 0)$ *and* $\mathbf{H}P^{4k+2}$ $(k \geq 0)$;
(4) *The Cayley projective plane* $\mathbf{Ca}P^2$.

Proof. These assertions follow from simple cohomological arguments if one uses Corollary 3.8. For an illustration, we indicate a proof of (2). If we assume that $\overline{\mathbf{C}P}^{2k}$ admits an almost complex structure, then since $\chi(\overline{\mathbf{C}P}^{2k}) = 2k+1$ and $\mathrm{Sign}(\overline{\mathbf{C}P}^{2k}) = -1$, by Corollary 3.8 one should have

$$2k + 1 - (-1)^k \langle \hat{A}(T\overline{\mathbf{C}P}^{2k})\hat{S}(T\overline{\mathbf{C}P}^{2k}), [\overline{\mathbf{C}P}^{2k}]\rangle \equiv 0 \mod 4\,,$$

which is in contradiction with the fact that

$$\langle \hat{A}(T\overline{\mathbf{C}P}^{2k})\hat{S}(T\overline{\mathbf{C}P}^{2k}), [\overline{\mathbf{C}P}^{2k}]\rangle = \mathrm{Sign}(\overline{\mathbf{C}P}^{2k}) = -1\,.$$

Thus $\overline{\mathbf{C}P}^{2k}$ does not admit an almost complex structure. □

We conclude this section with the following remarks.

Remark 3.10. (1) It was shown by Hopf [13] and Ehresmann [10] that S^4 cannot be given a complex structure. Borel and Serre [8] established a well known result that S^n admits almost complex structures if and only if $n = 2$ or 6.

(2) The complex projective space $\mathbf{C}P^n$ has a natural complex structure, thus it admits an almost complex structure. It is clear to see that $\overline{\mathbf{C}P}^{2k+1}$ admits an almost complex structure by complex conjugation. It was Kahn [14] who first found an unexpected phenomenon that $\overline{\mathbf{C}P}^{2}$ admits no almost complex structures. We extend this conclusion.

(3) Hirzebruch proved that $\mathbf{H}P^n$ does not admit any almost complex structures if $n \geq 4$. In Problem 15 in [11], Hirzebruch asked the problem of existence of almost complex structures on $\mathbf{H}P^2$ and $\mathbf{H}P^3$. It was Milnor who proved the nonexistence (cf. [12]).

(4) Borel and Hirzebruch [7, 19.6] showed that $\mathbf{Ca}P^2$ does not admit any almost complex structures.

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