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A Poincaré–Hopf type formula for Chern character numbers

Huitao Feng · Weiping Li · Weiping Zhang

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Abstract For two complex vector bundles admitting a homomorphism with isolated singularities between them, we establish a Poincaré–Hopf type formula for the difference of the Chern character numbers of these two vector bundles. As a consequence, we extend the original Poincaré–Hopf index formula to the case of complex vector fields.

1 Introduction and the statement of the main result

Let M be a closed, oriented, smooth manifold of dimension $2n$. Let E_+, E_- be two complex vector bundles over M.

Let $v \in \Gamma(\mathrm{Hom}(E_+, E_-))$ be a homomorphism between E_+ and E_-. Let $Z(v)$ denote the set of the points at which v is singular (that is, not invertible). We assume that the following basic assumption holds.

Basic Assumption 1.0 The point set $Z(v)$ consists of a finite number of points in M.

For any $p \in Z(v)$, we choose a small open ball $B(p)$ centered at p such that the closure $\overline{B(p)}$ contains no points in $Z(v)\backslash p$. Then, when restricted to the boundary $\partial B(p)$,

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H. Feng (✉)
School of Mathematics and Statistics, Chongqing University of Technology,
400050 Chongqing, People's Republic of China
e-mail: fht@nankai.edu.cn

W. Li
Department of Mathematics, Oklahoma State University, Stillwater, OK 74078-1058, USA
e-mail: wli@math.okstate.edu

W. Zhang
Chern Institute of Mathematics and LPMC, Nankai University,
300071 Tianjin, People's Republic of China
e-mail: weiping@nankai.edu.cn

the linear map

$$v|_{\partial B(p)}:E_+|_{\partial B(p)}\to E_-|_{\partial B(p)},\tag{1.1}$$

which we denote by v_p, is invertible. The map v_p determines an element in $K^1(S^{2n-1})=\mathbf{Z}$ which we denote by $\deg(v_p)\in\mathbf{Z}$.[1]

The main result in this paper is the following theorem:

Theorem 1.1 *Under the Basic Assumption 1.0, the following identity holds,*

$$\langle \mathrm{ch}(E+)-\mathrm{ch}(E_-),[M]\rangle=(-1)^{n-1}\sum_{p\in Z(v)}\deg\left(v_p\right).\tag{1.2}$$

Our original motivation is to establish an extension of the Poincaré–Hopf index formula for vector fields with isolated zero points (cf. [1, Theorem 11.25]*) to the case of complex vector fields, under the framework considered by Jacobowitz in* [3].

To be more precise, let $T_{\mathbf{C}}M=TM\otimes\mathbf{C}$ denote the complexification of the tangent vector bundle TM. Let $K=\xi+\sqrt{-1}\eta\in\Gamma(T_{\mathbf{C}}M)$ be a smooth section of $T_{\mathbf{C}}M$, with $\xi,\ \eta\in\Gamma(TM)$.

Let g^{TM} be a Riemannian metric on TM, then it induces canonically a complex symmetric bilinear form $h^{T_{\mathbf{C}}M}$ on $T_{\mathbf{C}}M$, such that

$$h^{T_{\mathbf{C}}M}(K,K)=|\xi|^2_{g^{TM}}-|\eta|^2_{g^{TM}}+2\sqrt{-1}\langle\xi,\eta\rangle_{g^{TM}}.\tag{1.3}$$

Jacobowitz proved in [3] the following vanishing result.

Proposition 1.2 [3] *If $h^{T_{\mathbf{C}}M}(K,K)$ is nowhere zero on M, then the Euler number of M vanishes: $\chi(M)=0$.*

If one takes $\eta=0$, then Proposition 1.2 reduces to the classical Hopf vanishing result: $\chi(M)=0$ if M admits a nowhere zero vector field.

Jacobowitz asked in [3] whether there is a counting formula for $\chi(M)$ of Poincaré–Hopf type, extending Proposition 1.2 to the case where $h^{T_{\mathbf{C}}M}(K,K)$ vanishes somewhere on M. In Sect. 3, we will establish such a formula as an application of Theorem 1.1, while Theorem 1.1 itself will be proved in Sect. 2.

2 A proof of Theorem 1.1

We will use the superconnection formalism developed in [5] to prove Theorem 1.1.

Due to the topological nature of both sides of (1.2), we first make some simplifying assumptions on the metrics and connections near the set of singularities $Z(v)$.

First of all, we assume that there is a Riemannian metric g^{TM} on TM such that for any $p\in Z(v)$, there is a coordinate system $(x_1,\dots,x_{2n})$, with $0\le x_i\le 1$ for $1\le i\le 2n$, centered around p such that

$$B_p(1)=\left\{(x_1,\dots,x_{2n})|\sum_{i=1}^{2n}x_i^2\le 1\right\}\subset M\backslash(Z(v)\backslash\{p\})\tag{2.1}$$

[1] One way to define $\deg(v_p)$ is that v_p in (1.1) defines a complex vector bundle $E_{v(p)}$ over a sphere $S^{2n}(v_p)$ with $\partial B(p)$ as an equator. Then one can define $\deg(v_p)=\langle\mathrm{ch}(E_{v(p)}),[S^{2n}(v_p)]\rangle$.

and

$$g^{TM}\Big|_{B_p(1)} = dx_1^2 + dx_2^2 + \cdots + dx_{2n}^2, \tag{2.2}$$

that is, the metric g^{TM} is Euclidean on each $B_p(1)$, $p \in Z(v)$.

On the other hand, on each $B_p(1)$, the bundles $E_\pm$ are trivial vector bundles. We equip these two trivial vector bundles over $B_p(1)$ the trivial metrics and trivial connections respectively. Moreover, we can deform v near $\partial B_p(1)$, so that $v_p : E_+|_{\partial B_p(1)} \to E_-|_{\partial B_p(1)}$ is *unitary*, while still keep the new homomorphism nonsingular on $M \backslash Z(v)$.

By partition of unity, we may then construct Hermitian metrics and connections $\nabla^{E_\pm}$ on $E_\pm$ over M such that the above simplifying assumptions hold on $\cup_{p\in Z(v)} B_p(1)$.

We now follow the formalism in [5].

Let $E = E_+ \oplus E_-$ be the $\mathbf{Z}_2$-graded complex vector bundle over M. Let $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$ be the $\mathbf{Z}_2$-graded connection on E.

Let $v : E_+ \to E_-$ extend to an (odd) endomorphism of E by acting as zero on E_-, with the notation unchanged. Let $v^* : E_- \to E_+$ (and thus also extends to an (odd) endomorphism of E) be the adjoint of v with respect to the Hermitian metrics on $E_\pm$ respectively.

Set $V = v + v^*$. Then V is an odd endomorphism of E. Moreover, V^2 is fiberwise positive over $M \backslash Z(v)$.

We fix a square root of $\sqrt{-1}$. Let $\varphi : \Omega^*(M) \to \Omega^*(M)$ be the rescaling on differential forms such that for any differential form α of degree k, $\varphi(\alpha) = (2\pi\sqrt{-1})^{-\frac{k}{2}}\alpha$. The final formulas below will not depend on the choice of this square root.

For any $t \in \mathbf{R}$, let $\mathbf{A}_t$ be the superconnection on E, in the sense of Quillen [5], defined by

$$\mathbf{A}_t = \nabla^E + tV. \tag{2.3}$$

Let $\mathrm{ch}(E, \mathbf{A}_t)$ be the associated Chern character form defined by

$$\mathrm{ch}\,(E, \mathbf{A}_t) = \varphi\, \mathrm{tr}_s \left[e^{-\mathbf{A}_t^2}\right]. \tag{2.4}$$

The following transgression formula has been proved in [5, (2)],

$$\frac{\partial \mathrm{ch}\,(E, \mathbf{A}_t)}{\partial t} = -\frac{1}{\sqrt{2\pi\sqrt{-1}}}\, d\, \varphi\, \mathrm{tr}_s \left[V e^{-\mathbf{A}_t^2}\right]. \tag{2.5}$$

Set for any $T > 0$,

$$\gamma(T) = \frac{1}{\sqrt{2\pi\sqrt{-1}}}\varphi \int_0^T \mathrm{tr}_s \left[V e^{-\mathbf{A}_t^2}\right] dt. \tag{2.6}$$

From (2.5) and (2.6), one gets

$$\mathrm{ch}\,(E, \mathbf{A}_0) - \mathrm{ch}\,(E, \mathbf{A}_T) = d\gamma(T). \tag{2.7}$$

Set $M_1 = M \backslash \bigcup_{p\in Z(v)} B_p(1)$.

Since V is invertible on M_1, by proceeding as in [5, §4], one sees that the following identity holds uniformly on M_1,

$$\lim_{T\to+\infty} \mathrm{ch}\,(E, \mathbf{A}_T) = 0. \tag{2.8}$$

Lemma 2.1 *The following identity holds,*

$$\langle \mathrm{ch}\,(E_+) - \mathrm{ch}\,(E_-)\,, [M]\rangle = -\sum_{p\in Z(v)} \lim_{T\to+\infty} \int_{\partial B_p(1)} \gamma(T). \tag{2.9}$$

Proof Since by our choice the connections $\nabla^{E_\pm}$ are the trivial connections when restricted to $\bigcup_{p\in Z(v)} B_p(1)$, one has

$$\langle \mathrm{ch}\,(E_+) - \mathrm{ch}\,(E_-)\,, [M]\rangle = \int_M \mathrm{ch}\,(E, \mathbf{A}_0) = \varphi \int_M \mathrm{tr}_s \left[e^{-(\nabla^E)^2}\right] = \varphi \int_{M_1} \mathrm{tr}_s \left[e^{-(\nabla^E)^2}\right]. \tag{2.10}$$

By (2.7), (2.8) and (2.10), we have

$$\begin{aligned} \langle \mathrm{ch}\,(E_+) - \mathrm{ch}\,(E_-)\,, [M]\rangle &= \lim_{T\to+\infty} \left(\int_{M_1} \mathrm{ch}\,(E, \mathbf{A}_0) - \int_{M_1} \mathrm{ch}\,(E, \mathbf{A}_T) \right) \\ &= \lim_{T\to+\infty} \int_{M_1} d\gamma(T) = \lim_{T\to+\infty} \int_{\partial M_1} \gamma(T) \\ &= -\sum_{p\in Z(v)} \lim_{T\to+\infty} \int_{\partial B_p(1)} \gamma(T), \end{aligned}$$

where the last equality comes from the orientation consideration. □

Recall that the map v_p is the restriction of v on $\partial B_p(1)$ (cf. (1.1)).

Lemma 2.2 *For any $p \in Z_v$, the following identity holds,*

$$\lim_{T\to+\infty} \int_{\partial B_p(1)} \gamma(T) = (-1)^n \deg(v_p). \tag{2.11}$$

Proof For any $p \in Z(v)$, since when restricted on the sphere $\partial B_p(1)$, the homomorphism v has been deformed to be unitary, we get that $v^* = v^{-1}$ and V^2 is the identity map acting on $E|_{\partial B_p(1)}$. Also, since ∇^E is the trivial connection over $B_p(1)$, we will use the simplified notation d for it. By (2.3), one has on $B_p(1)$ that

$$A_t = d + tV, \quad A_t^2 = d^2 + t[d, V] + t^2 V^2 = t^2 \mathrm{Id}_E + t dV.$$

One then deduces that

$$\begin{aligned}\int_{\partial B_p(1)} \gamma(T) &= \frac{1}{\sqrt{2\pi\sqrt{-1}}}\varphi \int_{\partial B_p(1)} \int_0^T \mathrm{tr}_s\left[V e^{-\mathbf{A}_t^2}\right] dt \\ &= \frac{1}{\sqrt{2\pi\sqrt{-1}}}\varphi \int_{\partial B_p(1)} \int_0^T e^{-t^2}\mathrm{tr}_s\left[V e^{-tdV}\right] dt \\ &= \frac{1}{(2\pi\sqrt{-1})^n}\frac{-1}{(2n-1)!}\int_0^T t^{2n-1}e^{-t^2}dt \int_{\partial B_p(1)} \Big(\mathrm{tr}_{E_+}\left[v^*dv\left(dv^*dv\right)^{n-1}\right] \\ &\quad -\mathrm{tr}_{E_-}\left[vdv^*\left(dvdv^*\right)^{n-1}\right]\Big) \\ &= \frac{1}{(2\pi\sqrt{-1})^n}\frac{2(-1)^n}{(2n-1)!}\int_0^T t^{2n-1}e^{-t^2}dt \int_{\partial B_p(1)} \mathrm{tr}_{E_+}\left[\left(v^{-1}dv\right)^{2n-1}\right].\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{T\to+\infty}\int_{\partial B_\epsilon(p)} \gamma(T) &= \frac{1}{(2\pi\sqrt{-1})^n}\frac{2(-1)^n}{(2n-1)!}\int_0^{+\infty} t^{2n-1}e^{-t^2}dt \int_{\partial B_p(1)} \mathrm{tr}_{E_+}\left[\left(v^{-1}dv\right)^{2n-1}\right] \\ &= \frac{1}{(2\pi\sqrt{-1})^n}\frac{(-1)^n(n-1)!}{(2n-1)!}\int_{\partial B_p(1)} \mathrm{tr}_{E_+}\left[\left(v^{-1}dv\right)^{2n-1}\right] \\ &= (-1)^n \deg(v_p),\end{aligned}$$

where one compares with [2, Propositions 1.2 and 1.4] for the last equality.

From Lemmas 2.1 and 2.2, one gets Theorem 1.1. □

We conclude this section with the following result which is complementary to Theorem 1.1.

Lemma 2.3 *Under the Basic Assumption 1.0, for any closed form* $\alpha \in \Omega^*(M)$ *without degree zero component, one has*

$$\langle [\alpha]\left(\mathrm{ch}\left(E_+\right) - \mathrm{ch}\left(E_-\right)\right), [M]\rangle = 0, \tag{2.12}$$

where $[\alpha] \in H^*(M, \mathbf{C})$ *is the de Rham cohomology class induced by* α.

Proof By the Poincaré lemma (cf. [1]), as α is closed and contains no zero degree component, on each $B_p(1)$, $p \in Z(v)$, there exists a form β_p such that $\alpha = d\beta_p$ on an open neighborhood of $B_p(1)$.

By partition of unity, one then constructs a differential form β on M such that $\beta = \beta_p$ on each $B_p(1)$, $p \in Z(v)$. Then,

$$\alpha - d\beta = 0 \tag{2.13}$$

on $\cup_{p\in Z(v)} B_p(1) = M \backslash M_1$.

On the other hand, by (2.4) and (2.5) one knows that for any $t \geq 0$, one has

$$\langle [\alpha] (\mathrm{ch}(E_+) - \mathrm{ch}(E_-)), [M] \rangle = \int_M (\alpha - d\beta)\varphi \, \mathrm{tr}_s \left[e^{-\mathbf{A}_t^2} \right]. \tag{2.14}$$

From (2.8), (2.13) and (2.14), and by taking $t \to +\infty$, one gets (2.12). □

3 A Poincaré–Hopf formula for complex vector fields

Let M be a closed and oriented manifold of dimension $2n$. Let g^{TM} be a Riemannian metric on TM. Let $T_{\mathbf{C}}M = TM \otimes \mathbf{C}$ be the complexification of TM. Then g^{TM} extends to a symmetric bilinear form $h^{T_{\mathbf{C}}M}$ on $T_{\mathbf{C}}M$.

Let $K = \xi + \sqrt{-1}\eta \in \Gamma(T_{\mathbf{C}}M)$ be a complex vector field on M, with $\xi, \eta \in \Gamma(TM)$. Then one has

$$h^{T_{\mathbf{C}}M}(K, K) = |\xi|^2_{g^{TM}} - |\eta|^2_{g^{TM}} + \sqrt{-1}\langle \xi, \eta \rangle_{g^{TM}}. \tag{3.1}$$

Let Z_K be the zero set of $h^{T_{\mathbf{C}}M}(K, K)$, that is,

$$Z_K = \left\{ x \in M : h^{T_{\mathbf{C}}M}(K(x), K(x)) = 0 \right\}. \tag{3.2}$$

In the rest of this section, we make the following assumption.

Basic Assumption 3.0 The set Z_K consists of a finite number of points.

Let $a_0 > 0$ be the injectivity radius of g^{TM}. Let $0 < \epsilon < \frac{a_0}{2}$.

For any $p \in Z_K$, let $B_p(\epsilon) = \{x \in M : d^{g^{TM}}(x, p) \leq \epsilon\}$ be the Riemannian ball centered at p. We may take ϵ small enough so that each $B_p(\epsilon)$ does not contain points in $Z_K \backslash \{p\}$.

Let $S(TB_p(\epsilon)) = S_+(TB_p(\epsilon)) \oplus S_-(TB_p(\epsilon))$ be the Hermitian bundle of spinors associated with $(TB_p(\epsilon), g^{TM}|_{B_p(\epsilon)})$. Let τ be the the involution on $S(TB_p(\epsilon))$ such that $\tau|_{S_\pm(TB_p(\epsilon))} = \pm \mathrm{Id}|_{S_\pm(TB_p(\epsilon))}$. Let $c(\cdot)$ denote the Clifford action on $S(TB_p(\epsilon))$.[2]

Let $v_K(p) : \Gamma(S_+(TB_p(\epsilon))) \to \Gamma(S_-(TB_p(\epsilon)))$ be defined by

$$v_K(p) = \tau c(\xi) + \sqrt{-1}c(\eta). \tag{3.3}$$

Then one can prove (see Lemma 3.2 below) that the restriction of $v_K(p)$ on the sphere $\partial B_p(\epsilon)$ is invertible. Thus it defines an integer $\deg(v_K(p)|_{\partial B_p(\epsilon)}) \in \mathbf{Z} = K^1(\partial B_p(\epsilon))$.

We can now state the main result of this section as follows.

Theorem 3.1 *Under the Basic Assumption 3.0,*

(i) *If* $n \geq 2$, *then the following identity holds,*

$$\chi(M) = -\sum_{p \in Z_K} \deg\left(v_K(p)|_{\partial B_p(\epsilon)}\right). \tag{3.4}$$

(ii) *If* $n = 1$, *set* $Z_{K,+} = \{x \in Z_K : \xi(x), \ \eta(x)$ form an oriented frame at $x\}$, *then*

$$\chi(M) = -\sum_{p \in Z_K \backslash Z_{K,+}} \deg\left(v_K(p)|_{\partial B_p(\epsilon)}\right). \tag{3.4$'$}$$

[2] For a thorough treatment of spin geometry involved here, see [4].

Proof For simplicity, we first assume that M is spin and denote by $S(TM) = S_+(TM) \oplus S_-(TM)$ the bundle of spinors associated with (TM, g^{TM}).

Let $v_K = \tau c(\xi) + \sqrt{-1}c(\eta) : S_+(TM) \to S_-(TM)$ be defined similarly as in (3.3), only that now it is defined on the whole manifold M.

Let $Z(v_K)$ denote the set of points at which v_K is not invertible.

Lemma 3.2 *One has, (i) If $n \geq 2$, then $Z(v_K) = Z_K$; (ii) If $n = 1$, then $Z(v_K) = Z_K \backslash Z_{K,+}$.*

Proof From (3.1) and (3.2), it is clear that $p \in Z_K$ if and only if $|\xi| = |\eta|$ and $\langle \xi, \eta \rangle = 0$.

Let $v_K^* : S_-(TM) \to S_+(TM)$ be the adjoint of v_K with respect to the natural Hermitian metrics on $S_\pm(TM)$. Set $V_K = v_K + v_K^* : S(TM) \to S(TM)$. Then v_K is not invertible if and only if V_K^2 is not strictly positive.

Clearly,

$$V_K = \tau c(\xi) + \sqrt{-1}c(\eta) : S(TM) \to S(TM). \tag{3.5}$$

From (3.5), one finds

$$V_K^2 = |\xi|^2 + |\eta|^2 + \sqrt{-1}\tau(c(\xi)c(\eta) - c(\eta)c(\xi)). \tag{3.6}$$

Now if at some $x \in M$, $|\xi| = |\eta|$ and $\langle \xi, \eta \rangle = 0$, then $V_K^2 = 2|\xi|^2 + 2\sqrt{-1}\tau c(\xi)c(\eta)$ which is clearly seen not invertible if $n \geq 2$ or if $n = 1$ but ξ and η do not form an oriented frame at x.[3]

Thus, one gets $Z_K \backslash Z_{K,+} \subset Z(v_K)$.

On the other hand, observe that if $|\xi| \neq |\eta|$, then $|\xi|^2 + |\eta|^2 > 2|\xi| \cdot |\eta|$, while it is clear that $2|\xi| \cdot |\eta| + \sqrt{-1}\tau(c(\xi)c(\eta) - c(\eta)c(\xi)) \geq 0$.

Thus if $|\xi(x)| \neq |\eta(x)|$, then x is not in $Z(v_K)$.

Now if at some $x \in M$, $|\xi| = |\eta|$ and $\langle \xi, \eta \rangle \neq 0$, one has

$$c(\xi)c(\eta) - c(\eta)c(\xi) = c(\xi)c\left(\eta - \frac{\langle \eta, \xi \rangle}{|\xi|^2}\xi\right) - c\left(\eta - \frac{\langle \eta, \xi \rangle}{|\xi|^2}\xi\right)c(\xi), \tag{3.7}$$

with

$$\left|\eta - \frac{\langle \eta, \xi \rangle}{|\xi|^2}\xi\right| < |\eta|. \tag{3.8}$$

From (3.6), (3.7), (3.8), one finds that if at some $x \in M$, $|\xi| = |\eta|$ and $\langle \xi, \eta \rangle \neq 0$, then $V_K^2 > 0$.

Thus, $Z(v_K) \subset Z_K$. Moreover, if $n = 1$, then one verifies directly that $Z(v_K) \subset Z_K \backslash Z_{K,+}$. The proof of Lemma 3.2 is completed. □

Back to the proof of Theorem 3.1. By Lemma 3.2, we know that the Basic Assumption 0.1 holds for $v_K : S_+(TM) \to S_-(TM)$. Thus one may apply Theorem 1.1 to it to get

$$\langle \mathrm{ch}\,(S_+(TM)) - \mathrm{ch}\,(S_-(TM))\,, [M] \rangle = (-1)^{n-1} \sum_{p \in Z(v_K)} \deg\left(v_K(p)|_{\partial B_p(\epsilon)}\right). \tag{3.9}$$

On the other hand, it is standard that (cf. [4])

$$\langle \mathrm{ch}\,(S_+(TM)) - \mathrm{ch}\,(S_-(TM))\,, [M] \rangle = (-1)^n \chi(M). \tag{3.10}$$

From (3.9) and (3.10), one gets (3.4).

[3] As one verifies in this case that either $\xi = \eta = 0$, or $c(\xi) - \sqrt{-1}\tau c(\eta) \neq 0$ while $(|\xi|^2 + \sqrt{-1}\tau c(\xi)c(\eta))(c(\xi) - \sqrt{-1}\tau c(\eta)) = 0$.

Thus we have proved Theorem 3.1 in the case where M is spin.

For the general case where M need not be spin, we may consider the Signature complex (cf. [4]) associated with (TM, g^{TM}) instead. Then the same argument above leads to formulas similar to (3.9) and (3.10), with the right hand sides both be multiplied by a factor 2^n, while in the left hand sides the Spin complex be replaced by the Signature complex. Thus one gets again (3.4). We leave the details to the interested reader.

The proof of Theorem 3.1 is completed. □

Remark 3.3 If $Z_K = \emptyset$, then one recovers (and at the same time gives a new proof of) the vanishing result of Jacobowitz [3] which has been stated in Proposition 1.2.

Remark 3.4 Theorem 3.1, in its most general form, should be regarded as a geometric result. As a simple amazing consequence (actually a consequence of Proposition 1.2), if $\chi(M) \neq 0$ and $K = \xi + \sqrt{-1}\eta \in \Gamma(T_{\mathbf{C}}M)$ is *nowhere zero* over M, then for any Riemannian metric g^{TM} on TM, there is at least one point $x \in M$, at which one has $|\xi|_{g^{TM}} = |\eta|_{g^{TM}}$ and $\langle \xi, \eta \rangle_{g^{TM}} = 0$. Moreover, if $n = 1$, then there exists at least *two* such points.[4]

Remark 3.5 One may also extend Theorem 3.1 to the case where TM is replaced by an arbitrary oriented Euclidean vector bundle. We leave the details to the interested reader.

Next, we show that Theorem 3.1 is indeed a generalization of the original Poincaré–Hopf index formula (cf. [1, Theorem 11.25]).

To do so, we take $\xi = 0$, then Z_K is the zero set of η, which we have assumed to consist of isolated points.

Without loss of generality we also assume that $|\eta| = 1$ on each $\partial B_p(\epsilon)$, $p \in Z_K$.

In view of the last equality in the proof of Lemma 2.2, one has

$$\deg\left(v_K(p)|_{\partial B_p(\epsilon)}\right) = \frac{1}{(2\pi\sqrt{-1})^n} \frac{(n-1)!}{(2n-1)!} \int_{\partial B_p(\epsilon)} \mathrm{tr}_{S_+(TM)}\left[\left(v^{-1}dv\right)^{2n-1}\right], \tag{3.11}$$

with

$$v = \sqrt{-1}c\left(\eta|_{\partial B_p(\epsilon)}\right). \tag{3.12}$$

Let $f_1, \ldots, f_{2n-1}$ be an orthonormal basis of $T(\partial B_p(\epsilon))$, let $f_1^*, \ldots, f_{2n-1}^*$ be the metric dual basis of $T^*(\partial B_p(\epsilon))$.

From (3.12), one deduces that (compare with [6, (27)])

$$\begin{aligned} &\mathrm{tr}_{S_+(TM)}\left[\left(v^{-1}dv\right)^{2n-1}\right] \\ &= -2^{n-1}(2n-1)!(\sqrt{-1})^n f_1^* \wedge \cdots \wedge f_{2n-1}^* \int^B \eta^* \wedge \left(\nabla_{f_1}^{TM}\eta\right)^* \wedge \cdots \wedge \left(\nabla_{f_{2n-1}}^{TM}\eta\right)^*, \end{aligned} \tag{3.13}$$

where ∇^{TM} is the Levi-Civita connection of g^{TM} and where $\int^B \eta^* \wedge (\nabla_{f_1}^{TM}\eta)^* \wedge \cdots \wedge (\nabla_{f_{2n-1}}^{TM}\eta)^*$ is the function on $\partial B_p(\epsilon)$ such that

[4] This is because one can switch ξ and η.

$$\eta^* \wedge \left(\nabla^{TM}_{f_1}\eta\right)^* \wedge \cdots \wedge \left(\nabla^{TM}_{f_{2n-1}}\eta\right)^* = \left(d\mathrm{vol}_{g^{TM}}\right) \int^B \eta^* \wedge \left(\nabla^{TM}_{f_1}\eta^*\right) \wedge \cdots \wedge \left(\nabla^{TM}_{f_{2n-1}}\eta\right)^* \tag{3.14}$$

on $\Lambda^{2n}(T^*M)|_{\partial B_p(\epsilon)}$.

Let $\eta_p : \partial B_p(\epsilon) \to S^{2n-1}(1)$ denote the canonical map induced by $\eta|_{\partial B_p(\epsilon)}$.

By (3.14), one finds

$$f_1^* \wedge \cdots \wedge f_{2n-1}^* \int^B \eta^* \wedge \left(\nabla^{TM}_{f_1}\eta\right)^* \wedge \cdots \wedge \left(\nabla^{TM}_{f_{2n-1}}\eta\right)^* = \eta_p^*\,\omega, \tag{3.15}$$

where ω is the volume form on $S^{2n-1}(1)$.

From (3.11), (3.13) and (3.15), one gets

$$\begin{aligned}\deg\left(v_K(p)|_{\partial B_p(\epsilon)}\right) &= -\frac{1}{(2\pi\sqrt{-1})^n}\frac{(n-1)!}{(2n-1)!}2^{n-1}(2n-1)!(\sqrt{-1})^n \int_{\partial B_p(\epsilon)} \eta_p^*\,\omega \\ &= \frac{-(n-1)!}{2\pi^n}\int_{\partial B_p(\epsilon)} \eta_p^*\,\omega = -\deg\left(\eta_p\right),\end{aligned} \tag{3.16}$$

where $\deg(\eta_p)$ denotes the Brouwer degree (cf. [1]) of the map $\eta_p : \partial B_p(\epsilon) \to S^{2n-1}(1)$.

From (3.4) and (3.16), one gets

$$\chi(M) = \sum_{p \in \text{zero set of } \eta} \deg\left(\eta_p\right),$$

which is exactly the original Poincaré–Hopf index formula (cf. [1, Theorem 11.25]).

Remark 3.6 Continuing Remark 3.4 and assume $n \geq 2$. Let $K = \xi + \sqrt{-1}\eta$ be such that the zero set of ξ is discrete and that $p \in M$ is a zero point of ξ such that $\deg(\xi_p) \neq \chi(M)$, while η vanishes on a closed ball of a sufficiently small positive radius around p and is nowhere zero outside this closed ball.[5] Then according to (3.16), $-\deg(v_K(p)) = \deg(\xi_p) \neq \chi(M)$. Combining this with Theorem 3.1, we see that for any Riemannian metric g^{TM}, there is $x \in M$ such that $|\xi|_{g^{TM}} = |\eta|_{g^{TM}} \neq 0$ and $\langle \xi, \eta \rangle_{g^{TM}} = 0$. This extends Remark 3.4 to the case where $K = \xi + \sqrt{-1}\eta$ might vanish on M.

Now we exhibit an example to illustrate the last line in Remark 3.4.

Example 3.7 Let $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ be the standard two sphere in the Euclidean space $\mathbf{R}^3$. Set $\xi = (-y, x, 0)$ and $\eta = (z, 0, -x)$. Clearly, as $x^2 + y^2 + z^2 = 1$, $\xi + \sqrt{-1}\eta$ is nowhere zero on S^2. Now $|\xi| = |\eta|$ together with $\langle \xi, \eta \rangle = 0$ imply that $x = \pm 1$, $y = z = 0$. Thus, Z_K consists of *two* points $p = (1, 0, 0)$, $q = (-1, 0, 0)$. One then verifies that at $q \in S^2$, $\xi = (0, -1, 0)$ and $\eta = (0, 0, 1)$ form an oriented frame of T_pS^2. Thus, by (3.4), one sees that the degree at p equals to -2, as the Euler number of S^2 is 2.

Finally, with the help of Example 3.7, we exhibit an application of Theorem 1.1 in the higher dimensional case.

[5] The existence of such a vector field is clear, as according to a famous theorem of Hopf, there always exists a vector field on M which vanishes only at p.

Example 3.8 We take a product $M = S^2 \times \cdots \times S^2$ with $m \geq 2$ copies of S^2. We use a subscript to denote the corresponding factor of S^2. So now let $\xi_i,\ \eta_i, 1 \leq i \leq m$, be the vector fields constructed in Example 3.7 on the ith factor S^2 (denoted by S_i^2). Let $v_{K,i}$ be the lifting to M of the corresponding map defined as in the proof of Theorem 3.1 on S_i^2. Then each $v_{K,i}$ maps $\Gamma(S_+(TM))$ to $\Gamma(S_-(TM))$. Set $v_K = \sum_{i=1}^m v_{K,i}$, then one verifies directly that v_K is singular only at the point $(p_1, \ldots, p_m) \in S^2 \times \cdots \times S^2$. By combining Theorem 1.1 with (3.9) and (3.10), one then gets that the degree of v_K at $(p_1, \ldots, p_m)$ equals to -2^m, as the Euler number of $S^2 \times \cdots \times S^2$ equals to 2^m. Conversely, one can compute the degree at $(p_1, \ldots, p_m)$ first, and then get the Euler number of $S^2 \times \cdots \times S^2$ by using Theorem 1.1.

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