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AN ANOMALY FORMULA FOR L^2-ANALYTIC TORSIONS ON MANIFOLDS WITH BOUNDARY

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Dedicated to Krzysztof P. Wojciechowski on his 50th birthday

We extend the definition, in the extended cohomology framework, of the L^2-analytic torsion for covering spaces due to Braverman-Carey-Farber-Mathai to the case of manifolds with boundary, and prove an associated anomaly formula. Our main result may be viewed as a common generalization of the anomaly formula for Ray-Singer analytic torsion for manifolds with boundary proved by Brüning-Ma, as well as the anomaly formula for L^2-analytic torsions for closed manifolds proved by Zhang. It generalizes also an earlier result of Lück-Schick, without the assumptions on the unitary representations as well as the technical "determinant class condition".

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1. Introduction

Let F be a unitary flat vector bundle on a closed Riemannian manifold X. Ray and Singer [27] defined an analytic torsion associated to (X, F) and proved that it does not depend on the Riemannian metric on X. Moreover, they conjectured that this analytic torsion coincides with the classical Reidemeister torsion defined using a triangulation on X (cf. Milnor [21]). This conjecture was later proved in the celebrated papers of Cheeger [10] and Müller [22]. Müller generalized this result in [23] to the case where F

is a unimodular flat vector bundle on X. Inspired by the considerations of Quillen [25], Bismut and Zhang [2] reformulated the above Cheeger-Müller theorem as an equality between the Reidemeister and Ray-Singer metrics defined on the determinant of cohomology, and proved an extension of it to the case of general flat vector bundles over X. The method used in [2] is different from those of Cheeger and Müller in that it makes use of a deformation by Morse functions introduced by Witten [30] on the de Rham complex. In particular, as an intermediate step, an important anomaly formula for Ray-Singer metrics has been established in [2], Theorem 0.1.

Recall that Ray and Singer [27] also defined the analytic torsion, in the unitary flat vector bundle case, for manifolds with boundary. Moreover, Cheeger [10] raised the question of computing the corresponding metric anomaly. This question was studied by Dai and Fang [11] for the case of unitary flat vector bundle, while a complete answer, valid for the general case of arbitrary flat vector bundles, is recently obtained by Brüning and Ma [4].

The purpose of this paper is to generalize the main results in [4] to the case of L^2-analytic torsions on infinite Galois covering spaces of manifolds with boundary. We recall that the L^2-torsions were first introduced, for closed manifolds, by Carey, Mathai and Lott in [9], [15] and [20], under the assumptions that the L^2-Betti numbers vanish and that certain technical "determinant class condition" (the more precise definition of "determinant class condition" indeed appears later in [7]) is satisfied. The later condition is satisfied if the Novikov-Shubin [24] invariants are positive. In [6] and [19], extensions to manifolds with boundary, in the case of unitary flat bundle case, have been studied. In [6], only the case of product metric near boundary has been considered, while in [19], Lück and Schick also considered the case of non-product metric near boundary.

Carey, Farber and Mathai [8] showed that the condition on the vanishing of the L^2-Betti numbers can be relaxed. This is achieved by constructing the determinant line of the reduced L^2-cohomology and defining the L^2-torsions as elements of the determinant line.

Recently, Braverman, Carey, Farber and Mathai [3] showed that if one considers the *extended* L^2-cohomology in the sense of Farber (cf. [13]) instead of the usually used *reduced* L^2-cohomology, then one can naturally define the L^2-analytic torsion as an L^2-element on the associated determinant lines, without requiring the "determinant class condition".

In this paper, we first generalize the construction in [3] to the case of manifolds with boundary, to define L^2-analytic torsions, in the case of manifolds with boundary, for arbitrary flat vector bundles and arbitrary Riemannian metric on the base manifold, without using the "determinant class condition". We then prove an anomaly formula of these L^2-analytic torsions. The main result can be thought of as a common generalization of the anomaly formula for Ray-Singer analytic torsion for manifolds with boundary proved by Brüning-Ma [4, 5], as well as the anomaly formula for L^2-analytic torsions for closed manifolds proved by Zhang [32]. It generalizes also [19], Theorem 7.6, without the assumptions on the flatness of the metrics on F, and on the technical "determinant class condition". In particular, it provides a positive answer to a question mentioned in [18], Page 190.

This paper is organized as follows. In Section 2, we recall from [3] the definition of the determinant line of extended cohomology of a finite length Hilbert cochain $\mathcal{A}$-complex with $\mathcal{A}$ a finite von Neumann algebra, as well as the definition of the L^2-torsion element lying in this determinant line. In Section 3, we construct the L^2-analytic torsion element, in the case of manifolds with boundary, by extending the construction in [3], and establish an anomaly formula for it.

2. L^2-torsion on the determinant of extended cohomology

In this section, we recall from [3] the definition of the L^2-torsion element which lies in the determinant of the extended cohomology associated to a finite length Hilbert cochain complex.

This section is organized as follows. In Section 2.1, we recall the definition of the extended cohomology of a finite length Hilbert cochain complex over a finite von Neumann algebra carrying a finite, normal and faithful trace. In Section 2.2, we recall the definition of the determinant of a finitely generated Hilbert module over a finite von Neumann algebra. In Section 2.3, we recall the definition of the L^2-torsion element of a finite length Hilbert cochain complex.

2.1. *Extended cohomology of a finite length Hilbert cochain complex*

Let $\mathcal{A}$ be a finite von Neumann algebra carrying a fixed finite, normal and faithful trace

$$\tau : \mathcal{A} \to \mathbf{C},$$

cf. [12], §I.6. Let * denote the canonical involution on $\mathcal{A}$ defined by taking adjoint. Let $l^2(\mathcal{A})$ denote the Hilbert space completion of $\mathcal{A}$ with respect to the inner product given by the trace

$$\langle a, b \rangle = \tau(b^* a). \tag{2.1}$$

A finitely generated Hilbert module over $\mathcal{A}$ is a Hilbert space M admitting a continuous left $\mathcal{A}$-structure (with respect to the norm topology on $\mathcal{A}$) such that there exists an isometric $\mathcal{A}$-linear embedding of M into $l^2(\mathcal{A}) \otimes H$, for some finite dimensional Hilbert space H.

Let (C^*, ∂) be a finite length Hilbert cochain complex over $\mathcal{A}$,

$$(C^*, \partial) : 0 \to C^0 \stackrel{\partial_0}{\to} C^1 \stackrel{\partial_1}{\to} \cdots \stackrel{\partial_{n-1}}{\to} C^n \to 0, \tag{2.2}$$

where each C^i, $0 \leq i \leq n$, is a finitely generated Hilbert module over $\mathcal{A}$ and the coboundary maps are bounded $\mathcal{A}$-linear operators. Since the image spaces of these coboundary maps need not be closed, the tautological cohomology of (C^*, ∂) need not be a Hilbert space. This is why in general one studies the *reduced* cohomology of (C^*, ∂), which is defined by

$$H^*(C^*, \partial) = \bigoplus_{i=0}^{n} H^i(C^*, \partial), \tag{2.3}$$

with

$$H^i(C^*, \partial) = \ker(\partial_i)/\overline{\mathrm{im}(\partial_{i-1})}, \quad 0 \leq i \leq n, \tag{2.4}$$

where one takes obviously that $\partial_{-1} = 0$ and $\partial_n = 0$.

On the other hand, there are still ways to extract more information from (C^*, ∂), rather than just from $H^*(C^*, \partial)$. One such is to consider the *extended* cohomology in the sense of Farber (cf. [13] and [3]), which is defined by

$$\mathcal{H}^*(C^*, \partial) = \bigoplus_{i=0}^{n} \mathcal{H}^i(C^*, \partial), \tag{2.5}$$

with

$$\mathcal{H}^i(C^*, \partial) = (\partial_{i-1} : C^{i-1} \to \ker(\partial_i)), \ 0 \leq i \leq n, \tag{2.6}$$

where $(\partial_{i-1} : C^{i-1} \to \ker(\partial_i))$, $0 \leq i \leq n$, lie in an abelian extended category. It consists of two parts: the projective part which is exactly the reduced cohomology defined in (2.3), as well as a torsion part

$$\mathcal{T}(\mathcal{H}^*(C^*, \partial)) = \oplus_{i=0}^{n} \mathcal{T}(\mathcal{H}^i(C^*, \partial))$$

defined as an element in the above abelian extended category, with

$$\mathcal{T}(\mathcal{H}^i(C^*, \partial)) = (\partial_{i-1} : C^{i-1} \to \overline{\mathrm{im}(\partial_{i-1})}), \ 0 \leq i \leq n. \tag{2.7}$$

More precisely, one has

$$\mathcal{H}^*(C^*, \partial) = H^*(C^*, \partial) \oplus \mathcal{T}(\mathcal{H}^*(C^*, \partial)), \tag{2.8}$$

with

$$\mathcal{H}^i(C^*, \partial) = H^i(C^*, \partial) \oplus \mathcal{T}(\mathcal{H}^i(C^*, \partial)), \quad 0 \le i \le n. \tag{2.9}$$

We refer to [13] and [3] for more details about the definition and basic properties of the above mentioned abelian extended category as well as the extended cohomology.

2.2. *The determinant of a finitely generated Hilbert module*

Let M be a finitely generated Hilbert module over $\mathcal{A}$. Let $GL(M)$ denote the set of all bounded $\mathcal{A}$-linear automorphisms of M. Let $\mathcal{C}_M$ denote the set of all inner products on M such that if $\langle\ ,\ \rangle \in \mathcal{C}_M$, then there exists $A \in GL(M)$ such that

$$\langle u, v\rangle = \langle Au, v\rangle_M, \quad \text{for any } u,\ v \in M, \tag{2.10}$$

with $\langle\ ,\ \rangle_M$ being the original inner product on M.

Following [8] and [3], we define the determinant line $\det M$ of M to be the real one dimensional vector space generated by symbols $\langle\ ,\ \rangle$, one for each element in $\mathcal{C}_M$ such that if $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ are two elements of $\mathcal{C}_M$ with

$$\langle u, v\rangle_2 = \langle Au, v\rangle_1, \quad \text{for any } u,\ v \in M, \tag{2.11}$$

for some $A \in GL(M)$, then as elements in $\det M$, one has

$$\langle\ ,\ \rangle_2 = \mathrm{Det}_\tau(A)^{-1/2} \cdot \langle\ ,\ \rangle_1, \tag{2.12}$$

where $\mathrm{Det}_\tau(A)$ is the Fuglede-Kadison determinant [14] of A.

For the sake of completeness, we recall the definition of $\mathrm{Det}_\tau(A)$ for any $A \in GL(M)$ and its basic properties from [8] and [3].

Let A_t, $0 \le t \le 1$, be a continuous piecewise smooth path $A_t \in GL(M)$ such that $A_0 = I$ and $A_1 = A$. The existence of such a path is clear as $GL(M)$ is known to be pathwise connected. Then define as in [8], (13) and [3], (2.7) that

$$\log \mathrm{Det}_\tau(A) = \int_0^1 \mathrm{Re}\left(\mathrm{Tr}_\tau[A_t^{-1}A_t']\right) dt, \tag{2.13}$$

where A_t' is the derivative of A_t with respect to t, while Tr_τ is the canonically induced trace on the commutant of M (cf. [8], Proposition 1.8).

It has been proved in [8] that the right hand side of (2.13) does not depend on the choice of the path A_t, $0 \le t \le 1$. Moreover, we recall the following basic properties taken from [8], Theorem 1.10 and [3], Theorem 2.11.

Proposition 2.1. *The function,*

$$\mathrm{Det}_\tau : GL(M) \to \mathbf{R}^{>0}, \tag{2.14}$$

called the Fuglede-Kadison determinant of A, satisfies,
(a) Det_τ *is a group homomorphism, that is,*

$$\mathrm{Det}_\tau(AB) = \mathrm{Det}_\tau(A) \cdot \mathrm{Det}_\tau(B), \quad \text{for} \quad A,\ B \in GL(M); \tag{2.15}$$

(b) *If I is the identity element in $GL(M)$, then*

$$\mathrm{Det}_\tau(\lambda I) = |\lambda|^{\tau(I)} \quad \text{for} \quad \lambda \in \mathbf{C},\ \lambda \neq 0; \tag{2.16}$$

(c) *One has*

$$\mathrm{Det}_{\lambda\tau}(A) = \mathrm{Det}_\tau(A)^\lambda \quad \text{for} \quad \lambda \in \mathbf{R}^{>0}; \tag{2.17}$$

(d) Det_τ *is continuous as a map $GL(M) \to \mathbf{R}^{>0}$, where $GL(M)$ is supplied with the norm topology;*
(e) *If A_t, $t \in [0,1]$, is a continuous piecewise smooth path in $GL(M)$, then*

$$\log\left[\frac{\mathrm{Det}_\tau(A_1)}{\mathrm{Det}_\tau(A_0)}\right] = \int_0^1 \mathrm{Re}\left(\mathrm{Tr}_\tau[A_t^{-1}A_t']\right) dt; \tag{2.18}$$

(f) *Let M, N be two finitely generated Hilbert modules over $\mathcal{A}$. Let $A \in GL(M)$, $B \in GL(N)$ and let*

$$\gamma : N \to M$$

be a bounded $\mathcal{A}$-linear homomorphism. We extend A, B, γ to obvious endomorphisms on $M \oplus N$ by taking $A|_N = 0$, $B|_M = 0$ and $\gamma|_M = 0$. Then $A + B + \gamma \in GL(M \oplus N)$ and

$$\mathrm{Det}_\tau(A + B + \gamma) = \mathrm{Det}_\tau(A) \cdot \mathrm{Det}_\tau(B). \tag{2.19}$$

Now we come back to the determinant line $\det M$. Clearly, $\det M$ has a canonical orientation as the transition coefficient $\mathrm{Det}_\tau(A)^{-1/2}$ is always positive.

Following [8], (2.3), for any bounded $\mathcal{A}$-linear isomorphism $f : M \to N$ between two finitely generated Hilbert modules over $\mathcal{A}$, there induces canonically an isomorphism of determinant lines $f_* : \det M \to \det N$, which preserves the orientations. Moreover, one has the following property which is recalled from [8], Proposition 2.5.

Proposition 2.2. *If $f \in GL(M)$, then the induced isomorphism f_* : $\det M \to \det M$ coincides with the multiplication by $\mathrm{Det}_\tau(f) \in \mathbf{R}^{>0}$.*

Remark 2.1. Following [8] and [3], one thinks of elements of $\det M$ as "densities" on M. In the $\mathcal{A} = \mathbf{C}$ case, this is dual to the considerations in [2] where one uses metrics on determinant lines instead of "volume forms".

2.3. *Extended cohomology and the torsion element of a finite length cochain complex of Hilbert modules*

Let (C^*, ∂) be a finite length Hilbert cochain complex over $\mathcal{A}$

$$(C^*, \partial) : 0 \to C^0 \stackrel{\partial_0}{\to} C^1 \stackrel{\partial_1}{\to} \cdots \stackrel{\partial_{n-1}}{\to} C^n \to 0 \tag{2.20}$$

as in (2.2). Let

$$\mathcal{H}^*(C^*, \partial) = \sum_{i=0}^{n} \mathcal{H}^i(C^*, \partial)$$

denote the corresponding extended cohomology defined in (2.5), which admits the splitting to projective and torsion parts as in (2.7)-(2.9).

Following [3], we define for each $0 \le i \le n$ that

$$\det \mathcal{H}^i(C^*, \partial) := \det H^i(C^*, \partial) \otimes \det \mathcal{T}(\mathcal{H}^i(C^*, \partial)) \tag{2.21}$$

with

$$\det \mathcal{T}(\mathcal{H}^i(C^*, \partial)) := \det \overline{\mathrm{im}(\partial_{i-1})} \otimes (\det C^{i-1})^* \otimes \det \ker(\partial_{i-1}). \tag{2.22}$$

Definition 2.1. (i) We define the determinant line of (C^*, ∂) to be

$$\det(C^*, \partial) = \bigotimes_{i=0}^{n} \left(\det C^i\right)^{(-1)^i}. \tag{2.23}$$

(ii) We define the determinant line of $\mathcal{H}^*(C^*, \partial)$ to be

$$\det \mathcal{H}^*(C^*, \partial) = \bigotimes_{i=0}^{n} \left(\det \mathcal{H}^i(C^*, \partial)\right)^{(-1)^i}. \tag{2.24}$$

The following result is recalled from [3], Proposition 7.2.

Proposition 2.3. *The cochain complex* (2.20) *defines a canonical isomorphism*

$$\nu_{(C^*, \partial)} : \det(C^*, \partial) \to \det \mathcal{H}^*(C^*, \partial). \tag{2.25}$$

For each $0 \leq i \leq n$, the (fixed) inner product on C^i determines an element $\sigma_i \in \det C^i$. They together determine an element

$$\sigma = \prod_{i=0}^{n} \sigma_i^{(-1)^i} \in \det(C^*, \partial). \tag{2.26}$$

Definition 2.2. (cf. [3], Definition 7.5) The positive element

$$\rho_{(C^*,\partial)} = \nu_{(C^*,\partial)}(\sigma) \in \det \mathcal{H}^*(C^*, \partial) \tag{2.27}$$

is called the torsion element of the cochain complex (C^*, ∂).

For any other $\mathbf{Z}$-graded inner product $\langle\ ,\ \rangle' \in \mathcal{C}_C$, that is, there exists $A_i \in GL(C^i)$ for any $0 \leq i \leq n$ such that

$$\langle u, v\rangle_i' = \langle A_i u, v\rangle \quad \text{for any} \quad u,\ v \in C^i, \tag{2.28}$$

let $\rho'_{(C^*,\partial)}$ denote the corresponding torsion element in $\det \mathcal{H}^*(C^*, \partial)$. Then one has the following anomaly formula for the torsion elements in $\det \mathcal{H}^*(C^*, \partial)$.

Proposition 2.4. *The following identity holds in* $\det \mathcal{H}^*(C^*, \partial)$*,*

$$\rho'_{(C^*,\partial)} = \rho_{(C^*,\partial)} \prod_{i=0}^{n} \mathrm{Det}_\tau(A_i)^{\frac{(-1)^{i+1}}{2}}. \tag{2.29}$$

Proof. Let σ_i' be the corresponding element in $\det C^i$. From (2.28), one has by definition (cf. (2.12))

$$\sigma_i' = \mathrm{Det}_\tau(A_i)^{-1/2}\sigma_i. \tag{2.30}$$

From Proposition 2.3 and from (2.26), (2.27) and (2.30), one gets (2.29).□

For any $0 \leq i \leq n$, let

$$\partial_i^* : C^{i+1} \to C^i$$

denote the adjoint of ∂^i with respect to the inner products on C^i and C^{i+1}.

Let

$$\partial = \sum_{i=1}^{n} \partial_i : C^* \to C^*, \qquad \partial^* = \sum_{i=1}^{n} \partial_i^* : C^* \to C^*$$

denote the induced homomorphisms on C^*. Then

$$\Box = (\partial + \partial^*)^2 \tag{2.31}$$

preserves each C^i. Let $\Box_i$ denote the restriction of $\Box$ on C^i.

Now consider the special case where the cochain complex (C^*, ∂) is acyclic, i.e., for any $0 \le i \le n$, $\mathrm{im}(\partial_i) = \ker(\partial_{i+1})$ (In particular, this implies that $\mathrm{im}(\partial_i)$ is closed in C^{i+1}). Then the torsion element $\rho_{(C^*,\partial)} = \nu_{(C^*,\partial)}(\sigma) \in \det \mathcal{H}^*(C^*, \partial) \simeq \mathbf{R}$ can be thought of as a positive real number.

The following result has been proved in [3], Proposition 7.8.

Proposition 2.5. *If the cochain complex (C^*, ∂) is acyclic, then the following identity holds,*

$$\log \rho_{(C^*,\partial)} = \frac{1}{2} \sum_{i=0}^{n} (-1)^{i+1} i \log \mathrm{Det}_\tau(\Box_i). \tag{2.32}$$

We refer to [3] for more complete discussions about the torsion elements in determinant lines.

3. Infinite covering spaces and the L^2-Ray-Singer torsion on the determinant of extended de Rham cohomology

In this section, we define the L^2-analytic torsion element for the infinite covering space of manifolds with boundary, and prove an anomaly formula for it.

This section is organized as follows. In Section 3.1, we define, in the case of manifolds with boundary, the extended de Rham cohomology associated to a lifted flat vector bundle on an infinite covering space. In Section 3.2, we define the L^2-analytic torsion element, in the manifolds with boundary case, as an element in the determinant of the extended de Rham cohomology. In Section 3.3, we state an anomaly formula, in the case of manifolds with boundary, about the L^2-analytic torsion element. In Section 3.4, we study the variational formula for the heat kernel. The anomaly formula is then proved in Section 3.5.

3.1. *Infinite covering spaces and the extended de Rham cohomology*

Let $\Gamma \to \widetilde{M} \xrightarrow{\pi} M$ be a Galois covering of a compact manifold M with boundary ∂M, with $\dim M = n$. Then $\widetilde{M}$ is a manifold with boundary $\partial\widetilde{M}$, which is a Γ-covering of ∂M. We make the assumption that Γ is an infinite group, as the case of finite group has been dealt with for example in [16] and [17].

Let (F, ∇^F) be a complex flat vector bundle over M carrying the flat connection ∇^F. Let g^F be a Hermitian metric on F. Let $(\widetilde{F}, \nabla^{\widetilde{F}})$ denote the naturally lifted flat vector bundle over $\widetilde{M}$ obtained as the pullback of (F, ∇^F) through the covering map π. Let $g^{\widetilde{F}}$ be the naturally lifted Hermitian metric on $\widetilde{F}$.

Let $\mathcal{N}(\Gamma)$ be the von Neumann algebra associated to Γ generated by the left regular representations on $l^2(\Gamma) \equiv l^2(\mathcal{N}(\Gamma))$. The canonical finite faithful trace on $\mathcal{N}(\Gamma)$ is given by the following formulas,

$$\begin{aligned} \tau_{\mathcal{N}(\Gamma)}(L_\alpha) =& 0, \text{ if } \alpha \neq 1, \\ & 1, \text{ if } \alpha = 1, \end{aligned} \tag{3.1}$$

where L_α denote the left action of $\alpha \in \Gamma$ on $l^2(\Gamma)$. It induces canonically a trace on the commutant of any finitely generated Hilbert $\mathcal{N}(\Gamma)$-module (cf. [8], Proposition 1.8), which will be denoted by $\mathrm{Tr}_{\mathcal{N}}$.

For any $0 \leq i \leq n$, denote

$$\Omega^i(\widetilde{M}, \widetilde{F}) = \Gamma(\Lambda^i(T^*\widetilde{M}) \otimes \widetilde{F}), \qquad \Omega^*(\widetilde{M}, \widetilde{F}) = \bigoplus_{i=0}^{n} \Omega^i(\widetilde{M}, \widetilde{F}). \tag{3.2}$$

Let $d^{\widetilde{F}}$ denote the natural exterior differential on $\Omega^*(\widetilde{M}, \widetilde{F})$ induced from $\nabla^{\widetilde{F}}$ which maps each $\Omega^i(\widetilde{M}, \widetilde{F})$, $0 \leq i \leq n$, into $\Omega^{i+1}(\widetilde{M}, \widetilde{F})$.

Let g^{TM} be a Riemannian metric on TM. Let $g^{T\partial M}$ be its restricted metric on $T\partial M$. Let $g^{T\widetilde{M}}$ be the lifted Riemannian metric on $T\widetilde{M}$ and denote by $\langle \cdot, \cdot \rangle_{\Lambda(T^*\widetilde{M}) \otimes \widetilde{F}}$ the induced Hermitian metric on $\Lambda(T^*\widetilde{M}) \otimes \widetilde{F}$. Let $o(T\widetilde{M})$ be the orientation bundle of $T\widetilde{M}$, and let $dv_{\widetilde{M}}$ be the Riemannian volume element on $(T\widetilde{M}, g^{T\widetilde{M}})$, then we can view $dv_{\widetilde{M}}$ as a section of $\Lambda^n(T^*\widetilde{M}) \otimes o(T\widetilde{M})$. The metrics $g^{T\widetilde{M}}$, $g^{\widetilde{F}}$ determine a canonical inner product on each $\Omega^i(\widetilde{M}, \widetilde{F})$, $0 \leq i \leq n$ as follow,

$$\langle \sigma, \sigma' \rangle := \int_X \langle \sigma, \sigma' \rangle_{\Lambda(T^*\widetilde{M}) \otimes \widetilde{F}} dv_{\widetilde{M}} \quad \text{for } \sigma, \sigma' \in \Omega(\widetilde{M}, \widetilde{F}). \tag{3.3}$$

Let $L^2(\Omega^i(\widetilde{M}, \widetilde{F}))$, $0 \leq i \leq n$, denote the Hilbert spaces obtained from the corresponding L^2-completion.

Let $g^{T\partial\widetilde{M}}$ be the metric on $T\partial\widetilde{M}$ lifted from $g^{T\partial M}$. We identify the normal bundle $N_{\partial\widetilde{M}}$ to $\partial\widetilde{M}$ in $\widetilde{M}$ with the orthogonal complement of $T\partial\widetilde{M}$ in $T\widetilde{M}|_{\partial\widetilde{M}}$.

Denote by $e_n = \widetilde{e}_{\mathfrak{n}}$ the inward pointing unit normal vector field along $\partial\widetilde{M}$. We also put, with $i(\cdot)$ the notation of interior multiplication,

$$\Omega^i_a(\widetilde{M}, \widetilde{F}) = \{\sigma \in \Omega^i(\widetilde{M}, \widetilde{F}); \quad i(\widetilde{e}_{\mathfrak{n}})\sigma = i(\widetilde{e}_{\mathfrak{n}})(d^{\widetilde{F}}\sigma) = 0 \text{ on } \partial\widetilde{M}\}. \tag{3.4}$$

Let $d_a^{\widetilde{F}}$ be the closure of $d^{\widetilde{F}}$ with respect to the (absolute) boundary condition (3.4). Then

$$d_a^{\widetilde{F}} : L^2(\Omega^*(\widetilde{M}, \widetilde{F})) \to L^2(\Omega^*(\widetilde{M}, \widetilde{F}))$$

is an unbounded operator. Let

$$d_a^{\widetilde{F}*} : L^2(\Omega^*(\widetilde{M}, \widetilde{F})) \to L^2(\Omega^*(\widetilde{M}, \widetilde{F}))$$

be the adjoint of it. Set

$$\widetilde{D}_a = d_a^{\widetilde{F}} + d_a^{\widetilde{F}*}. \tag{3.5}$$

For any $\mathcal{I} \subseteq \mathbf{R}$ and $0 \le i \le n$, denote by

$$L^2_{a,\mathcal{I}}(\Omega^i(\widetilde{M}, \widetilde{F})) \subseteq L^2(\Omega^i(\widetilde{M}, \widetilde{F})) \tag{3.6}$$

the image of the spectral projection $P_{\mathcal{I},i} : L^2(\Omega^i(\widetilde{M}, \widetilde{F})) \to L^2(\Omega^i(\widetilde{M}, \widetilde{F}))$ of $\widetilde{D}_a^2|_{L^2(\Omega^i(\widetilde{M}, \widetilde{F}))}$ corresponding to $\mathcal{I}$.

The following result generalizes a theorem of Shubin [28], Theorem 5.1 which has been recalled in [32], Theorem 3.1.

Theorem 3.1. *Fix $\varepsilon > 0$. Then for any $0 \le i \le n$,*

(i) $L^2_{a,[0,\varepsilon]}(\Omega^i(\widetilde{M}, \widetilde{F})) \subset \Omega^i_a(\widetilde{M}, \widetilde{F})$, *i.e.,* $L^2_{a,[0,\varepsilon]}(\Omega^i(\widetilde{M}, \widetilde{F}))$ *consists of smooth forms verifying the boundary condition* (3.4)*;*

(ii) *When carrying the induced metric from that of* $L^2(\Omega^i(\widetilde{M}, \widetilde{F}))$, $L^2_{a,[0,\varepsilon]}(\Omega^i(\widetilde{M}, \widetilde{F}))$ *is a finitely generated Hilbert module over* $\mathcal{N}(\Gamma)$.

Proof. (i) As in [28], we make use of elliptic estimates. Fix any $\lambda > 0$, from the standard elliptic estimate, one knows that

$$(\widetilde{D}_a^2 + \lambda)^{-1} : L^2_{a,[0,\varepsilon]}(\Omega^i(\widetilde{M}, \widetilde{F})) \to L^2_{a,[0,\varepsilon]}(\Omega^i(\widetilde{M}, \widetilde{F})) \tag{3.7}$$

is a well-defined, onto map which increases the degree of differentiability by two. By applying this to powers of $(\widetilde{D}_a^2 + \lambda)^{-1}$, we see then any element in $L^2_{a,[0,\varepsilon]}(\Omega^i(\widetilde{M}, \widetilde{F}))$ is smooth and verifies the boundary condition (3.4).

(ii) By simple smooth deformations and the homotopy invariance of the finite rank property (cf. [28]), we need only to deal with case where g^{TM} and g^F are of product structure near ∂M. Now in the case where all structures are a product near the boundary, one can proceed as in [17] to reduce the problem to the double of $\widetilde{M}$, on which one can apply the result of Shubin [28], Theorem 5.1. $\square$

Now consider the finite length cochain complex of $\mathcal{N}(\Gamma)$-Hilbert modules

$$(L^2_{a,[0,\varepsilon]}(\Omega^*(\widetilde{M},\widetilde{F})),d_a^{\widetilde{F}}):0\to L^2_{a,[0,\varepsilon]}(\Omega^0(\widetilde{M},\widetilde{F}))\xrightarrow{d_a^{\widetilde{F}}}L^2_{a,[0,\varepsilon]}(\Omega^1(\widetilde{M},\widetilde{F}))$$
$$\to\cdots\xrightarrow{d_a^{\widetilde{F}}}L^2_{a,[0,\varepsilon]}(\Omega^n(\widetilde{M},\widetilde{F}))\to 0. \quad (3.8)$$

It is easy to verify that the extended cohomology of $(L^2_{a,[0,\varepsilon]}(\Omega^*(\widetilde{M},\widetilde{F})),d_a^{\widetilde{F}})$ is independent of $\varepsilon>0$. For if $\varepsilon'>\varepsilon>0$, the sub-complex $(L^2_{a,(\varepsilon,\varepsilon']}(\Omega^*(\widetilde{M},\widetilde{F})),d_a^{\widetilde{F}})$ of $(L^2_{a,[0,\varepsilon']}(\Omega^*(\widetilde{M},\widetilde{F})),d_a^{\widetilde{F}})$ is acyclic. Moreover, it is easy to verify that this extended cohomology, up to bounded $\mathcal{N}(\Gamma)$-linear isomorphisms, does not depend on the choice of the metrics g^{TM} and g^F on TM and F respectively. We denote it by $\mathcal{H}^{(2)}_{a,\mathrm{dR}}(\Omega^*(\widetilde{M},\widetilde{F}),d_a^{\widetilde{F}})$.

Definition 3.1. The extended cohomology $\mathcal{H}^{(2)}_{a,\mathrm{dR}}(\Omega^*(\widetilde{M},\widetilde{F}),d_a^{\widetilde{F}})$ defined above is called the L^2-extended (absolute) de Rham cohomology associated to $\widetilde{M}$ and F.

3.2. *L^2-Ray-Singer torsion on the determinant of the extended de Rham cohomology*

We continue the discussion of the above subsection.

In view of Definition 2.2, for any $\varepsilon>0$, the finite length cochain complex of $\mathcal{N}(\Gamma)$-Hilbert modules $(L^2_{a,[0,\varepsilon]}(\Omega^*(\widetilde{M},\widetilde{F})),d_a^{\widetilde{F}})$ in (3.8) determines a torsion element in $\det\mathcal{H}^{(2)}_{a,\mathrm{dR}}(\Omega^*(\widetilde{M},\widetilde{F}),d_a^{\widetilde{F}})$. We denote this torsion element by $T_{a,[0,\varepsilon]}(\widetilde{M},F,g^{TM},g^F)$.

By proceeding as in [3], Section 12.2, for any $s\in\mathbf{C}$ with $\mathrm{Re}(s)>\frac{n}{2}$ and for $0\le i\le n$, set

$$\zeta^i_{a,(\varepsilon,+\infty)}(s)=\frac{1}{\Gamma(s)}\int_0^{+\infty}t^{s-1}\mathrm{Tr}_{\mathcal{N}}\left[\exp\left(-t\widetilde{D}_a^2|_{L^2_{a,(\varepsilon,+\infty)}(\Omega^i(\widetilde{M},\widetilde{F}))}\right)\right]dt. \quad (3.9)$$

Then $\zeta^i_{a,(\varepsilon,+\infty)}(s)$ is analytic in s for $\mathrm{Re}(s)>\frac{n}{2}$. Moreover, by using [19], Lemma 1.3, one finds that $\zeta^i_{(\varepsilon,+\infty)}(s)$ can be extended to a meromorphic function on $\mathbf{C}$ which is holomorphic at $s=0$.

Let

$$T_{a,(\varepsilon,+\infty)}(\widetilde{M},F,g^{TM},g^F)\in\mathbf{R}^+$$

be defined by

$$\log T_{a,(\varepsilon,+\infty)}(\widetilde{M},F,g^{TM},g^F)=\frac{1}{2}\sum_{i=0}^n(-1)^i i\left.\frac{\partial\zeta^i_{a,(\varepsilon,+\infty)}(s)}{\partial s}\right|_{s=0}. \quad (3.10)$$

By proceeding as in [3], Lemma 12.4, one knows that the product

$$T_{a,[0,\varepsilon]}(\widetilde{M}, F, g^{TM}, g^F) \cdot T_{a,(\varepsilon,+\infty)}(\widetilde{M}, F, g^{TM}, g^F)$$

in $\det \mathcal{H}^{(2)}_{a,\mathrm{dR}}(\Omega^*(\widetilde{M}, \widetilde{F}), d^{\widetilde{F}}_a)$ does not depend on $\varepsilon > 0$.

Definition 3.2. The L^2-Ray-Singer (or L^2-analytic) torsion element associated to $(\widetilde{M}, F, g^{TM}, g^F)$ is the positive element in the determinant of the extended de Rham cohomology $\mathcal{H}^{(2)}_{a,\mathrm{dR}}(\Omega^*(\widetilde{M}, \widetilde{F}), d^{\widetilde{F}}_a)$ defined by

$$\begin{aligned} &T^{(2)}_{a,RS}(\widetilde{M}, F, g^{TM}, g^F) \\ &\qquad = T_{a,[0,\varepsilon]}(\widetilde{M}, F, g^{TM}, g^F) \cdot T_{a,(\varepsilon,+\infty)}(\widetilde{M}, F, g^{TM}, g^F). \end{aligned} \tag{3.11}$$

3.3. *An anomaly formula for the L^2-Ray-Singer torsion elements*

We continue the discussion of the above subsection.

For $\Gamma = \{1\}$, the above construction gives us the usual torsion element $T_{a,RS}(\widetilde{M}, F, g^{TM}, g^F)$ which is dual to the Ray-Singer metric discussed in [2], [4], Def. 1.2 and [5], Def. 4.3.

For convenience of notation we use $\mathrm{l.i.m.}_{t\to 0} F_t$ to denote the constant term in an asymptotic expansion F_t with respect to the parameter t.

We can now state the main result of this paper.

Let g^{TM}_u (resp. g^F_u), $0 \le u \le 1$, be a smooth path of metrics on TM (resp. F). Let $*_u$ be the usual Hodge star operator associated to g^{TM} for the $F = \mathbf{C}$ case (cf. [31], Chapter 4).

Theorem 3.2. *The following identity holds,*

$$\begin{aligned} \frac{\partial}{\partial u}\left(\log T^{(2)}_{a,RS}(\widetilde{M}, F, g^{TM}_u, g^F_u)\right) &= \frac{\partial}{\partial u}\left(\log T_{a,RS}(M, F, g^{TM}_u, g^F_u)\right) \\ &= -\frac{1}{2}\mathrm{l.i.m.}_{t\to 0}\mathrm{Tr}_s\left[\left(*_u^{-1}\frac{\partial *_u}{\partial u} + (h^F_u)^{-1}\frac{\partial h^F_u}{\partial u}\right)e^{-tD^2_{u,a}}\right]. \end{aligned} \tag{3.12}$$

Remark 3.1. If M is a compact manifold without boundary, then Theorem 3.2 is [32], (3.80). If we assume moreover $\Gamma = \{1\}$, then it is [2], Theorem 4.14

Remark 3.2. If $g^F_u = g^F$ is a fixed flat metric on F (i.e. (F, ∇^F, g^F) is an unitary flat bundle), the first equation of (3.12) was obtained in [19], Theorem 7.6 under certain technical "determinant class condition". Thus Theorem 3.2 generalizes [19], Theorem 7.6, without the assumptions on the flatness of g^F, and on the technical "determinant class condition".

The second equation of (3.12) is [10], Theorem 3.27 and [27], Theorem 7.3 when $g^F_u = g^F$ is a fixed flat metric on F. For a general family of metrics (g^{TM}_u, g^F_u), the second equation of (3.12) was proved in [5], Theorem 4.5. From [4], Theorem 0.2 and [5], Theorem 0.1, §5.5, we get immediately the anomaly formula for $\log T^{(2)}_{a,RS}(\widetilde{M}, F, g^{TM}, g^F)$ which differs by a factor $-\frac{1}{2}$, as the torsion element is dual to the Ray-Singer metric. We left the details to the readers.

3.4. *Variational formula for the heat kernel*

The results in this subsection were essentially obtained in [10], Theorems 3.10, 3.27 and [27], Prop. 6.1, Theorem 7.3 when $g^F_u = g^F$ is a fixed flat metric on F. In [5], §4.2, it is observed that their proof works also for any Hermitian metric on F. Our main point here is a reformulation of these results in the spirit of the proof of [1], Theorem 1.18 in the covering case.

Let $*^{\widetilde{F}}$ be the Hodge operator

$$*^{\widetilde{F}} : \Lambda(T^*\widetilde{M}) \otimes \widetilde{F} \to \Lambda(T^*\widetilde{M}) \otimes \widetilde{F}^* \otimes o(T\widetilde{M})$$

defined by

$$(\sigma \wedge *^{\widetilde{F}} \sigma')_F = \langle \sigma, \sigma' \rangle_{\Lambda(T^*\widetilde{M})\otimes \widetilde{F}} \, dv_{\widetilde{M}}.$$

A direct verification shows that, when acting on $\Omega^i(\widetilde{M}, \widetilde{F})$, one has

$$d^{\widetilde{F}*}_u = (-1)^i (*^{\widetilde{F}}_u)^{-1} d^{\widetilde{F}\otimes o(T\widetilde{M})} *^{\widetilde{F}}_u . \tag{3.13}$$

We only consider orthonormal frames $\{e_i\}^n_{i=1}$ of $T\widetilde{M}$ with the property that near the boundary V, $e_n =: \widetilde{e}_{\mathfrak{n}}$ is the inward pointing unit normal at any boundary point and $\{e_i\}^{n-1}_{i=1}$ is an orthonormal basis of $T\partial\widetilde{M}$. Let $\{e^i\}$ be the corresponding dual frame of $T^*\widetilde{M}$.

Let $e^{-t\widetilde{D}^2_a}(x,z)$, $(x, z \in \widetilde{M})$, be the smooth kernel of the operator $e^{-t\widetilde{D}^2_a}$ with respect to $dv_{\widetilde{M}}(z)$. Then

$$e^{-t\widetilde{D}^2_a}(x,z) \in \oplus^n_{k=0} (\Lambda^k(T^*\widetilde{M}) \otimes \widetilde{F})_x \otimes (\Lambda^k(T^*\widetilde{M}) \otimes \widetilde{F})^*_z \, .$$

We denote by $e^{-t\widetilde{D}^2_a}(x,z)_k$ the component of $e^{-t\widetilde{D}^2_a}(x,z)$ on $(\Lambda^k(T^*\widetilde{M}) \otimes \widetilde{F})_x \otimes (\Lambda^k(T^*\widetilde{M}) \otimes \widetilde{F})^*_z$. By using the metric $\langle \cdot, \cdot \rangle_{\Lambda(T^*\widetilde{M})\otimes\widetilde{F}}$, we will identify $(\Lambda(T^*\widetilde{M}) \otimes \widetilde{F})^*$ to $\Lambda(T^*\widetilde{M}) \otimes \widetilde{F}$, thus the operations $d^{\widetilde{F}}_z$, $d^{\widetilde{F}*}_z$ act naturally on $e^{-t\widetilde{D}^2_a}(x,z)$.

Lemma 3.1. *For $\sigma \in \Omega(\widetilde{M}, \widetilde{F}) \cap L^2(\Omega(\widetilde{M}, \widetilde{F}))$, we have*

$$\begin{aligned} &e^{-t\widetilde{D}_a^2} d^{\widetilde{F}} \sigma = d_a^{\widetilde{F}} e^{-t\widetilde{D}_a^2} \sigma, \\ &e^{-t\widetilde{D}_a^2} d^{\widetilde{F}*} \sigma = d_a^{\widetilde{F}*} e^{-t\widetilde{D}_a^2} \sigma + \int_{\partial \widetilde{M}} e^{-t\widetilde{D}_a^2}(\cdot, y) i(e_n) \sigma(y) dv_{\partial \widetilde{M}}(y). \end{aligned} \tag{3.14}$$

Especially,

$$d_x^{\widetilde{F}} e^{-t\widetilde{D}_a^2}(x,z)_k = d_z^{\widetilde{F}*} e^{-t\widetilde{D}_a^2}(x,z)_{k+1}. \tag{3.15}$$

Proof. At first, by the identification of the orientation bundle $o(T\widetilde{M})$ and $o(T\partial\widetilde{M})$ in [5], §1.3, for $\sigma, \sigma' \in \Omega(\widetilde{M}, \widetilde{F}) \cap L^2(\Omega(\widetilde{M}, \widetilde{F}))$,

$$\begin{aligned} \langle d^{\widetilde{F}} \sigma, \sigma' \rangle &= \int_{\widetilde{M}} ((d^{\widetilde{F}} \sigma) \wedge *^{\widetilde{F}} \sigma')_{\widetilde{F}} = \langle \sigma, d^{\widetilde{F}*} \sigma' \rangle + \int_{\partial \widetilde{M}} (\sigma \wedge *^{\widetilde{F}} \sigma')_{\widetilde{F}} \\ &= \langle \sigma, d^{\widetilde{F}*} \sigma' \rangle - \int_{\partial \widetilde{M}} \langle e^n \wedge \sigma, \sigma' \rangle (y) dv_{\partial \widetilde{M}}(y). \end{aligned} \tag{3.16}$$

As $d_a^{\widetilde{F}*}$, $d_a^{\widetilde{F}}$ commute with $\widetilde{D}_a^2$, they also commute with $e^{-t\widetilde{D}_a^2}$. Thus for

$$\sigma \in \Omega(\widetilde{M}, \widetilde{F}) \cap L^2(\Omega(\widetilde{M}, \widetilde{F})), \qquad \sigma' \in \Omega_a(\widetilde{M}, \widetilde{F}) \cap L^2(\Omega(\widetilde{M}, \widetilde{F})),$$

by (3.16)

$$\begin{aligned} &\langle d_a^{\widetilde{F}} e^{-t\widetilde{D}_a^2} \sigma, \sigma' \rangle = \langle e^{-t\widetilde{D}_a^2} \sigma, d_a^{\widetilde{F}*} \sigma' \rangle = \langle \sigma, e^{-t\widetilde{D}_a^2} d_a^{\widetilde{F}*} \sigma' \rangle \\ &= \langle \sigma, d_a^{\widetilde{F}*} e^{-t\widetilde{D}_a^2} \sigma' \rangle = \langle d^{\widetilde{F}} \sigma, e^{-t\widetilde{D}_a^2} \sigma' \rangle = \langle e^{-t\widetilde{D}_a^2} d^{\widetilde{F}} \sigma, \sigma' \rangle. \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} \langle d_a^{\widetilde{F}*} e^{-t\widetilde{D}_a^2} \sigma, \sigma' \rangle &= \langle \sigma, e^{-t\widetilde{D}_a^2} d_a^{\widetilde{F}} \sigma' \rangle = \langle \sigma, d_a^{\widetilde{F}} e^{-t\widetilde{D}_a^2} \sigma' \rangle \\ &= \langle d^{\widetilde{F}*} \sigma, e^{-t\widetilde{D}_a^2} \sigma' \rangle - \int_{\partial \widetilde{M}} \langle \sigma, e^n \wedge (e^{-t\widetilde{D}_a^2} \sigma') \rangle (y) dv_{\partial \widetilde{M}}(y) \\ &= \langle e^{-t\widetilde{D}_a^2} d^{\widetilde{F}*} \sigma, \sigma' \rangle - \left\langle \int_{\partial \widetilde{M}} e^{-t\widetilde{D}_a^2}(\cdot, y) i(e_n) \sigma(y) dv_{\partial \widetilde{M}}(y), \sigma' \right\rangle. \end{aligned} \tag{3.18}$$

From (3.17), (3.18), we get (3.14).

Now for $\sigma \in \Omega_a(\widetilde{M}, \widetilde{F}) \cap L^2(\Omega(\widetilde{M}, \widetilde{F}))$, by (3.16) and (3.17),

$$\begin{aligned} \int_{\widetilde{M}} (d_z^{\widetilde{F}*} e^{-t\widetilde{D}_a^2}(x,z)) \sigma(z) dv_{\widetilde{M}}(z) &= \int_{\widetilde{M}} e^{-t\widetilde{D}_a^2}(x,z) (d^{\widetilde{F}} \sigma)(z) dv_{\widetilde{M}}(z) \\ &= (d^{\widetilde{F}} e^{-t\widetilde{D}_a^2} \sigma)(x) = \int_{\widetilde{M}} d_x^{\widetilde{F}} e^{-t\widetilde{D}_a^2}(x,z) \sigma(z) dv_{\widetilde{M}}(z). \end{aligned} \tag{3.19}$$

From (3.19), we get (3.15). The proof of Lemma 3.1 is complete. □

Let g_u^{TM} (resp. g_u^F), $0 \le u \le 1$, be a smooth path of metrics on TM (resp. F). When dealing with objects associated with (g_u^{TM}, g_u^F), we will use a subscript "u" to indicate. While at $u = 0$ we usually omit this subscript indication. In particular, we will use $\langle\, , \rangle_{\widetilde{M},u}$, $\langle\, , \rangle_{\partial\widetilde{M},u}$ to denote the product on $\widetilde{M}$, $\partial\widetilde{M}$ with respect to $dv_{\widetilde{M},u}$, $dv_{\partial\widetilde{M},u}$. Then one has

$$Q_u := (*_u^{\widetilde{F}})^{-1} \frac{\partial *_u^{\widetilde{F}}}{\partial u} = (*_u)^{-1} \frac{\partial *_u}{\partial u} + (g_u^{\widetilde{F}})^{-1} \frac{\partial g_u^{\widetilde{F}}}{\partial u}. \tag{3.20}$$

In what follow, all operations are applied to the variable z when we do not specify them.

Lemma 3.2.

$$\begin{aligned}
&(*_{u,w}^F)^{-1} \tfrac{\partial}{\partial u} \left(*_{u,w}^F e^{-t\widetilde{D}_{u,a}^2}(x,w) \right) \\
&\quad = \int_0^t \Big\{ - \Big\langle [d^F, [d_u^{\widetilde{F}*}, Q_u]] e^{-(t-s)\widetilde{D}_{u,a}^2}(x,z), e^{-s\widetilde{D}_{u,a}^2}(z,w) \Big\rangle_{\widetilde{M},u} \\
&\qquad + \Big\langle i(e_n) Q_u d^F e^{-(t-s)\widetilde{D}_{u,a}^2}(x,z), e^{-s\widetilde{D}_{u,a}^2}(z,w) \Big\rangle_{\partial\widetilde{M},u} \\
&\qquad + \Big\langle i(e_n) Q_u e^{-(t-s)\widetilde{D}_{u,a}^2}(x,z), d_u^{\widetilde{F}*} e^{-s\widetilde{D}_{u,a}^2}(z,w) \Big\rangle_{\partial\widetilde{M},u} \Big\}.
\end{aligned} \tag{3.21}$$

Proof. We only need to prove (3.21) for $u = 0$. At first, by [5], (4.10), we have

$$i(e_{n,u}) d_u^{\widetilde{F}*} \sigma|_{\partial\widetilde{M}} = 0 \quad \text{if } i(e_{n,u})\sigma|_{\partial\widetilde{M}} = 0. \tag{3.22}$$

We know also that for $\sigma \in \Omega(\widetilde{M}, \widetilde{F}) \cap L^2(\Omega(\widetilde{M}, \widetilde{F}))$

$$\begin{aligned}
&\lim_{s\to 0} \int_{\widetilde{M}} e^{-s\widetilde{D}_{u,a}^2}(x,z)\sigma(z) dv_{\widetilde{M},0}(z) \\
&\quad = \lim_{s\to 0} \int_{\widetilde{M}} \Big(e^{-s\widetilde{D}_{u,a}^2}(x,z) \wedge *_u^{\widetilde{F}} ((*_u^{\widetilde{F}})^{-1} *_0^{\widetilde{F}} \sigma)(z) \Big)_F \\
&\quad = \lim_{s\to 0} \Big(e^{-s\widetilde{D}_{u,a}^2} (*_u^{\widetilde{F}})^{-1} *_0^{\widetilde{F}} \sigma \Big)(x) = ((*_u^{\widetilde{F}})^{-1} *_0^{\widetilde{F}} \sigma)(x).
\end{aligned} \tag{3.23}$$

By (3.16), (3.22) and (3.23), we get

$$
\begin{aligned}
e^{-t\widetilde{D}^2_{u,a}}&(x,w) - *^{-1}_{u,w} *_{0,w} e^{-t\widetilde{D}^2_{0,a}}(x,w)\\
&= -\int_0^t \frac{\partial}{\partial s}\left\langle e^{-(t-s)\widetilde{D}^2_{u,a}}(x,z), e^{-s\widetilde{D}^2_{0,a}}(z,w)\right\rangle_{\widetilde{M},0}\\
&= \int_0^t \Bigg[\left\langle \frac{\partial}{\partial t} e^{-(t-s)\widetilde{D}^2_{u,a}}(x,z), e^{-s\widetilde{D}^2_{0,a}}(z,w)\right\rangle_{\widetilde{M},0}\\
&\quad + \left\langle e^{-(t-s)\widetilde{D}^2_{u,a}}(x,z), \widetilde{D}^2_{0,a} e^{-s\widetilde{D}^2_{0,a}}(z,w)\right\rangle_{\widetilde{M},0}\Bigg]\\
= \int_0^t &\Bigg[\left\langle (\frac{\partial}{\partial t} + \widetilde{D}^2_{0,a}) e^{-(t-s)\widetilde{D}^2_{u,a}}(x,z), e^{-s\widetilde{D}^2_{0,a}}(z,w)\right\rangle_{\widetilde{M},0}\\
&- \left\langle d^{\widetilde{F}} e^{-(t-s)\widetilde{D}^2_{u,a}}(x,z), e_0^n \wedge e^{-s\widetilde{D}^2_{0,a}}(z,w)\right\rangle_{\partial\widetilde{M},0}\\
&\quad - \left\langle e^{-(t-s)\widetilde{D}^2_{u,a}}(x,z), e_0^n \wedge d_0^{\widetilde{F}*} e^{-s\widetilde{D}^2_{0,a}}(z,w)\right\rangle_{\partial\widetilde{M},0}\Bigg].
\end{aligned}
\tag{3.24}
$$

From our definition of $e^{-t\widetilde{D}^2_{u,a}}$, we have

$$
\begin{aligned}
&\Big((\frac{\partial}{\partial t} + \widetilde{D}^2_{u,a}) e^{-t\widetilde{D}^2_{u,a}}\Big)(x,z) = 0, \qquad (3.25)\\
&\Big(i(e_{n,u}) e^{-t\widetilde{D}^2_{u,a}}\Big)(x,z) = \Big(i(e_{n,u}) d^{\widetilde{F}} e^{-t\widetilde{D}^2_{u,a}}\Big)(x,z) = 0, \text{ for } x \in \partial\widetilde{M}.
\end{aligned}
$$

From the explicit construction of the operator $e^{-t\widetilde{D}^2_{u,a}}$, as observed in [19], p. 560, it is differentiable with respect to u, as $\widetilde{M}$ has bounded geometry. We get

$$
\begin{aligned}
&\Big((\frac{\partial}{\partial t} + \widetilde{D}^2_{u,a}) \frac{\partial}{\partial u} e^{-t\widetilde{D}^2_{u,a}} + (\frac{\partial}{\partial u}\widetilde{D}^2_{u,a}) e^{-t\widetilde{D}^2_{u,a}}\Big)(x,z) = 0,\\
&\Big(i\left(\tfrac{\partial}{\partial u} e_{n,u}\right) e^{-t\widetilde{D}^2_{u,a}} + i(e_{n,u}) \tfrac{\partial}{\partial u} e^{-t\widetilde{D}^2_{u,a}}\Big)(x,z) = 0 \text{ for } x \in \partial\widetilde{M},\\
&\Big(i\left(\tfrac{\partial}{\partial u} e_{n,u}\right) d^{\widetilde{F}} e^{-t\widetilde{D}^2_{u,a}} + i(e_{n,u}) d^{\widetilde{F}} \tfrac{\partial}{\partial u} e^{-t\widetilde{D}^2_{u,a}}\Big)(x,z) = 0 \text{ for } x \in \partial\widetilde{M}.
\end{aligned}
\tag{3.26}
$$

By (3.26), differentiating (3.24) with respect to u and setting $u = 0$

gives

$$\begin{aligned}
&(*^F_{0,w})^{-1}\tfrac{\partial}{\partial u}(*^F_{u,w}e^{-t\widetilde{D}^2_{u,a}})(x,w)\\
&\quad=\int_0^t\Big[-\Big\langle(\frac{\partial}{\partial u}\widetilde{D}^2_{u,a})e^{-(t-s)\widetilde{D}^2_{u,a}}(x,z),e^{-s\widetilde{D}^2_{0,a}}(z,w)\Big\rangle_{\widetilde{M},0}\\
&\qquad+\Big\langle i\left(\tfrac{\partial}{\partial u}e_{n,u}\right)d^{\widetilde{F}}e^{-(t-s)\widetilde{D}^2_{0,a}}(x,z),e^{-s\widetilde{D}^2_{0,a}}(z,w)\Big\rangle_{\partial\widetilde{M},0}\\
&\qquad+\Big\langle i\left(\tfrac{\partial}{\partial u}e_{n,u}\right)e^{-(t-s)\widetilde{D}^2_{0,a}}(x,z),d^{\widetilde{F}*}e^{-s\widetilde{D}^2_{0,a}}(z,w)\Big\rangle_{\partial\widetilde{M},0}\Big].
\end{aligned}\tag{3.27}$$

From (3.13) and (3.20), one gets

$$\frac{\partial}{\partial u}d_u^{\widetilde{F}*}=[d_u^{\widetilde{F}*},Q_u],\qquad \frac{\partial}{\partial u}\widetilde{D}^2_{u,a}=[d^F,[d_u^{\widetilde{F}*},Q_u]].\tag{3.28}$$

Set

$$\dot{g}_u^{TX}:=(g_u^{TX})^{-1}(\frac{\partial}{\partial u}g_u^{TX}).\tag{3.29}$$

Observe that $\frac{\partial}{\partial u}e_{j,u}$, $e_{j,u}\in T\partial\widetilde{M}$ for $j<n$, thus we compute that

$$\tfrac{\partial}{\partial u}e_{n,u}=-\sum_{j=1}^n\langle\dot{g}_u^{TX}e_{j,u},e_{n,u}\rangle_{g_u^{TX}}e_{j,u}+\frac12\langle\dot{g}_u^{TX}e_{n,u},e_{n,u}\rangle e_{n,u}.\tag{3.30}$$

By [2], Prop. 4.15, we have,

$$*_u^{-1}\frac{\partial *_u}{\partial u}=-\frac12\sum_{1\le j,k\le n}\left\langle e_j,\dot{g}_u^{TX}e_k\right\rangle_{g_u^{TX}}(e^j\wedge i(e_k)-i(e_j)\wedge e^k).\tag{3.31}$$

From (3.30) and (3.31), we get at $u=0$,

$$i(\tfrac{\partial}{\partial u}e_{n,u})=\left[i(e_n),*_u^{-1}\frac{\partial *_u}{\partial u}\right]+\frac12\langle\dot{g}_0^{TX}e_n,e_n\rangle i(e_n).\tag{3.32}$$

From (3.25), (3.27), (3.28) and (3.32), we get (3.21). □

Let N denote the number operator on $\Omega^*(\widetilde{M},\widetilde{F})$ acting by multiplication by i on $\Omega^i(\widetilde{M},\widetilde{F})$. It extends to obvious actions on L^2-completions.

Let $\mathrm{Tr}_{\mathcal{N},s}[\cdot]=\mathrm{Tr}_{\mathcal{N}}[(-1)^N\cdot]$ be the supertrace in the sense of Quillen[26], taking on bounded $\mathcal{N}(\Gamma)$-linear operators acting on $\Omega^*(\widetilde{M},\widetilde{F})$ as well as their L^2-completions. In what follows we will also adopt the notation in [26] of supercommutators.

Theorem 3.3. *We have the following identity,*

$$\tfrac{\partial}{\partial u}\mathrm{Tr}_{\mathcal{N},s}\left[Ne^{-t\widetilde{D}^2_{u,a}}\right]=t\tfrac{\partial}{\partial t}\mathrm{Tr}_{\mathcal{N},s}\left[Q_ue^{-t\widetilde{D}^2_{u,a}}\right].\tag{3.33}$$

Proof. Let U be a fundamental domain of the covering $\pi : \widetilde{M} \to M$, and let $U_1 = \overline{U} \cap \pi^{-1}(\partial M)$. Observe first that for any Γ-equivariant smooth operator P acting on $\Omega(\widetilde{M}, \widetilde{F})$, if we denote by $P(x,z)$ the smooth kernel of P with respect to $dv_{\widetilde{M},u}(z)$, then

$$\mathrm{Tr}_{\mathcal{N},s}[P] = \int_U (*_u^{\widetilde{F}} P(x,x))_F. \tag{3.34}$$

Thus when we apply (3.34) to (3.21), and reverse the order of integration on the right hand side of (3.34), then use (3.15), (3.31) and the fact that N preserves the boundary condition, we get

$$\begin{aligned}\frac{\partial}{\partial u}\mathrm{Tr}_{\mathcal{N},s}\left[N\exp\left(-t\widetilde{D}_u^2\right)\right] &= -t\mathrm{Tr}_{\mathcal{N},s}\left[[d^{\widetilde{F}},[d^{\widetilde{F}*},Q_u]]Ne^{-t\widetilde{D}^2_{u,a}}\right]\\ &+ t\int_{U_1}\mathrm{Tr}_s[((i(e_n)Q_u d^{\widetilde{F}})_{x'}Ne^{-t\widetilde{D}^2_{u,a}})(w,x')|_{w=x'}]dv_{\partial\widetilde{M},u}(x')\\ &+ t\int_{U_1}\mathrm{Tr}_s[N(Q_u e^n d^{\widetilde{F}*})_{x'}e^{-t\widetilde{D}^2_{u,a}}(x',w)|_{w=x'}]dv_{\partial\widetilde{M},u}(x').\end{aligned} \tag{3.35}$$

By (3.15) and (3.31),

$$\begin{aligned}&\mathrm{Tr}_s\left[(Q_u e^n d^{\widetilde{F}*})_{x'}Ne^{-t\widetilde{D}^2_{u,a}}(x',w)|_{w=x'}\right]\\ &\qquad = -\mathrm{Tr}_s\left[(i(e_n)Q_u N d^{\widetilde{F}}e^{-t\widetilde{D}^2_{u,a}})(x',x')\right].\end{aligned} \tag{3.36}$$

From (3.14) and the fact that $d^{\widetilde{F}}e^{-s\widetilde{D}^2_{u,a}}$, $d^{\widetilde{F}*}e^{-s\widetilde{D}^2_{u,a}}$ are smooth Γ-equivariant operators, we see that for any Γ-equivariant differential operator P which changes the $\mathbf{Z}_2$-grading on $\Omega(\widetilde{M}, \widetilde{F})$, we have

$$\begin{aligned}\mathrm{Tr}_{\mathcal{N},s}\left[d^{\widetilde{F}}Pe^{-t\widetilde{D}^2_{u,a}}\right] &= \mathrm{Tr}_{\mathcal{N},s}\left[e^{-s\widetilde{D}^2_{u,a}}d^{\widetilde{F}}Pe^{-(t-s)\widetilde{D}^2_{u,a}}\right]\\ &= \mathrm{Tr}_{\mathcal{N},s}\left[d_a^{\widetilde{F}}e^{-s\widetilde{D}^2_{u,a}}Pe^{-(t-s)\widetilde{D}^2_{u,a}}\right]\\ = -\mathrm{Tr}_{\mathcal{N},s}\left[Pe^{-(t-s)\widetilde{D}^2_{u,a}}d_a^{\widetilde{F}}e^{-s\widetilde{D}^2_{u,a}}\right] &= -\mathrm{Tr}_{\mathcal{N},s}\left[Pd_a^{\widetilde{F}}e^{-t\widetilde{D}^2_{u,a}}\right],\end{aligned} \tag{3.37}$$

and in the same way

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{N},s}\left[d^{\widetilde{F}*}Pe^{-t\widetilde{D}^2_{u,a}}\right] &= \mathrm{Tr}_{\mathcal{N},s}\left[e^{-s\widetilde{D}^2_{u,a}}d^{\widetilde{F}*}Pe^{-(t-s)\widetilde{D}^2_{u,a}}\right] \\
&= \mathrm{Tr}_{\mathcal{N},s}\Big[d_a^{\widetilde{F}*}e^{-s\widetilde{D}^2_{u,a}}Pe^{-(t-s)\widetilde{D}^2_{u,a}} \\
&\quad + \int_{\partial\widetilde{M}} e^{-sD_a^2}(\cdot,z)i(e_n)Pe^{-(t-s)\widetilde{D}^2_{u,a}}(z,\cdot)dv_{\partial\widetilde{M}}(z)\Big] \\
&= -\mathrm{Tr}_{\mathcal{N},s}\left[Pd_a^{\widetilde{F}*}e^{-t\widetilde{D}^2_{u,a}}\right] \\
&\quad + \int_{U_1}\mathrm{Tr}_s\left[i(e_n)Pe^{-t\widetilde{D}^2_{u,a}}(x',x')\right]dv_{\partial\widetilde{M},u}(x').
\end{aligned} \tag{3.38}$$

We also have

$$[d^{\widetilde{F}},N] = -d^{\widetilde{F}}, \quad [d^{\widetilde{F}*},N] = d^{\widetilde{F}*}. \tag{3.39}$$

From (3.35)-(3.39), we get

$$\begin{aligned}
&\tfrac{\partial}{\partial u}\mathrm{Tr}_{\mathcal{N},s}\left[Ne^{-t\widetilde{D}^2_{u,a}}\right] \\
&= -\mathrm{Tr}_{\mathcal{N},s}\left[Q_u(Nd^{\widetilde{F}}d^{\widetilde{F}*}+d^{\widetilde{F}*}Nd^{\widetilde{F}}-d^{\widetilde{F}}Nd^{\widetilde{F}*}-d^{\widetilde{F}*}d^{\widetilde{F}}N)e^{-t\widetilde{D}^2_{u,a}}\right] \\
&\qquad = -\mathrm{Tr}_{\mathcal{N},s}\left[Q_u\widetilde{D}^2_{u,a}e^{-t\widetilde{D}^2_{u,a}}\right] = \tfrac{\partial}{\partial t}\mathrm{Tr}_{\mathcal{N},s}\left[Q_ue^{-t\widetilde{D}^2_{u,a}}\right].
\end{aligned} \tag{3.40}$$

The proof of Theorem is complete. □

3.5. *A proof of Theorem 3.2*

First, by proceeding as in the beginning of [32], Section 3.4, one gives a slightly more flexible formula of the L^2-Ray-Singer torsion element $T^{(2)}_{a,RS}(\widetilde{M},F,g^{TM},g^F)$ defined in (3.11).

For any $c>0$, let

$$(C^*,d_a^{\widetilde{F}}) \subset (\Omega_a^*(\widetilde{M},\widetilde{F}),d^{\widetilde{F}})$$

be a finite length $\mathcal{N}(\Gamma)$-Hilbert cochain subcomplex of $(L^2_a(\Omega^*(\widetilde{M},\widetilde{F})),d_a^{\widetilde{F}})$ such that $(L^2_{a,[0,c]}(\Omega^*(\widetilde{M},\widetilde{F})),d_a^{\widetilde{F}})$ is a subcomplex of $(C^*,d_a^{\widetilde{F}})$. That is, as $\mathcal{N}(\Gamma)$-Hilbert cochain complexes, one has

$$(L^2_{a,[0,c]}(\Omega^*(\widetilde{M},\widetilde{F})),d_a^{\widetilde{F}}) \subseteq (C^*,d_a^{\widetilde{F}}). \tag{3.41}$$

Let $d_{C^*}^{\widetilde{F}*}:C^*\to C^*$ be the formal adjoint of $d_a^{\widetilde{F}}:C^*\to C^*$ with respect to the induced Hilbert metric on C^* from that of $L^2(\Omega^*(\widetilde{M},\widetilde{F}))$. Set

$$\begin{aligned}
D_{C^*} &= d_a^{\widetilde{F}}+d_{C^*}^{\widetilde{F}*}, \\
D_{C^*}^2 &= (d_a^{\widetilde{F}}+d_{C^*}^{\widetilde{F}*})^2 = d_{C^*}^{\widetilde{F}*}d_a^{\widetilde{F}}+d_a^{\widetilde{F}}d_{C^*}^{\widetilde{F}*}:C^*\to C^*.
\end{aligned} \tag{3.42}$$

Then $D^2_{C^*}$ preserves the $\mathbf{Z}$-grading of C^*. Moreover, one has

$$D^2_{C^*} = \widetilde{D}^2_a : L^2_{a,[0,c]}(\Omega^*(\widetilde{M},\widetilde{F})) \to L^2_{a,[0,c]}(\Omega^*(\widetilde{M},\widetilde{F})). \tag{3.43}$$

For any $0 \le i \le n$, let $D^2_{C^i}$ denote the restriction of $D^2_{C^*}$ on C^i.

By (3.41) it is clear that the extended cohomology of $(C^*, d^{\widetilde{F}}_a)$ is identical to that of $(L^2_{a,[0,c]}(\Omega^*(\widetilde{M},\widetilde{F})), d^{\widetilde{F}}_a)$. That is, one has

$$\mathcal{H}^*(C^*, d^{\widetilde{F}}_a) \equiv \mathcal{H}^{(2)}_{a,\mathrm{dR}}(\Omega^*(\widetilde{M},\widetilde{F}), d^{\widetilde{F}}_a). \tag{3.44}$$

From (3.44), one sees that $(C^*, d^{\widetilde{F}}_a)$ induces canonically an L^2-torsion element in $\det \mathcal{H}^{(2)}_{a,\mathrm{dR}}(\Omega^*(\widetilde{M},\widetilde{F}), d^{\widetilde{F}}_a)$. We denote it by

$$T_{(C^*, d^{\widetilde{F}}_a)} \in \det \mathcal{H}^{(2)}_{a,\mathrm{dR}}(\Omega^*(\widetilde{M},\widetilde{F}), d^{\widetilde{F}}_a). \tag{3.45}$$

For any $s \in \mathbf{C}$ with $\mathrm{Re}(s) > \frac{n}{2}$ and for $0 \le i \le n$, set

$$\zeta^i_{C^*,\perp}(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \Big(\mathrm{Tr}_{\mathcal{N}} \Big[\exp\Big(-t \widetilde{D}^2_a|_{L^2(\Omega^i(\widetilde{M},\widetilde{F}))} \Big) \Big] - \mathrm{Tr}_{\mathcal{N}} \left[\exp\left(-t D^2_{C^i} \right) \right] \Big)\, dt. \tag{3.46}$$

Then one sees easily that each $\zeta^i_{C^*,\perp}(s)$, $0 \le i \le n$, is a holomorphic function for $\mathrm{Re}(s) > \frac{n}{2}$ and can be extended to a meromorphic function on $\mathbf{C}$ which is holomorphic at $s = 0$. Let $T_{(C^*, d^{\widetilde{F}}_a),\perp} \in \mathbf{R}^+$ be defined by

$$\log T_{(C^*, d^{\widetilde{F}}_a),\perp} = \frac{1}{2} \sum_{i=0}^{n} (-1)^i i \left. \frac{\partial \zeta^i_{C^*,\perp}(s)}{\partial s} \right|_{s=0}. \tag{3.47}$$

The following analogue of [32], Proposition 3.6 can be proved in the same way as there.

Proposition 3.1. *There holds in* $\det \mathcal{H}^{(2)}_{a,\mathrm{dR}}(\Omega^*(\widetilde{M},\widetilde{F}), d^{\widetilde{F}})$ *the following identity,*

$$T^{(2)}_{a,RS}(\widetilde{M}, F, g^{TM}, g^F) = T_{(C^*, d^{\widetilde{F}}_a)} \cdot T_{(C^*, d^{\widetilde{F}}_a),\perp}. \tag{3.48}$$

We now come to the proof of Theorem 3.2.

Let g^{TM}_u (resp. g^F_u), $0 \le u \le 1$, be a smooth path of metrics on TM (resp. F) such that $g^{TM}_0 = g^{TM}$, $g^{TM}_1 = g'^{TM}$ (resp. $g^F_0 = g^F$, $g^F_1 = g'^F$).

We now state the following analogue of [32], Proposition 3.7.

Proposition 3.2. *For any $u_0 \in [0,1]$, there exists $k_0 > 0$ such that for any $k > k_0$, one can construct a family of finite length $\mathcal{N}(\Gamma)$-Hilbert cochain subcomplex $(C^*(u), d^{\widetilde{F}}_{u,a})$ of $(\Omega^*_{u,a}(\widetilde{M},\widetilde{F}), d^{\widetilde{F}})$ such that*

(i) *One has the inclusion relation of cochain complexes*

$$(L^2_{u,a,[0,1]}(\Omega^*(\widetilde{M},\widetilde{F})), d^{\widetilde{F}}_{u,a}) \subseteq (C^*(u), d^{\widetilde{F}}_{u,a}); \tag{3.49}$$

(ii) *The cochain complex* $(C^*(u), d^{\widetilde{F}}_{u,a})$ *depends smoothly on* $u \in [0,1]$, *and* $(C^*(u_0), d^{\widetilde{F}}_{u_0,a}) = (L^2_{u_0,a,[0,k]}(\Omega^*(\widetilde{M},\widetilde{F})), d^{\widetilde{F}})$.

Proof. Proposition 3.2 can be proved in the same way as in [32], Proposition 3.7 where we take $u_0 = 0$, with easy modifications with respect to the appearance of the boundary ∂M. The only places need to take more care are listed as follows:

1. One notes here that the analogue of [32], (3.32) still holds here, as by Theorem 3.1, $\mathrm{Im}(P_{[0,k],u})$ consists of smooth forms. Thus $d^{\widetilde{F}}_{u,a}$ acts on them just as usual $d^{\widetilde{F}}$, not depending on u. By setting $d^{\widetilde{F}}$ to be $d^{\widetilde{F}}_{u,a}$ in an analogue of [32], (3.35), one can complete the proof of (i) easily.

2. For the proof of (ii), one needs to modify the proof of [32], Lemma 3.8. Here, one needs to take care about the analogue of [32], (3.39). For such an analogue holds, we need to assume that $x \in \Omega^*_{0,a}(\widetilde{M},\widetilde{F})$. Indeed, if we fix a Γ-invariant first Sobolev norm denoted by $\|\cdot\|_1$, then it is easy to see that there exist A_1, $B_1 > 0$ such that for any smooth form $x \in \Omega^*(\widetilde{M},\widetilde{F})$ and any $u \in [0,1]$, one has

$$\|\widetilde{D}_u x\|_{0,u} \leq A_1\|x\|_1 + B_1\|x\|_0, \tag{3.50}$$

while there exist A_2, $B_2 > 0$ such that for any $x \in \Omega^*_{0,a}(\widetilde{M},\widetilde{F})$, one has

$$A_2\|x\|_1 - B_2\|x\|_0 \leq \|\widetilde{D}x\|_0. \tag{3.51}$$

From (3.50) and (3.51), one sees that there exist A, $B > 0$ such that $x \in \Omega^*_{0,a}(\widetilde{M},\widetilde{F})$, one has

$$\|\widetilde{D}_u x\|_{0,u} \leq A\|\widetilde{D}x\|_0 + B\|x\|_0, \tag{3.52}$$

which is exactly the analogue of [32], (3.39) we need.

One can then proceed as in [32], Proof of Lemma 3.8 to complete the proof of (ii). □

We now come back to the proof of Theorem 3.2 for $u = 0$.

By (3.48), one gets that for any $0 \leq u \leq 1$,

$$T^{(2)}_{RS}(\widetilde{M}, F, g^{TM}_u, g^F_u) = T_{(C^*(u),d^{\widetilde{F}})} \cdot T_{(C^*(u),d^{\widetilde{F}}),\perp}. \tag{3.53}$$

For any $s \in \mathbf{C}$ with $\mathrm{Re}(s) > \frac{n}{2}$ and $0 \le u \le 1$, set

$$\theta_u(s) = \sum_{i=0}^{n} (-1)^i i \zeta^i_{C^*(u),\perp}(s). \tag{3.54}$$

From (3.46) and (3.54), one can rewrite $\theta_u(s)$ as

$$\theta_u(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \left(\mathrm{Tr}_{\mathcal{N},s} \left[N \exp\left(-t\widetilde{D}^2_{u,a} \right) \right] \right. \\ \left. - \mathrm{Tr}_{\mathcal{N},s} \left[N \exp\left(-tD^2_{C^*(u)} \right) \right] \right) dt. \tag{3.55}$$

For any $0 \le u \le 1$, let $P_{C^*(u)}$ denote the orthogonal projection from $L^2_u(\Omega^*(\widetilde{M}, \widetilde{F}))$ onto $C^*(u)$. Then by Proposition 3.2, $P_{C^*(u)}$ depends smoothly on $u \in [0,1]$. Moreover, one has

$$d^{\widetilde{F}} P_{C^*(u)} = P_{C^*(u)} d^{\widetilde{F}} P_{C^*(u)}. \tag{3.56}$$

Let $d^{\widetilde{F}*}_{C^*(u)} : C^*(u) \to C^*(u)$ be the formal adjoint of

$$d^{\widetilde{F}}_{C^*(u)} = P_{C^*(u)} d^{\widetilde{F}} P_{C^*(u)} : C^*(u) \to C^*(u). \tag{3.57}$$

Then in view of (3.56), one has

$$d^{\widetilde{F}*}_{C^*(u)} = P_{C^*(u)} d^{\widetilde{F}*}_{u,a} P_{C^*(u)} = P_{C^*(u)} d^{\widetilde{F}*}_{u,a}. \tag{3.58}$$

Set

$$\widetilde{D}_{C^*(u)} = d^{\widetilde{F}}_{C^*(u)} + d^{\widetilde{F}*}_{C^*(u)}. \tag{3.59}$$

One has, similar as in (3.39), that

$$[\widetilde{D}_{C^*(u)}, N] = -d^{\widetilde{F}}_{C^*(u)} + d^{\widetilde{F}*}_{C^*(u)}. \tag{3.60}$$

In order to have a formula for $\frac{\partial}{\partial u} d^{\widetilde{F}*}_{C^*(u)}$ similar to (3.28), by using (3.28) and (3.58), we compute

$$\begin{aligned} \tfrac{\partial}{\partial u} d^{\widetilde{F}*}_{C^*(u)} = \tfrac{\partial}{\partial u}\left(P_{C^*(u)} d^{\widetilde{F}*}_u \right) &= \left(\tfrac{\partial}{\partial u} P_{C^*(u)} \right) d^{\widetilde{F}*}_u + P_{C^*(u)} \tfrac{\partial}{\partial u} d^{\widetilde{F}*}_u \\ &= \left(\tfrac{\partial}{\partial u} P_{C^*(u)} \right) d^{\widetilde{F}*}_u + P_{C^*(u)} [d^{\widetilde{F}*}_u, Q_u] \\ &= \left(\tfrac{\partial}{\partial u} P_{C^*(u)} \right) d^{\widetilde{F}*}_u + P_{C^*(u)} d^{\widetilde{F}*}_u Q_u - P_{C^*(u)} Q_u d^{\widetilde{F}*}_u \\ &= [d^{\widetilde{F}*}_{C^*(u)}, Q_u] + \left(\tfrac{\partial}{\partial u} P_{C^*(u)} \right) d^{\widetilde{F}*}_u + Q_u P_{C^*(u)} d^{\widetilde{F}*}_u - P_{C^*(u)} Q_u d^{\widetilde{F}*}_u. \end{aligned} \tag{3.61}$$

Since $C^*(u)$, $0 \leq u \leq 1$, are finitely generated Hilbert modules, by using (3.60), (3.61), as in [32], (3.63), one deduces

$$\begin{aligned}
&\frac{\partial}{\partial u}\mathrm{Tr}_{\mathcal{N},s}\left[N\exp\left(-t\widetilde{D}^2_{C^*(u)}\right)\right] = -t\mathrm{Tr}_{\mathcal{N},s}\left[N\frac{\partial \widetilde{D}^2_{C^*(u)}}{\partial u}\exp\left(-t\widetilde{D}^2_{C^*(u)}\right)\right]\\
&\quad = -t\mathrm{Tr}_{\mathcal{N},s}\left[\left[N,\widetilde{D}_{C^*(u)}\right]\frac{\partial d^{\widetilde{F}*}_{C^*(u)}}{\partial u}\exp\left(-t\widetilde{D}^2_{C^*(u)}\right)\right]\\
&\quad = t\frac{\partial}{\partial t}\mathrm{Tr}_{\mathcal{N},s}\left[Q_u\exp\left(-t\widetilde{D}^2_{C^*(u)}\right)\right] - t\mathrm{Tr}_{\mathcal{N},s}\left[\left(d^{\widetilde{F}}_{C^*(u)} - d^{\widetilde{F}*}_{C^*(u)}\right)\right.\\
&\quad \left.\left(\frac{\partial P_{C^*(u)}}{\partial u}d^{\widetilde{F}*}_u + Q_uP_{C^*(u)}d^{\widetilde{F}*}_u - P_{C^*(u)}Q_ud^{\widetilde{F}*}_u\right)\exp\left(-t\widetilde{D}^2_{C^*(u)}\right)\right]\\
&\quad = t\frac{\partial}{\partial t}\mathrm{Tr}_{\mathcal{N},s}\left[Q_u\exp\left(-t\widetilde{D}^2_{C^*(u)}\right)\right] - t\mathrm{Tr}_{\mathcal{N},s}\left[\left(d^{\widetilde{F}}_{C^*(u)} - d^{\widetilde{F}*}_{C^*(u)}\right)\right.\\
&\left.\left(P_{C^*(u)}\frac{\partial P_{C^*(u)}}{\partial u}d^{\widetilde{F}*}_uP_{C^*(u)} + Q_u[P_{C^*(u)}, d^{\widetilde{F}*}_u]\right)\exp\left(-t\widetilde{D}^2_{C^*(u)}\right)\right]. \qquad (3.62)
\end{aligned}$$

Denote for $0 \leq u \leq 1$ that

$$\begin{aligned}
f(u) = &\left(d^{\widetilde{F}}_{C^*(u)} - d^{\widetilde{F}*}_{C^*(u)}\right)\cdot\ldots\\
&\ldots\cdot\left(P_{C^*(u)}\frac{\partial P_{C^*(u)}}{\partial u}d^{\widetilde{F}*}_uP_{C^*(u)} + Q_u[P_{C^*(u)}, d^{\widetilde{F}*}_u]\right).
\end{aligned} \qquad (3.63)$$

Since $C^*(u)$ contains $L^2_{u,[0,1]}(\Omega^*(\widetilde{M},\widetilde{F}))$ for $0 \leq u \leq 1$ (cf. (3.49)), one sees that when $t \to +\infty$,

$$\mathrm{Tr}_{\mathcal{N},s}\left[Q_u\exp\left(-t\widetilde{D}^2_{u,a}\right)\right] - \mathrm{Tr}_{\mathcal{N},s}\left[Q_u\exp\left(-t\widetilde{D}^2_{C^*(u)}\right)\right]$$

is of exponential decay.

On the other hand, since, when restricted to the subcomplex $(L^2_{u,a,[0,1]}(\Omega^*(\widetilde{M},\widetilde{F})), d^{\widetilde{F}}_{u,a})$ of $(C^*(u), d^{\widetilde{F}})$, $d^{\widetilde{F}*}_u$ commutes with $P_{C^*(u)}$, while

$$P_{C^*(u)}\frac{\partial P_{C^*(u)}}{\partial u}P_{C^*(u)} = 0, \qquad (3.64)$$

from (3.63), (3.64) one gets

$$f(u)|_{L^2_{u,[0,1]}(\Omega^*(\widetilde{M},\widetilde{F}))} = 0. \qquad (3.65)$$

From (3.49) and (3.65), one sees that as $t \to +\infty$,

$$\mathrm{Tr}_{\mathcal{N},s}\left[f(u)\exp\left(-t\widetilde{D}^2_{C^*(u)}\right)\right]$$

is of exponential decay.

By (3.33), (3.55), (3.62), (3.63) and (3.65), we have for $\mathrm{Re}(s)$ large enough that

$$\begin{aligned}
\frac{\partial \theta_u(s)}{\partial u} = & \frac{1}{\Gamma(s)} \int_0^{+\infty} t^s \frac{\partial}{\partial t} \Big(\mathrm{Tr}_{\mathcal{N},s} \Big[Q_u \exp\Big(-t \widetilde{D}^2_{u,a}\Big)\Big] \\
& \qquad - \mathrm{Tr}_{\mathcal{N},s} \Big[Q_u \exp\Big(-t \widetilde{D}^2_{C^*(u)}\Big)\Big]\Big) dt \\
& - \frac{1}{\Gamma(s)} \int_0^{+\infty} t^s \mathrm{Tr}_{\mathcal{N},s} \Big[f(u) \exp\Big(-t \widetilde{D}^2_{C^*(u)}\Big)\Big] dt \\
= & \frac{-s}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \Big(\mathrm{Tr}_{\mathcal{N},s} \Big[Q_u \exp\Big(-t \widetilde{D}^2_{u,a}\Big)\Big] \\
& \qquad - \mathrm{Tr}_{\mathcal{N},s} \Big[Q_u \exp\Big(-t \widetilde{D}^2_{C^*(u)}\Big)\Big]\Big) dt \\
& - \frac{1}{\Gamma(s)} \int_0^{+\infty} t^s \mathrm{Tr}_{\mathcal{N},s} \Big[f(u) \exp\Big(-t \widetilde{D}^2_{C^*(u)}\Big)\Big] dt. \qquad (3.66)
\end{aligned}$$

Now by using the finite propagation speed of solutions of hyperbolic equations (cf. [29], §2.8, §6.1), we know from [19], Theorem 2.26 that as $t \to 0^+$, for any positive integer l one has an asymptotic expansion

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{N},s} \Big[Q_u \exp\Big(-t \widetilde{D}^2_{u,a}\Big)\Big] &= \mathrm{Tr}_s \left[Q_u \exp\left(-t D^2_{u,a}\right)\right] + o(u^{l/2}) \qquad (3.67) \\
&= \sum_{j=-n}^{l} M_{j,u} t^{j/2} + o(u^{l/2}).
\end{aligned}$$

From (3.66) and (3.67), one finds that for any $0 \leq u \leq 1$, one has

$$\begin{aligned}
\frac{\partial}{\partial u} \left(\left. \frac{\partial \theta_u(s)}{\partial s} \right|_{s=0} \right) = & -M_{0,u} + \mathrm{Tr}_{\mathcal{N},s} \left[Q_u P_{C^*(u)} \right] \\
& - \int_0^{+\infty} \mathrm{Tr}_{\mathcal{N},s} \Big[f(u) \exp\Big(-t \widetilde{D}^2_{C^*(u)}\Big)\Big] dt. \qquad (3.68)
\end{aligned}$$

Now observe that we are applying Proposition 3.2 for $u_0 = 0$, thus one has, as in [32], (3.70),

$$(C^*(0), d^{\widetilde{F}}) = (L^2_{0,a,[0,k]}(\Omega^*(\widetilde{M}, \widetilde{F})), d_a^{\widetilde{F}}). \qquad (3.69)$$

Thus one again has the fact that $d_u^{\widetilde{F}*}$ commutes with $P_{C^*(u)}$, which, together with (3.64), implies that

$$f(0) = 0. \qquad (3.70)$$

From (3.47), (3.54), (3.68) and (3.70), one finds

$$\left.\frac{\partial \log T_{(C^*(u),d^{\tilde{F}}),\perp}}{\partial u}\right|_{u=0} = -\frac{M_{0,0}}{2} + \frac{1}{2}\mathrm{Tr}_{\mathcal{N},s}\left[Q_0 P_{C^*(0)}\right]. \tag{3.71}$$

Now let us consider the variation of $T_{(C^*(u),d^{\tilde{F}})}$ near $u=0$.
Observe that for any

$$\omega,\ \omega' \in C^*(0) = L^2_{a,[0,k]}(\Omega^*(\widetilde{M},\widetilde{F})),$$

the induced inner product of them in $C^*(u)$ is given by

$$\begin{aligned}\langle P_{C^*(u)}\omega, P_{C^*(u)}\omega'\rangle_u = \langle \omega, P_{C^*(u)}\omega'\rangle_u &= \int_{\widetilde{M}}\left(\omega \wedge *^{\widetilde{F}}_u P_{C^*(u)}\omega'\right)_{\widetilde{F}} \\ &= \left\langle \omega, \left(*^{\widetilde{F}}\right)^{-1} *^{\widetilde{F}}_u P_{C^*(u)}\omega' \right\rangle. \end{aligned} \tag{3.72}$$

Set for $0 \le u \le 1$ that

$$A_u = P_{C^*(0)}\left(*^{\widetilde{F}}\right)^{-1} *^{\widetilde{F}}_u P_{C^*(u)}P_{C^*(0)} : C^*(0) \to C^*(0). \tag{3.73}$$

From (2.27)-(2.29), (3.45), (3.72) and (3.73), one finds,

$$\log\frac{T_{(C^*(u),d^{\tilde{F}})}}{T_{(C^*(0),d^{\tilde{F}})}} = -\frac{1}{2}\sum_{i=0}^{n}(-1)^i \log \mathrm{Det}_{\tau_{\mathcal{N}(\Gamma)}}\left(A_u|_{C^i(0)}\right). \tag{3.74}$$

From (2.18) and (3.74), one deduces

$$\frac{\partial}{\partial u}\log\frac{T_{(C^*(u),d^{\tilde{F}})}}{T_{(C^*(0),d^{\tilde{F}})}} = -\frac{1}{2}\mathrm{Tr}_{\mathcal{N},s}\left[A_u^{-1}\frac{\partial A_u}{\partial u}\right]. \tag{3.75}$$

By (3.73), one sees directly that

$$A_u|_{u=0} = \mathrm{Id}|_{C^*(0)}. \tag{3.76}$$

From (3.20), (3.64), (3.73), (3.75) and (3.76), one finds

$$\begin{aligned}\left.\frac{\partial}{\partial u}\right|_{u=0}\log\frac{T_{(C^*(u),d^{\tilde{F}})}}{T_{(C^*(0),d^{\tilde{F}})}} &= -\frac{1}{2}\mathrm{Tr}_{\mathcal{N},s}\left[P_{C^*(0)}\left(*^{\widetilde{F}}\right)^{-1}\left.\frac{\partial *^{\widetilde{F}}_u}{\partial u}\right|_{u=0} P_{C^*(0)}\right] \\ &= -\frac{1}{2}\mathrm{Tr}_{\mathcal{N},s}\left[Q_0 P_{C^*(0)}\right]. \end{aligned} \tag{3.77}$$

From (3.53), (3.71) and (3.77), one gets

$$\left.\frac{\partial}{\partial u}\right|_{u=0}\log\frac{T^{(2)}_{RS}(\widetilde{M},F,g^{TM}_u,g^F_u)}{T^{(2)}_{RS}(\widetilde{M},F,g^{TM},g^F)} = -\frac{M_{0,0}}{2}. \tag{3.78}$$

Since (3.78) holds for arbitrary (g^{TM}, g^F), one gets indeed that for any $0 \leq u \leq 1$,

$$\frac{\partial}{\partial u} \log \frac{T^{(2)}_{RS}(\widetilde{M}, F, g^{TM}_u, g^F_u)}{T^{(2)}_{RS}(\widetilde{M}, F, g^{TM}, g^F)} = -\frac{M_{0,u}}{2}. \tag{3.79}$$

Now by using (3.67), one sees that for any $0 \leq u \leq 1$, $M_{0,u}$ is exactly the same quantity appears in [4], (7), (8) and [5], Theorem 4.5, where a similar result is proved for the usual Ray-Singer metrics.

The proof of Theorem 3.2 is complete. □

Remark 3.3. If for any $u \in [0,1]$, $\mathrm{Spec}(\widetilde{D}^2_{u,a})$ contains a non-empty gap, then the proof of Theorem 3.2 can be simplified a lot. Here we did not make this assumption as usually $\mathrm{Spec}(\widetilde{D}^2_{u,a})$, $u \in [0,1]$, may not be discrete when Γ is an infinite group.

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References

1. J.-M. Bismut, H. Gillet and C. Soulé, Analytic torsion and holomorphic determinant bundles III. *Comm. Math. Phys.* 115 (1998), 301-351.
2. J.-M. Bismut and W. Zhang, An extension of a theorem by Cheeger and Müller. *Astérisque* Tom. 205, Paris, (1992).
3. M. Braverman, A. Carey, M. Farber and V. Mathai, L^2 torsion without the determinant class condition and extended L^2 cohomology. *Comm. Contemp. Math.* 7 (2005), 421-462.
4. J. Brüning and X. Ma, An anomaly formula for Ray-Singer metrics on manifolds with boundary. *C. R. Acad. Sci. Paris, Ser. I*, 335 (2002), 603-608.
5. J. Brüning and X. Ma, An anomaly formula for Ray-Singer metrics on manifolds with boundary. To appear in *Geom. Funct. Anal.*
6. D. Burghelea, L. Friedlander and T. Kappeler, Torsions for manifolds with boundary and glueing formulas. *Math. Nachr.* 208 (1999), 31-91.
7. D. Burghelea, L. Friedlander, T. Kappeler and P. McDonald, Analytic and Reidemeister torsion for representations in finite type Hilbert modules. *Geom. Funct. Anal.* 6 (1996), 751-859.
8. A. Carey, M. Farber and V. Mathai, Determinant lines, von Neumann algebras and L^2 torsion. *J. Reine Angew. Math.* 484 (1997), 153-181.
9. A. Carey and V. Mathai, L^2-torsion invariants. *J. Funct. Anal.* 110 (1992), 337-409.

10. J. Cheeger, Analytic torsion and the heat equation. *Ann. of Math.* 109 (1979), 259-332.
11. X. Dai and H. Fang, Analytic torsion and R-torsion for manifolds with boundary. *Asian J. Math.* 4 (2000), 695-714.
12. J. Dixmier, von Neumann Algebras. *North-Holland Publishing Co.*, Amsterdam-New York, 1981. xxxviii+437 pp.
13. M. Farber, Von Neumann categories and extended L^2-cohomology. *K-Theory* 15 (1998), 347-405.
14. B. Fuglede and R. V. Kadison, Determinant theory in finite factors. *Ann. of Math.* 55 (1952), 520-530.
15. J. Lott, Heat kernels on covering spaces and topological invariants. *J. Diff. Geom.* 35 (1992), 471-510.
16. J. Lott and M. Rothenberg, Analytic torsion for group actions. *J. Diff. Geom.* 34 (1991), 431-481.
17. W. Lück, Analytic and topological torsion for manifolds with boundary and symmetry. *J. Diff. Geom.* 37 (1993), 263-322.
18. W. Lück, *L^2-Invariants: Theory and Applications to Geometry and K-Theory.* Springer-Verlag Berlin Heidelberg 2002.
19. W. Lück and T. Schick, L^2-torsion of hyperbolic manifolds of finite volume. *Geom. Funct. Anal.* 9 (1999), 518-567.
20. V. Mathai, L^2-analytic torsion. *J. Funct. Anal.* 107 (1992), 369-386.
21. J. Milnor, Whitehead torsion. *Bull. Amer. Math. Soc.* 72 (1966), 358-426.
22. W. Müller, Analytic torsion and the R-torsion of Riemannian manifolds. *Adv. in Math.* 28 (1978), 233-305.
23. W. Müller, Analytic torsion and the R-torsion for unimodular representations. *J. Amer. Math. Soc.* 6 (1993), 721-753.
24. S. P. Novikov and M. A. Shubin, Morse theory and von Neumann invariants on non-simply connected manifolds. *Uspehi. Mat. Nauk* 41 (1986), 222-223.
25. D. Quillen, Determinants of Cauchy-Riemann operators over a Riemann surface. *Funct. Anal. Appl.* 14 (1985), 31-34.
26. D. Quillen, Superconnections and the Chern character. *Topology* 24 (1985), 89-95.
27. D. B. Ray and I. M. Singer, *R*-torsion and the Laplacian on Riemannian manifolds. *Adv. in Math.* 7 (1971), 145-210.
28. M. Shubin, De Rham theorem for extended L^2-cohomology. *Voronezh Winter Mathematical School,* Amer. Math. Soc., 1998, pp. 217-231.
29. Taylor M., *Partial Differential Equations I. Basic Theory.* Applied Mathematical sciences 115, Springer 1996.
30. E. Witten, Supersymmetry and Morse theory. *J. Diff. Geom.* 17 (1982), 661-692.
31. W. Zhang, *Lectures on Chern-Weil Theory and Witten Deformations,* Nankai Tracks in Mathematics, Vol. 4. World Scientific, Singapore, 2001.
32. W. Zhang, An extended Cheeger-Müller theorem for covering spaces. *Topology* 44 (2005), 1093-1131.

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