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An extended Cheeger–Müller theorem for covering spaces

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Abstract

We generalize a theorem of Bismut–Zhang, which extends the Cheeger–Müller theorem on Ray–Singer torsion and Reidemeister torsion, to the case of infinite Galois covering spaces. Our result is stated in the framework of extended cohomology, and generalizes in this case a recent result of Braverman–Carey–Farber–Mathai. It does not use the determinant class condition and thus also (potentially) generalizes several results on L^2-torsions due to Burghelea, Friedlander, Kappeler and McDonald. We combine the framework developed by Braverman–Carey–Farber–Mathai on the determinant of extended cohomology with the heat kernel method developed in the original paper of Bismut–Zhang to prove our result.

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1. Introduction

Let F be a unitary flat vector bundle on a closed Riemannian manifold X. In [31], Ray and Singer defined an analytic torsion associated to (X, F) and proved that it does not depend on the Riemannian metric on X. Moreover, they conjectured that this analytic torsion coincides with the classical Reidemeister torsion defined using a triangulation on X (cf. [25]). This conjecture was later proved in the celebrated

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papers of Cheeger [12] and Müller [26]. Müller generalized this result in [27] to the case where F is a unimodular flat vector bundle on X. In [4], inspired by the considerations of Quillen [29], Bismut and Zhang reformulated the above Cheeger–Müller theorem as an equality between the Reidemeister and Ray–Singer metrics defined on the determinant of cohomology, and proved an extension of it to the case of general flat vector bundles over X. The method used in [4] is different from those of Cheeger and Müller in that it makes use of a deformation by Morse functions introduced by Witten [35] on the de Rham complex.

The purpose of this paper is to generalize the main results in [4] to the case of L^2-torsions on infinite Galois covering spaces of closed manifolds. We recall that the L^2-torsions were first introduced by Carey, Lott and Mathai [10,19,22], under the assumptions that the L^2-Betti numbers vanish and that certain technical "determinant class condition" is satisfied. The later condition is satisfied if the Novikov–Shubin invariants introduced in [28] are positive.

In [9], Carey, Farber and Mathai showed that the condition on the vanishing of the L^2-Betti numbers can be relaxed. This is achieved by constructing the determinant line of the reduced L^2-cohomology and defining the L^2-torsions as elements of the determinant line. They also reformulated the result of Burghelea, Friedlander, Kappeller and Mcdonald [8] on the equality between the L^2-Reidemeister torsion and L^2-Ray–Singer torsion for unitary representations, under the "determinant class condition", as an equality between two L^2-elements on the determinant line of the reduced L^2-cohomology.

In [11], Carey, Mathai and Mishchenko introduced what they called "relative torsion" in order to avoid the "determinant class condition" in the consideration of L^2-torsions. This concept was later used by Burghelea et al. [7] to generalize the main result in [8] to the case of nonunitary representations. It is pointed out in [7] that the main result in [7] also extends the generalized Cheeger–Müller theorem proved in [4] to the case of infinite covering spaces, under the "determinant class condition".

Recently, Braverman, Carey, Farber and Mathai [6] showed that if one considers the *extended* L^2-cohomology in the sense of Farber (cf. [13]) instead of the usually used *reduced* L^2-cohomology, then one can naturally define the L^2-Reidemeister and L^2-Ray–Singer torsions as L^2-elements on the associated determinant lines, without requiring the "determinant class condition". By combining with the main result on relative torsion in [7], they established an *extended* Cheeger–Müller theorem for these L^2-elements on odd dimensional infinite covering spaces for unimodular representations, which holds without the "determinant class condition".

In this paper, we will show that one can indeed prove a full extension of the generalized Cheeger–Müller theorem proved in [4] to the case of infinite covering spaces without requiring the "determinant class condition", in the framework of [6]. Moreover, we show that one can prove such a result by a direct adaptation of the strategy and method in [4] to this new situation. Thus, the proof will be purely analytical and avoid for example the use of the concept of relative torsion. The key ingredients to this proof include the basic L^2-estimates of the deformed de Rham–Witten complex developed in [8,32], the extended de Rham theorem established by Shubin [33], as well as the finite propagation speed technique which is crucial in adapting the local index computations in [4] into the infinite covering spaces situation. Moreover, as observed in [5,15], one does not need the full strength of the L^2-Helffer–Sjöstrand analysis of the L^2-Witten complex developed in [8]. This simplifies much of the matter.

As in [4], in order to establish the above-mentioned extended Cheeger–Müller theorem for covering spaces, one should first establish an anomaly formula for the L^2-Ray–Singer torsion on the determinant of the extended de Rham cohomology. Such a formula will also be established in the present paper, see Theorem 3.4 for details.

We should also mention that the results in [6–9,33] hold for more general finite type Hilbert modules over a finite von Neumann algebra on a closed manifold, here we will concentrate on the infinite covering spaces situation which corresponds to a special kind of Hilbert modules over a closed manifold.

This paper is organized as follows. In Section 2, we recall from [6] the definition of the determinant line of extended cohomology of a finite length Hilbert cochain $\mathscr{A}$-complex with $\mathscr{A}$ a finite von Neumann algebra, as well as the definition of the L^2-torsion element lying in this determinant line. We also construct the L^2-Milnor torsion element lying in the determinant line associated to the extended cohomology of an L^2-Thom–Smale cochain complex associated to a lifted Morse function on an infinite covering space satisfying the Thom–Smale transversality conditions. In Section 3, we recall from [6] the definition of the L^2-Ray–Singer torsion element lying in the determinant line of extended cohomology of de Rham complexes on infinite covering spaces, and establish an anomaly formula for it. In Section 4, we recall from [33] the de Rham theorem for the extended cohomologies, and state the main result of this paper, which is an extension of [4, Theorem 0.2], in Theorem 4.2. We prove this result modulus two intermediate results. These two intermediate results are then proved in Sections 5 and 6 respectively.

2. L^2-Milnor torsion on the determinant of extended cohomology

In this section, we define what we call the L^2-Milnor torsion element on an infinite covering space. Following [6], the element lies in the determinant of the extended cohomology of an L^2-Thom–Smale cochain complex.

This section is organized as follows. In Section 2.1, we recall the definition of the extended cohomology of a finite length Hilbert cochain complex over a finite von Neumann algebra carrying a finite, normal and faithful trace. In Section 2.2, we recall the definition of the determinant of a finitely generated Hilbert module over a finite von Neumann algebra. In Section 2.3, we recall the definition of the L^2-torsion element of a finite length Hilbert cochain complex. In Section 2.4, we define the L^2-Milnor torsion element.

2.1. Extended cohomology of a finite length Hilbert cochain complex

Let $\mathscr{A}$ be a finite von Neumann algebra carrying a fixed finite, normal and faithful trace $\tau : \mathscr{A} \to \mathbf{C}$. Let * denote the canonical involution on $\mathscr{A}$ defined by taking adjoint. Let $l^2(\mathscr{A})$ denote the Hilbert space completion of $\mathscr{A}$ with respect to the inner product given by the trace

$$\langle a, b\rangle = \tau(b^* a). \tag{2.1}$$

A finitely generated Hilbert module over $\mathscr{A}$ is a Hilbert space M admitting a continuous left $\mathscr{A}$-structure (with respect to the norm topology on $\mathscr{A}$) such that there exists an isometric $\mathscr{A}$-linear embedding of M into $l^2(\mathscr{A}) \otimes H$, for some finite dimensional Hilbert space H.

Let (C^*, ∂) be a finite length Hilbert cochain complex over $\mathscr{A}$,

$$(C^*, \partial) : 0 \to C^0 \stackrel{\partial_0}{\to} C^1 \stackrel{\partial_1}{\to} \cdots \stackrel{\partial_{n-1}}{\to} C^n \to 0, \tag{2.2}$$

where each C^i, $0 \leqslant i \leqslant n$, is a finitely generated Hilbert module over $\mathscr{A}$ and the coboundary maps are bounded $\mathscr{A}$-linear operators. Since the image spaces of these coboundary maps need not be closed, the tautological cohomology of (C^*, ∂) need not be a Hilbert space. This is why in general one studies the *reduced* cohomology of (C^*, ∂), which is defined by

$$H^*(C^*, \partial) = \bigoplus_{i=0}^{n} H^i(C^*, \partial) \quad \text{with } H^i(C^*, \partial) = \ker(\partial_i)/\overline{\mathrm{im}(\partial_{i-1})}, \ 0 \leqslant i \leqslant n, \tag{2.3}$$

where one takes obviously that $\partial_{-1} = 0$ and $\partial_n = 0$.

On the other hand, there are still ways to extract more information from (C^*, ∂), rather than just from $H^*(C^*, \partial)$. One such is to consider the *extended* cohomology in the sense of Farber (cf. [13,6]), which is defined by

$$\mathscr{H}^*(C^*, \partial) = \bigoplus_{i=0}^{n} \mathscr{H}^i(C^*, \partial) \quad \text{with } \mathscr{H}^i(C^*, \partial) = (\partial_{i-1} : C^{i-1} \to \ker(\partial_i)), \ 0 \leqslant i \leqslant n, \tag{2.4}$$

where $(\partial_{i-1} : C^{i-1} \to \ker(\partial_i))$, $0 \leqslant i \leqslant n$, lie in an abelian extended category. It constitutes of two parts: the projective part which is exactly the reduced cohomology defined in (2.3), as well as a torsion part $\mathscr{T}(\mathscr{H}^*(C^*, \partial)) = \oplus_{i=0}^{n} \mathscr{T}(\mathscr{H}^i(C^*, \partial))$ defined as an element in the above abelian extended category, with

$$\mathscr{T}(\mathscr{H}^i(C^*, \partial)) = (\partial_{i-1} : C^{i-1} \to \overline{\mathrm{im}(\partial_{i-1})}), \ 0 \leqslant i \leqslant n. \tag{2.5}$$

More precisely, one has

$$\mathscr{H}^*(C^*, \partial) = H^*(C^*, \partial) \oplus \mathscr{T}(\mathscr{H}^*(C^*, \partial)) \tag{2.6}$$

with

$$\mathscr{H}^i(C^*, \partial) = H^i(C^*, \partial) \oplus \mathscr{T}(\mathscr{H}^i(C^*, \partial)), \ 0 \leqslant i \leqslant n. \tag{2.7}$$

We refer to [13,6] for more details about the definition and basic properties of the above-mentioned abelian extended category as well as the extended cohomology.

2.2. The determinant of a finitely generated Hilbert module

Let M be a finitely generated Hilbert module over $\mathscr{A}$. Let $GL(M)$ denote the set of all bounded $\mathscr{A}$-linear automorphisms of M. Let $\mathscr{C}_M$ denote the set of all inner products on M such that if $\langle\ ,\ \rangle \in \mathscr{C}_M$, then there exists $A \in GL(M)$ such that

$$\langle u, v\rangle = \langle Au, v\rangle_M, \quad \text{for any } u, v \in M \tag{2.8}$$

with $\langle\ ,\ \rangle_M$ being the original inner product on M.

Following [9,6], we define the determinant line $\det M$ of M to be the real one dimensional vector space generated by symbols $\langle\ ,\ \rangle$, one for each element in $\mathscr{C}_M$ such that if $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ are two elements of $\mathscr{C}_M$ with

$$\langle u, v\rangle_2 = \langle Au, v\rangle_1, \quad \text{for any } u, v \in M, \tag{2.9}$$

for some $A \in GL(M)$, then as elements in $\det M$, one has

$$\langle\ ,\ \rangle_2 = \mathrm{Det}_\tau(A)^{-1/2} \cdot \langle\ ,\ \rangle_1, \tag{2.10}$$

where $\mathrm{Det}_\tau(A)$ is the Fuglede–Kadison determinant [14] of A.

For the sake of self-completeness, we recall the definition of $\mathrm{Det}_\tau(A)$ for any $A \in GL(M)$ and its basic properties from [9,6].

Let A_t, $0 \leqslant t \leqslant 1$, be a continuous piecewise smooth path $A_t \in GL(M)$ such that $A_0 = I$ and $A_1 = A$. The existence of such a path is clear as $GL(M)$ is known to be pathwise connected. Then define as in [9, (13); 6, (2.7)] that

$$\log \mathrm{Det}_\tau(A) = \int_0^1 \mathrm{Re}(\mathrm{Tr}_\tau[A_t^{-1}A_t'])\,\mathrm{d}t, \tag{2.11}$$

where A_t' is the derivative of A_t with respect to t, while Tr_τ is the canonically induced trace on the commutant of M (cf. [9, Proposition 1.8]).

It has been proved in [9] that the right-hand side of (2.11) does not depend on the choice of the path A_t, $0 \leqslant t \leqslant 1$. Moreover, we recall the following basic properties taken from [9, Theorem 1.10; 6, Theorem 2.11].

Proposition 2.1. *The function,*

$$\mathrm{Det}_\tau : GL(M) \to \mathbf{R}^{>0}, \tag{2.12}$$

called the Fuglede–Kadison determinant of A, *satisfies,*

(a) Det_τ *is a group homomorphism, that is,*

$$\mathrm{Det}_\tau(AB) = \mathrm{Det}_\tau(A) \cdot \mathrm{Det}_\tau(B), \quad \textit{for } A, B \in GL(M). \tag{2.13}$$

(b) *If* I *is the identity element in* $GL(M)$, *then*

$$\mathrm{Det}_\tau(\lambda I) = |\lambda|^{\tau(I)} \quad \textit{for } \lambda \in \mathbf{C},\ \lambda \neq 0. \tag{2.14}$$

(c) *One has*

$$\mathrm{Det}_{\lambda\tau}(A) = \mathrm{Det}_\tau(A)^\lambda \quad \textit{for } \lambda \in \mathbf{R}^{>0}. \tag{2.15}$$

(d) Det_τ *is continuous as a map* $GL(M) \to \mathbf{R}^{>0}$, *where* $GL(M)$ *is supplied with the norm topology.*

(e) *If* A_t, $t \in [0,1]$, *is a continuous piecewise smooth path in* $GL(M)$, *then*

$$\log\left[\frac{\mathrm{Det}_\tau(A_1)}{\mathrm{Det}_\tau(A_0)}\right] = \int_0^1 \mathrm{Re}(\mathrm{Tr}_\tau[A_t^{-1}A_t'])\,\mathrm{d}t. \tag{2.16}$$

(f) *Let* M, N *be two finitely generated Hilbert modules over* $\mathscr{A}$. *Let* $A \in GL(M)$, $B \in GL(N)$ *and let* $\gamma : N \to M$ *be a bounded* $\mathscr{A}$*-linear homomorphism. We extend* A, B, γ *to obvious endomorphisms on* $M \oplus N$ *by taking* $A|_N = 0$, $B|_M = 0$ *and* $\gamma|_M = 0$. *Then* $A + B + \gamma \in GL(M \oplus N)$ *and*

$$\mathrm{Det}_\tau(A + B + \gamma) = \mathrm{Det}_\tau(A) \cdot \mathrm{Det}_\tau(B). \tag{2.17}$$

Now come back to the determinant line $\det M$. Clearly, $\det M$ has a canonical orientation as the transition coefficient $\mathrm{Det}_\tau(A)^{-1/2}$ is always positive.

Following [9, 2.3], for any bounded $\mathscr{A}$-linear isomorphism $f : M \to N$ between two finitely generated Hilbert modules over $\mathscr{A}$, there induces canonically an isomorphism of determinant lines $f_* : \det M \to$

$\det N$, which preserves the orientations. Moreover, one has the following property which is recalled from [9, Proposition 2.5].

Proposition 2.2. *If $f \in GL(M)$, then the induced isomorphism $f_* : \det M \to \det M$ coincides with the multiplication by* $\mathrm{Det}_\tau(f) \in \mathbf{R}^{>0}$.

Remark 2.3. Following [9,6], one thinks of elements of $\det M$ as "densities" on M. In the $\mathscr{A} = \mathbf{C}$ case, this is dual to the considerations in [4] where one uses metrics on determinant lines instead of "volume forms".

2.3. Extended cohomology and the torsion element of a finite length cochain complex of Hilbert modules

Let (C^*, ∂) be a finite length Hilbert cochain complex over $\mathscr{A}$

$$(C^*, \partial) : 0 \to C^0 \xrightarrow{\partial_0} C^1 \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{n-1}} C^n \to 0 \tag{2.18}$$

as in (2.2). Let $\mathscr{H}^*(C^*, \partial) = \sum_{i=0}^n \mathscr{H}^i(C^*, \partial)$ denote the corresponding extended cohomology defined in (2.4), which admits the splitting to projective and torsion parts as in (2.5)–(2.7).

Following [6], we define for each $0 \leqslant i \leqslant n$ that

$$\det \mathscr{H}^i(C^*, \partial) := \det H^i(C^*, \partial) \otimes \det \mathscr{T}(\mathscr{H}^i(C^*, \partial)) \tag{2.19}$$

with

$$\det \mathscr{T}(\mathscr{H}^i(C^*, \partial)) := \det \overline{\mathrm{im}(\partial_{i-1})} \otimes (\det C^{i-1})^* \otimes \det \ker(\partial_{i-1}). \tag{2.20}$$

Definition 2.4. (i) We define the determinant line of (C^*, ∂) to be

$$\det(C^*, \partial) = \bigotimes_{i=0}^{n} (\det C^i)^{(-1)^i}. \tag{2.21}$$

(ii) We define the determinant line of $\mathscr{H}^*(C^*, \partial)$ to be

$$\det \mathscr{H}^*(C^*, \partial) = \bigotimes_{i=0}^{n} (\det \mathscr{H}^i(C^*, \partial))^{(-1)^i}. \tag{2.22}$$

The following result is recalled from [6, Proposition 7.2].

Proposition 2.5. *The cochain complex* (2.18) *defines a canonical isomorphism*

$$\nu_{(C^*,\partial)} : \det(C^*, \partial) \to \det \mathscr{H}^*(C^*, \partial). \tag{2.23}$$

For each $0 \leqslant i \leqslant n$, the (fixed) inner product on C^i determines an element $\sigma_i \in \det C^i$. They together determine an element

$$\sigma = \prod_{i=0}^{n} \sigma_i^{(-1)^i} \in \det(C^*, \partial). \tag{2.24}$$

Definition 2.6 (*[6, Definition 7.5]*). The positive element

$$\rho_{(C^*,\partial)} = \nu_{(C^*,\partial)}(\sigma) \in \det \mathscr{H}^*(C^*, \partial) \tag{2.25}$$

is called the torsion element of the cochain complex (C^*, ∂).

For any other **Z**-graded inner product $\langle\ ,\ \rangle' \in \mathscr{C}_C$, that is, there exists $A_i \in GL(C^i)$ for any $0 \leqslant i \leqslant n$ such that

$$\langle u, v\rangle_i' = \langle A_i u, v\rangle \quad \text{for any } u, v \in C^i, \tag{2.26}$$

let $\rho'_{(C^*,\partial)}$ denote the corresponding torsion element in $\det \mathscr{H}^*(C^*, \partial)$. Then one has the following anomaly formula for the torsion elements in $\det \mathscr{H}^*(C^*, \partial)$.

Proposition 2.7. *The following identity holds in* $\det \mathscr{H}^*(C^*, \partial)$:

$$\rho'_{(C^*,\partial)} = \rho_{(C^*,\partial)} \prod_{i=0}^{n} \mathrm{Det}_\tau(A_i)^{(-1)^{i+1}/2}. \tag{2.27}$$

Proof. Let σ_i' be the corresponding element in $\det C^i$. From (2.26), one has by definition (cf. (2.10))

$$\sigma_i' = \mathrm{Det}_\tau(A_i)^{-1/2}\sigma_i. \tag{2.28}$$

From Proposition 2.5 and from (2.24), (2.25) and (2.28), one gets (2.27). □

For any $0 \leqslant i \leqslant n$, let $\partial_i^* : C^{i+1} \to C^i$ denote the adjoint of ∂^i with respect to the inner products on C^i and C^{i+1}.

Let $\partial = \sum_{i=1}^n \partial_i : C^* \to C^*$, $\partial^* = \sum_{i=1}^n \partial_i^* : C^* \to C^*$ denote the induced homomorphisms on C^*. Then

$$\square = (\partial + \partial^*)^2 \tag{2.29}$$

preserves each C^i. Let $\square_i$ denote the restriction of $\square$ on C^i.

Now consider the special case where the cochain complex (C^*, ∂) is acyclic, i.e., for any $0 \leqslant i \leqslant n$, $\mathrm{im}(\partial_i) = \ker(\partial_{i+1})$ (in particular, this implies that $\mathrm{im}(\partial_i)$ is closed in C^{i+1}). Then the torsion element $\rho_{(C^*,\partial)} = \nu_{(C^*,\partial)}(\sigma) \in \det \mathscr{H}^*(C^*, \partial) \simeq \mathbf{R}$ can be thought of as a positive real number.

The following result has been proved in [6, Proposition 7.8].

Proposition 2.8. *If the cochain complex* (C^*, ∂) *is acyclic, then the following identity holds*:

$$\log \rho_{(C^*,\partial)} = \frac{1}{2} \sum_{i=0}^{n} (-1)^{i+1} i \log \mathrm{Det}_\tau(\square_i). \tag{2.30}$$

We refer to [6] for more complete discussions about the torsion elements in determinant lines.

2.4. L^2-Milnor torsion for covering spaces

Let $\Gamma \to \widetilde{M} \xrightarrow{\pi} M$ be a Galois covering of a closed smooth manifold M, with $\dim M = n$. We make the assumption that Γ is an infinite group, as the case of finite group has been dealt with for example in [20,21,5].

Let (F, ∇^F) be a complex flat vector bundle over M carrying the flat connection ∇^F. Let g^F be a Hermitian metric on F. Let $(\widetilde{F}, \nabla^{\widetilde{F}})$ denote the naturally lifted flat vector bundle over $\widetilde{M}$ obtained as the pullback of (F, ∇^F) through the covering map π. Let $g^{\widetilde{F}}$ be the naturally lifted Hermitian metric on $\widetilde{F}$.

Let (F^*, ∇^{F^*}) be the dual complex flat vector bundle of (F, ∇^F) carrying the flat connection ∇^{F^*}. Let g^{F^*} be the dual metric on F^*. Let $(\widetilde{F}^*, \nabla^{\widetilde{F}^*})$ and $g^{\widetilde{F}^*}$ denote the corresponding lifted objects on $\widetilde{M}$.

Let $f : M \to \mathbf{R}$ be a Morse function. Let g^{TM} be a Riemannian metric on TM such that the corresponding gradient vector field $-X = -\nabla f \in \Gamma(TM)$ satisfies the Smale transversality conditions (cf. [34]), that is, the unstable cells (of $-X$) intersect transversally with the stable cells. Let $\widetilde{f}$ (resp. $g^{T\widetilde{M}}$) denote the lifted Morse function of f on $\widetilde{M}$ (resp. lifted Riemannian metric on $T\widetilde{M}$). Then the corresponding gradient vector field $-\widetilde{X} = -\nabla\widetilde{f} \in \Gamma(T\widetilde{M})$ still satisfies the Smale transversality conditions. Set

$$B = \{x \in M;\, X(x) = 0\}, \quad \widetilde{B} = \{\widetilde{x} \in \widetilde{M};\, \widetilde{X}(\widetilde{x}) = 0\}. \tag{2.31}$$

For any $\widetilde{x} \in \widetilde{B}$, let $W^{\mathrm{u}}(\widetilde{x})$ (resp. $W^{\mathrm{s}}(\widetilde{x})$) denote the unstable (resp. stable) cell at $\widetilde{x}$, with respect to $-\widetilde{X}$. We also choose an orientation $O_{\widetilde{x}}^-$ (resp. $O_{\widetilde{x}}^+$) on $W^{\mathrm{u}}(\widetilde{x})$ (resp. $W^{\mathrm{s}}(\widetilde{x})$) in a Γ-invariant way.

Let $\widetilde{x}, \widetilde{y} \in \widetilde{B}$ satisfy the Morse index relation $\mathrm{ind}(\widetilde{y}) = \mathrm{ind}(\widetilde{x}) - 1$, then $\Gamma(\widetilde{x}, \widetilde{y}) = W^{\mathrm{u}}(\widetilde{x}) \cap W^{\mathrm{s}}(\widetilde{y})$ consists a finite number of integral curves γ of $-\widetilde{X}$. Moreover, for each $\gamma \in \Gamma(\widetilde{x}, \widetilde{y})$, by using the orientations chosen above, on can define a number $n_\gamma(\widetilde{x}, \widetilde{y}) = \pm 1$ as in [4, (1.28)].

If $\widetilde{x} \in \widetilde{B}$, let $[W^{\mathrm{u}}(\widetilde{x})]$ be the complex line generated by $W^{\mathrm{u}}(\widetilde{x})$. Set

$$C_*(W^{\mathrm{u}}, \widetilde{F}^*) = \bigoplus_{\widetilde{x} \in \widetilde{B}} [W^{\mathrm{u}}(\widetilde{x})] \otimes \widetilde{F}^*_{\widetilde{x}}, \tag{2.32}$$

$$C_i(W^{\mathrm{u}}, \widetilde{F}^*) = \bigoplus_{\widetilde{x} \in \widetilde{B}, \mathrm{ind}(\widetilde{x}) = i} [W^{\mathrm{u}}(\widetilde{x})] \otimes \widetilde{F}^*_{\widetilde{x}}. \tag{2.33}$$

If $\widetilde{x} \in \widetilde{B}$, the flat vector bundle $\widetilde{F}^*$ is canonically trivialized on $W^{\mathrm{u}}(\widetilde{x})$. In particular, if $\widetilde{x}, \widetilde{y} \in \widetilde{B}$ satisfy $\mathrm{ind}(\widetilde{y}) = \mathrm{ind}(\widetilde{x}) - 1$, and if $\gamma \in \Gamma(\widetilde{x}, \widetilde{y})$, $f^* \in \widetilde{F}^*_{\widetilde{x}}$, let $\tau_\gamma(f^*)$ be the parallel transport of $f^* \in \widetilde{F}^*_{\widetilde{x}}$ into $\widetilde{F}^*_{\widetilde{y}}$ along γ with respect to the flat connection $\nabla^{\widetilde{F}^*}$.

Clearly, for any $\widetilde{x} \in \widetilde{B}$, there is only a finite number of $\widetilde{y} \in \widetilde{B}$, satisfying together that $\mathrm{ind}(\widetilde{y}) = \mathrm{ind}(\widetilde{x}) - 1$ and $\Gamma(\widetilde{x}, \widetilde{y}) \neq \emptyset$.

If $\widetilde{x} \in \widetilde{B}$, $f^* \in \widetilde{F}^*_{\widetilde{x}}$, set

$$\partial(W^{\mathrm{u}}(\widetilde{x}) \otimes f^*) = \sum_{\widetilde{y} \in \widetilde{B}, \mathrm{ind}(\widetilde{y}) = \mathrm{ind}(\widetilde{x}) - 1} \sum_{\gamma \in \Gamma(\widetilde{x}, \widetilde{y})} n_\gamma(\widetilde{x}, \widetilde{y}) W^{\mathrm{u}}(\widetilde{y}) \otimes \tau_\gamma(f^*). \tag{2.34}$$

Then ∂ maps $C_i(W^{\mathrm{u}}, \widetilde{F}^*)$ into $C_{i-1}(W^{\mathrm{u}}, \widetilde{F}^*)$. Moreover, one has

$$\partial^2 = 0. \tag{2.35}$$

That is, $(C_*(W^{\mathrm{u}}, \widetilde{F}^*), \partial)$ forms a chain complex. We call it the L^2-Thom–Smale complex associated to $(\widetilde{M}, F, -X)$.

If $\widetilde{x} \in \widetilde{B}$, let $[W^{\mathrm{u}}(\widetilde{x})]^*$ be the dual line to $W^{\mathrm{u}}(\widetilde{x})$. Let $(C^*(W^{\mathrm{u}}, \widetilde{F}), \widetilde{\partial})$ be the complex which is dual to $(C_*(W^{\mathrm{u}}, \widetilde{F}^*), \partial)$. For $0 \leqslant i \leqslant n$, one has

$$C^i(W^{\mathrm{u}}, \widetilde{F}) = \bigoplus_{\widetilde{x}\in\widetilde{B}, \mathrm{ind}(\widetilde{x})=i} [W^{\mathrm{u}}(\widetilde{x})]^* \otimes \widetilde{F}_{\widetilde{x}}. \tag{2.36}$$

It is easy to verify that both ∂ and $\widetilde{\partial}$ are Γ-equivariant with respect to the natural Γ action on $C_i(W^{\mathrm{u}}, \widetilde{F}^*)$'s and $C^i(W^{\mathrm{u}}, \widetilde{F})$'s for $0 \leqslant i \leqslant n$, which is induced from the canonical deck action of Γ on $\widetilde{M}$.

Let $W^{\mathrm{u}}(\widetilde{x})^* \in [W^{\mathrm{u}}(\widetilde{x})]^*$ be such that $\langle W^{\mathrm{u}}(\widetilde{x}), W^{\mathrm{u}}(\widetilde{x})^*\rangle = 1$.

We now introduce an inner product on each $[W^{\mathrm{u}}(\widetilde{x})]^* \otimes \widetilde{F}_{\widetilde{x}}$ such that for any $f, f' \in \widetilde{F}_{\widetilde{x}}$,

$$\langle W^{\mathrm{u}}(\widetilde{x})^* \otimes f, W^{\mathrm{u}}(\widetilde{x})^* \otimes f'\rangle = \langle f, f'\rangle_{g^{\widetilde{F}_{\widetilde{x}}}}. \tag{2.37}$$

Let $l^2(\Gamma)$ denote the Hilbert space obtained through the L^2-completion of the group algebra of Γ with respect to the canonical trace on it.

For any $0 \leqslant i \leqslant n$, let $C^i(W^{\mathrm{u}}, \widetilde{F})$ carry the inner product obtained from those defined in (2.37) so that the splitting (2.36) is orthogonal. Then $C^i(W^{\mathrm{u}}, \widetilde{F})$ is a Hilbert space, which is isomorphic to the direct sum of n_i-copies of $l^2(\Gamma)$, where $n_i = \#\{x \in B : \mathrm{ind}(x) = i\}$ is the number of critical points of $f : M \to \mathbf{R}$ with Morse index i.

Let $\mathcal{N}(\Gamma)$ be the von Neumann algebra associated to Γ generated by the left regular representations on $l^2(\Gamma) \equiv l^2(\mathcal{N}(\Gamma))$. The canonical finite faithful trace on $\mathcal{N}(\Gamma)$ is given by the following formulas:

$$\tau_{\mathcal{N}(\Gamma)}(L_\alpha) = 0 \quad \text{if } \alpha \neq 1, \tag{2.38}$$

while

$$\tau_{\mathcal{N}(\Gamma)}(L_\alpha) = 1 \quad \text{if } \alpha = 1, \tag{2.39}$$

where L_α denote the left action of $\alpha \in \Gamma$ on $l^2(\Gamma)$. It induces canonically a trace on the commutant of any finitely generated Hilbert $\mathcal{N}(\Gamma)$-module (cf. [9, Proposition 1.8]), which will be denoted by $\mathrm{Tr}_{\mathcal{N}}$.

Then each $C^i(W^{\mathrm{u}}, \widetilde{F})$, $0 \leqslant i \leqslant n$, as well as $C^*(W^{\mathrm{u}}, \widetilde{F}) = \bigoplus_{i=0}^n C^i(W^{\mathrm{u}}, \widetilde{F})$, becomes a Hilbert $\mathcal{N}(\Gamma)$-module. Moreover, the coboundary map $\widetilde{\partial}$ is $\mathcal{N}(\Gamma)$-linear.

In summary, $(C^*(W^{\mathrm{u}}, \widetilde{F}), \widetilde{\partial})$ is a finite length Hilbert cochain complex over $\mathcal{N}(\Gamma)$ in the sense of Section 2.1. We call it the L^2-Thom–Smale cochain complex associated to $(\widetilde{M}, F, g^F, -X)$.

Definition 2.9. The torsion element in the determinant line of the extended cohomology of the L^2-Thom–Smale cochain complex $(C^*(W^{\mathrm{u}}, \widetilde{F}), \widetilde{\partial})$, in the sense of Definition 2.6, is called the L^2-Milnor torsion element associated to $(\widetilde{M}, F, g^F, -X)$, and is denoted by $\rho_{(\widetilde{M}, F, g^F, -X)}$.

From the anomaly formula (2.27), one deduces easily the following result.

Proposition 2.10. *If g_1^F is another Hermitian metric on the flat vector bundle F over M. Let $\rho_{(\widetilde{M}, F, g_1^F, -X)}$ denote the corresponding torsion element in* $\det \mathcal{H}(C^*(W^{\mathrm{u}}, \widetilde{F}), \widetilde{\partial})$, *then the following anomaly* *formula holds*:

$$\rho_{(\widetilde{M},F,g_1^F,-X)} = \rho_{(\widetilde{M},F,g^F,-X)} \prod_{x\in B} \det \left(\left(g^{F|_x} \right)^{-1} g_1^{F|_x} \right)^{(-1)^{\mathrm{ind}(x)+1}/2} \tag{2.40}$$

3. Infinite covering spaces and the L^2-Ray–Singer torsion on the determinant of extended de Rham cohomology

In this section, we recall the definition of the L^2-Ray–Singer torsion element in the infinite covering space case and prove an anomaly formula for it.

This section is organized as follows. In Section 3.1, we recall the definition of the extended de Rham cohomology associated to a lifted flat vector bundle on an infinite covering space. In Section 3.2, we define the L^2-Ray–Singer torsion element as an element in the determinant of the extended de Rham cohomology. In Section 3.3, we state an anomaly formula about the L^2-Ray–Singer torsion element. This anomaly formula is then proved in Section 3.4.

3.1. Infinite covering spaces and the extended de Rham cohomology

We make the same assumptions and use the same notations as in Section 2.4. Thus we have an infinite Γ covering space $\widetilde{M} \to M$, with $\dim M = n$, and a flat vector bundle (F, ∇^F) over M, etc. However, we do not use the Morse function and make transversality assumptions as in Section 2.4.

For any $0 \leqslant i \leqslant n$, denote

$$\Omega^i(\widetilde{M}, \widetilde{F}) = \Gamma(\Lambda^i(T^*\widetilde{M}) \otimes \widetilde{F}), \quad \Omega^*(\widetilde{M}, \widetilde{F}) = \bigoplus_{i=0}^{n} \Omega^i(\widetilde{M}, \widetilde{F}). \tag{3.1}$$

Let $d^{\widetilde{F}}$ denote the natural exterior differential on $\Omega^*(\widetilde{M}, \widetilde{F})$ induced from $\nabla^{\widetilde{F}}$ which maps each $\Omega^i(\widetilde{M}, \widetilde{F})$, $0 \leqslant i \leqslant n$, into $\Omega^{i+1}(\widetilde{M}, \widetilde{F})$.

The lifted Riemannian metric $g^{T\widetilde{M}}$ determines a canonical inner product on each $\Omega^i(\widetilde{M}, \widetilde{F})$, $0 \leqslant i \leqslant n$. Let $L^2(\Omega^i(\widetilde{M}, \widetilde{F}))$, $0 \leqslant i \leqslant n$, denote the Hilbert spaces obtained from the corresponding L^2-completion.

Then we can consider the L^2- de Rham complex

$$\begin{aligned}(L^2(\Omega^*(\widetilde{M}, \widetilde{F})), d^{\widetilde{F}}) : 0 \to L^2(\Omega^0(\widetilde{M}, \widetilde{F})) &\overset{d^{\widetilde{F}}}{\to} L^2(\Omega^1(\widetilde{M}, \widetilde{F})) \\ &\to \cdots \overset{d^{\widetilde{F}}}{\to} L^2(\Omega^n(\widetilde{M}, \widetilde{F})) \to 0.\end{aligned} \tag{3.2}$$

Let $d^{\widetilde{F}*} : \Omega^*(\widetilde{M}, \widetilde{F}) \to \Omega^*(\widetilde{M}, \widetilde{F})$ denote the formal adjoint of $d^{\widetilde{F}}$. Set

$$\widetilde{D} = d^{\widetilde{F}} + d^{\widetilde{F}*}, \quad \widetilde{D}^2 = (d^{\widetilde{F}} + d^{\widetilde{F}*})^2 = d^{\widetilde{F}*} d^{\widetilde{F}} + d^{\widetilde{F}} d^{\widetilde{F}*}. \tag{3.3}$$

Then the Laplacian $\widetilde{D}^2$ preserves the $\mathbf{Z}$-grading of $\Omega^*(\widetilde{M}, \widetilde{F})$.

For any $\mathscr{I} \subseteq \mathbf{R}$ and $0 \leqslant i \leqslant n$, denote by

$$L^2_{\mathscr{I}}(\Omega^i(\widetilde{M}, \widetilde{F})) \subseteq L^2(\Omega^i(\widetilde{M}, \widetilde{F})) \tag{3.4}$$

the image of the spectral projection $P_{\mathscr{I},i} : L^2(\Omega^i(\widetilde{M}, \widetilde{F})) \to L^2(\Omega^i(\widetilde{M}, \widetilde{F}))$ of $\widetilde{D}^2|_{L^2(\Omega^i(\widetilde{M},\widetilde{F}))}$ corresponding to $\mathscr{I}$.

We recall the following important result due to Shubin [33, Theorem 5.1].

Theorem 3.1. *Fix $\varepsilon > 0$. Then for any $0 \leqslant i \leqslant n$,*

(i) $L^2_{[0,\varepsilon]}(\Omega^i(\widetilde{M}, \widetilde{F})) \subset \Omega^i(\widetilde{M}, \widetilde{F})$, *i.e.*, $L^2_{[0,\varepsilon]}(\Omega^i(\widetilde{M}, \widetilde{F}))$ *consists of smooth forms*;

(ii) *when carrying the induced metric from that of $L^2(\Omega^i(\widetilde{M}, \widetilde{F}))$, $L^2_{[0,\varepsilon]}(\Omega^i(\widetilde{M}, \widetilde{F}))$ is a finitely generated Hilbert module over $\mathscr{N}(\Gamma)$.*

Now consider the finite length cochain complex of $\mathscr{N}(\Gamma)$-Hilbert modules

$$(L^2_{[0,\varepsilon]}(\Omega^*(\widetilde{M}, \widetilde{F})), d^{\widetilde{F}}) : 0 \to L^2_{[0,\varepsilon]}(\Omega^0(\widetilde{M}, \widetilde{F})) \xrightarrow{d^{\widetilde{F}}} L^2_{[0,\varepsilon]}(\Omega^1(\widetilde{M}, \widetilde{F})) \to \cdots \xrightarrow{d^{\widetilde{F}}} L^2_{[0,\varepsilon]}(\Omega^n(\widetilde{M}, \widetilde{F})) \to 0. \tag{3.5}$$

It is easy to verify that the extended cohomology of $(L^2_{[0,\varepsilon]}(\Omega^*(\widetilde{M}, \widetilde{F})), d^{\widetilde{F}})$ is independent of $\varepsilon > 0$. For if $\varepsilon' > \varepsilon > 0$, the subcomplex $(L^2_{(\varepsilon,\varepsilon']}(\Omega^*(\widetilde{M}, \widetilde{F})), d^{\widetilde{F}})$ of $(L^2_{[0,\varepsilon']}(\Omega^*(\widetilde{M}, \widetilde{F})), d^{\widetilde{F}})$ is acyclic. Moreover, it is easy to verify that this extended cohomology, up to bounded $\mathscr{N}(\Gamma)$-linear isomorphisms, does not depend on the choice of the metrics g^{TM} and g^F on TM and F, respectively. We denote it by $\mathscr{H}^{(2)}_{\mathrm{dR}}(\Omega^*(\widetilde{M}, \widetilde{F}), d^{\widetilde{F}})$.

Definition 3.2. The extended cohomology $\mathscr{H}^{(2)}_{\mathrm{dR}}(\Omega^*(\widetilde{M}, \widetilde{F}), d^{\widetilde{F}})$ defined above is called the L^2-extended de Rham cohomology associated to $\widetilde{M}$ and F.

3.2. L^2-Ray–Singer torsion on the determinant of the extended de Rham cohomology

We continue the discussion of the above subsection.

In view of Definition 2.6, for any $\varepsilon > 0$, the finite length cochain complex of $\mathscr{N}(\Gamma)$-Hilbert modules $(L^2_{[0,\varepsilon]}(\Omega^*(\widetilde{M}, \widetilde{F})), d^{\widetilde{F}})$ in (3.5) determines a torsion element in $\det \mathscr{H}^{(2)}_{\mathrm{dR}}(\Omega^*(\widetilde{M}, \widetilde{F}), d^{\widetilde{F}})$. We denote this torsion element by $T_{[0,\varepsilon]}(\widetilde{M}, F, g^{TM}, g^F)$.

Also, following [6, Section 12.2], for any $s \in \mathbf{C}$ with $\mathrm{Re}(s) > n/2$ and for $0 \leqslant i \leqslant n$, set

$$\zeta^i_{(\varepsilon,+\infty)}(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \mathrm{Tr}_{\mathscr{N}}[\exp(-t\widetilde{D}^2|_{L^2_{(\varepsilon,+\infty)}(\Omega^i(\widetilde{M},\widetilde{F}))})]\,\mathrm{d}t. \tag{3.6}$$

Then $\zeta^i_{(\varepsilon,+\infty)}(s)$ is analytic in s for $\mathrm{Re}(s) > n/2$ and can be extended to a meromorphic function on $\mathbf{C}$ which is holomorphic at $s = 0$ (cf. [8,19,22]). Let $T_{(\varepsilon,+\infty)}(\widetilde{M}, F, g^{TM}, g^F) \in \mathbf{R}^+$ be defined by

$$\log T_{(\varepsilon,+\infty)}(\widetilde{M}, F, g^{TM}, g^F) = \frac{1}{2} \sum_{i=0}^{n} (-1)^i i \left.\frac{\partial \zeta^i_{(\varepsilon,+\infty)}(s)}{\partial s}\right|_{s=0}. \tag{3.7}$$

By [6, Lemma 12.4], the product $T_{[0,\varepsilon]}(\widetilde{M}, F, g^{TM}, g^F) \cdot T_{(\varepsilon,+\infty)}(\widetilde{M}, F, g^{TM}, g^F)$ in $\det \mathscr{H}^{(2)}_{\mathrm{dR}}(\Omega^*(\widetilde{M}, \widetilde{F}), d^{\widetilde{F}})$ does not depend on $\varepsilon > 0$.

Definition 3.3 (*[6, Definition 12.5]*). The L^2-Ray–Singer torsion element associated to $(\widetilde{M}, F, g^{TM}, g^F)$ is the positive element in the determinant of the extended de Rham cohomology $\mathscr{H}^{(2)}_{\mathrm{dR}}(\Omega^*(\widetilde{M}, \widetilde{F}), d^{\widetilde{F}})$ defined by

$$T^{(2)}_{\mathrm{RS}}(\widetilde{M}, F, g^{TM}, g^F) = T_{[0,\varepsilon]}(\widetilde{M}, F, g^{TM}, g^F) \cdot T_{(\varepsilon,+\infty)}(\widetilde{M}, F, g^{TM}, g^F). \tag{3.8}$$

In [6, Section 13], Braverman, Carey, Farber and Mathai showed that if dim $M = n$ is odd and (F, ∇^F, g^F) is unimodular, then $T^{(2)}_{\mathrm{RS}}(\widetilde{M}, F, g^{TM}, g^F)$ does not depend on g^{TM}. They proved this result by using the L^2-Cheeger–Müller type theorem they proved in this situation. In the next subsections, we will give a direct proof of a general anomaly formula extending both the above result as well as the Bismut–Zhang anomaly formula [4, Theorem 0.1] of Ray–Singer metrics in the $\Gamma = \{e\}$ case.

3.3. An anomaly formula for the L^2-Ray–Singer torsion elements

We continue the discussion of the above subsection.

Let $\theta(F, g^F) \in \Omega^1(M)$ be defined by (cf. [4, Definiton 4.1])

$$\theta(F, g^F) = \mathrm{Tr}_F[(g^F)^{-1}\nabla^F g^F]. \tag{3.9}$$

Then $\theta(F, g^F)$ is a closed one form on M (cf. [4, Proposition 4.6]).

Let ∇^{TM} denote the Levi–Civita connection associated to the Riemannian metric g^{TM} on *TM*. Let $R^{TM} = (\nabla^{TM})^2$ be the curvature of ∇^{TM}. Let $e(TM, \nabla^{TM}) \in \Omega^n(M, o(TM))$ be the associated Euler form defined by (cf. [4, (3.17); 37, Chapter 3])

$$e(TM, \nabla^{TM}) = \mathrm{Pf}\left(\frac{R^{TM}}{2\pi}\right). \tag{3.10}$$

Let g'^{TM} be another Riemannian metric on *TM* and ∇'^{TM} be the associated Levi–Civita connection. Let $\widetilde{e}(TM, \nabla^{TM}, \nabla'^{TM})$ be the Chern–Simons class of $n-1$ smooth forms on M valued in $o(TM)$, which is defined modulo exact $n-1$ forms, such that

$$d\widetilde{e}(TM, \nabla^{TM}, \nabla'^{TM}) = e(TM, \nabla'^{TM}) - e(TM, \nabla^{TM}) \tag{3.11}$$

(cf. [4, (4.10)]). Of course, if n is odd,

$$\widetilde{e}(TM, \nabla^{TM}, \nabla'^{TM}) = 0. \tag{3.12}$$

Let g'^F be another metric on F. Let ρ_F, ρ'_F denote the volume element on det F induced by g^F, g'^F, respectively. Then $\rho'_F/\rho_F \in \mathbf{R}^+$. One verifies easily that

$$\log\frac{\rho'_F}{\rho_F} = -\frac{1}{2}\log\det{}_F((g^F)^{-1}g'^F). \tag{3.13}$$

From (3.9) and (3.13), one deduces that

$$d\log\frac{\rho'_F}{\rho_F} = \frac{1}{2}(\theta(F, g^F) - \theta(F, g'^F)) \tag{3.14}$$

(cf. [4, (4.12)]).

Let $T^{(2)}_{\mathrm{RS}}(\widetilde{M}, F, g'^{TM}, g'^{F}) \in \det \mathscr{H}^{(2)}_{\mathrm{dR}}(\Omega^*(\widetilde{M}, \widetilde{F}), d^{\widetilde{F}})$ denote the L^2-Ray–Singer torsion element associated to g'^{TM} and g'^{F}. Then the positive real number

$$\frac{T^{(2)}_{\mathrm{RS}}(\widetilde{M}, F, g'^{TM}, g'^{F})}{T^{(2)}_{\mathrm{RS}}(\widetilde{M}, F, g^{TM}, g^{F})} \in \mathbf{R}^+$$

is well-defined.

We can now state the anomaly formula for L^2-Ray–Singer torsion elements as follows.

Theorem 3.4. *The following identity holds*:

$$\log \frac{T^{(2)}_{\mathrm{RS}}(\widetilde{M}, F, g'^{TM}, g'^{F})}{T^{(2)}_{\mathrm{RS}}(\widetilde{M}, F, g^{TM}, g^{F})} = \int_M \left(\log \frac{\rho'_F}{\rho_F}\right) e(TM, \nabla^{TM}) + \frac{1}{2} \int_M \theta(F, g'^{F}) \widetilde{e}(TM, \nabla^{TM}, \nabla'^{TM}). \tag{3.15}$$

In particular, if $\dim M = n$ *is odd, then*

$$\log \frac{T^{(2)}_{\mathrm{RS}}(\widetilde{M}, F, g'^{TM}, g'^{F})}{T^{(2)}_{\mathrm{RS}}(\widetilde{M}, F, g^{TM}, g^{F})} = 0. \tag{3.16}$$

Theorem 3.4 will be proved in the next subsection.

Remark 3.5. Eq. (3.15) generalizes the anomaly formula of Bismut–Zhang [4, Theorem 0.1] to the infinite covering space case. Also, when (g^F, ∇^F) and (g'^F, ∇^F) are unimodular, (3.16) is a special case of [6, Theorem 13.8].

3.4. A proof of Theorem 3.4

We first give a slightly more flexible formula of the L^2-Ray–Singer torsion element $T^{(2)}_{\mathrm{RS}}(\widetilde{M}, F, g^{TM}, g^{F})$ defined in (3.8).

For any $a > 0$, let $(C^*, d^{\widetilde{F}})$ be a finite length $\mathscr{N}(\Gamma)$-Hilbert cochain subcomplex of $(L^2(\Omega^*(\widetilde{M}, \widetilde{F})), d^{\widetilde{F}})$ such that $(L^2_{[0,a]}(\Omega^*(\widetilde{M}, \widetilde{F})), d^{\widetilde{F}})$ is a subcomplex of $(C^*, d^{\widetilde{F}})$. That is, as $\mathscr{N}(\Gamma)$-Hilbert cochain complexes, one has

$$(L^2_{[0,a]}(\Omega^*(\widetilde{M}, \widetilde{F})), d^{\widetilde{F}}) \subseteq (C^*, d^{\widetilde{F}}). \tag{3.17}$$

Let $d^{\widetilde{F}*}_{C^*} : C^* \to C^*$ be the formal adjoint of $d^{\widetilde{F}} : C^* \to C^*$ with respect to the induced Hilbert metric on C^* from that of $L^2(\Omega^*(\widetilde{M}, \widetilde{F}))$. Set

$$D_{C^*} = d^{\widetilde{F}} + d^{\widetilde{F}*}_{C^*}, \quad D^2_{C^*} = (d^{\widetilde{F}} + d^{\widetilde{F}*}_{C^*})^2 = d^{\widetilde{F}*}_{C^*} d^{\widetilde{F}} + d^{\widetilde{F}} d^{\widetilde{F}*}_{C^*} : C^* \to C^*. \tag{3.18}$$

Then $D^2_{C^*}$ preserves the $\mathbf{Z}$-grading of C^*. Moreover, one has

$$D^2_{C^*} = \widetilde{D}^2 : L^2_{[0,a]}(\Omega^*(\widetilde{M}, \widetilde{F})) \to L^2_{[0,a]}(\Omega^*(\widetilde{M}, \widetilde{F})). \tag{3.19}$$

For any $0\leqslant i\leqslant n$, let $D^2_{C^i}$ denote the restriction of $D^2_{C^*}$ on C^i.

By (3.17) it is clear that the extended cohomology of $(C^*, d^{\widetilde{F}})$ is identical to that of $(L^2_{[0,a]}(\Omega^*(\widetilde{M},\widetilde{F})), d^{\widetilde{F}})$. That is, one has

$$\mathscr{H}^*(C^*, d^{\widetilde{F}}) \equiv \mathscr{H}^{(2)}_{\mathrm{dR}}(\Omega^*(\widetilde{M},\widetilde{F}), d^{\widetilde{F}}). \tag{3.20}$$

From (3.20), one sees that $(C^*, d^{\widetilde{F}})$ induces canonically an L^2-torsion element in $\det \mathscr{H}^{(2)}_{\mathrm{dR}}(\Omega^*(\widetilde{M},\widetilde{F}), d^{\widetilde{F}})$. We denote it by

$$T_{(C^*, d^{\widetilde{F}})} \in \det \mathscr{H}^{(2)}_{\mathrm{dR}}(\Omega^*(\widetilde{M},\widetilde{F}), d^{\widetilde{F}}). \tag{3.21}$$

For any $s\in\mathbf{C}$ with $\mathrm{Re}(s)>n/2$ and for $0\leqslant i\leqslant n$, set

$$\zeta^i_{C^*,\perp}(s) = \frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}(\mathrm{Tr}_{\mathscr{N}}[\exp(-t\widetilde{D}^2|_{L^2(\Omega^i(\widetilde{M},\widetilde{F}))})] - \mathrm{Tr}_{\mathscr{N}}[\exp(-tD^2_{C^i})])\,\mathrm{d}t. \tag{3.22}$$

If we rewrite the right-hand side of (3.22) as

$$\begin{aligned}&\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}(\mathrm{Tr}_{\mathscr{N}}[\exp(-t\widetilde{D}^2|_{L^2(\Omega^i(\widetilde{M},\widetilde{F}))})] - \mathrm{Tr}_{\mathscr{N}}[\exp(-t\widetilde{D}^2|_{L^2_{[0,a]}(\Omega^i(\widetilde{M},\widetilde{F}))})])\,\mathrm{d}t\\ &\quad+\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}(\mathrm{Tr}_{\mathscr{N}}[\exp(-t\widetilde{D}^2|_{L^2_{[0,a]}(\Omega^i(\widetilde{M},\widetilde{F}))})] - \mathrm{Tr}_{\mathscr{N}}[\exp(-tD^2_{C^i})])\,\mathrm{d}t,\end{aligned} \tag{3.23}$$

then in view of (3.17) and (3.19), one sees that each $\zeta^i_{C^*,\perp}(s)$, $0\leqslant i\leqslant n$, is a holomorphic function for $\mathrm{Re}(s)>n/2$ and can be extended to a meromorphic function on $\mathbf{C}$ which is holomorphic at $s=0$. Let $T_{(C^*,d^{\widetilde{F}}),\perp}\in\mathbf{R}^+$ be defined by

$$\log T_{(C^*,d^{\widetilde{F}}),\perp} = \frac{1}{2}\sum_{i=0}^n(-1)^i i\left.\frac{\partial\zeta^i_{C^*,\perp}(s)}{\partial s}\right|_{s=0}. \tag{3.24}$$

Proposition 3.6. *The following identity holds in* $\det \mathscr{H}^{(2)}_{\mathrm{dR}}(\Omega^*(\widetilde{M},\widetilde{F}), d^{\widetilde{F}})$:

$$T^{(2)}_{\mathrm{RS}}(\widetilde{M}, F, g^{TM}, g^F) = T_{(C^*,d^{\widetilde{F}})}\cdot T_{(C^*,d^{\widetilde{F}}),\perp}. \tag{3.25}$$

Proof. By (3.17), one can split the cochain complex $(C^*, d^{\widetilde{F}})$ to the direct sum of $(L^2_{[0,a]}(\Omega^*(\widetilde{M},\widetilde{F})), d^{\widetilde{F}})$ and its orthogonal complement, which is clearly acyclic. Proposition 3.6 then follows easily from Proposition 2.8, Definition 3.3 and formulas (3.6), (3.7) and (3.21)–(3.24). □

We now come to the proof of Theorem 3.4.

Let g^{TM}_u (resp. g^F_u), $0\leqslant u\leqslant 1$, be a smooth path of metrics on *TM* (resp. *F*) such that $g^{TM}_0=g^{TM}$, $g^{TM}_1=g'^{TM}$ (resp. $g^F_0=g^F$, $g^F_1=g'^F$).

When dealing with objects associated with (g^{TM}_u, g^F_u), we will use a subscript "*u*" to indicate. While at $u=0$ we usually omit this subscript indication.

Proposition 3.7. *For any $u \in [0,1]$, one can construct a finite length $\mathcal{N}(\Gamma)$-Hilbert cochain subcomplex $(C^*(u), d^{\widetilde{F}})$ of $(L^2_u(\Omega^*(\widetilde{M}, \widetilde{F})), d^{\widetilde{F}})$ such that*

(i) *One has the inclusion relation of cochain complexes*

$$(L^2_{u,[0,1]}(\Omega^*(\widetilde{M}, \widetilde{F})), d^{\widetilde{F}}) \subseteq (C^*(u), d^{\widetilde{F}}). \tag{3.26}$$

(ii) *The cochain complex $(C^*(u), d^{\widetilde{F}})$ depends smoothly on $u \in [0,1]$.*

Proof. We will use a trick due to Fangbing Wu [36] (cf. [18, Section 2.4]).

For any $k > 0$ and $0 \leqslant u \leqslant 1$, let $P_{[0,k],u}$ denote the orthogonal projection from $L^2_u(\Omega^*(\widetilde{M}, \widetilde{F}))$ onto $L^2_{[0,k]}(\Omega^*(\widetilde{M}, \widetilde{F})) \subset L^2_u(\Omega^*(\widetilde{M}, \widetilde{F}))$, where we view $L^2_{[0,k]}(\Omega^*(\widetilde{M}, \widetilde{F})) \subset L^2(\Omega^*(\widetilde{M}, \widetilde{F}))$ as a (closed) subspace in $L^2_u(\Omega^*(\widetilde{M}, \widetilde{F}))$.

It is clear that for any fixed $k > 0$, $P_{[0,k],u}$ depends smoothly on $u \in [0,1]$.

Let $\chi : \mathbf{R} \to [0,1]$ be a smooth function such that $\chi(t) = 0$ if $t \leqslant 2$ while $\chi(t) = 1$ if $t \geqslant 5$.

Then $\chi(\widetilde{D}^2_u) : L^2_u(\Omega^*(\widetilde{M}, \widetilde{F})) \to L^2_u(\Omega^*(\widetilde{M}, \widetilde{F}))$ depends smoothly on $u \in [0,1]$. Moreover, for any $0 \leqslant u \leqslant 1$, in view of Theorem 3.1, one sees easily that the closure of the image of

$$\mathrm{Id} - \chi(\widetilde{D}^2_u) : L^2_u(\Omega^*(\widetilde{M}, \widetilde{F})) \to L^2_u(\Omega^*(\widetilde{M}, \widetilde{F}))$$

is a finitely generated $\mathcal{N}(\Gamma)$-Hilbert module.

One then checks easily that as $k \to +\infty$,

$$(\mathrm{Id} - \chi(\widetilde{D}^2_u))P_{[0,k],u} \to \mathrm{Id} - \chi(\widetilde{D}^2_u) \tag{3.27}$$

in the operator norm. Moreover, the convergence is uniform with respect to $u \in [0,1]$.

Let now $k > 10$ be fixed such that for any $u \in [0,1]$, one has

$$\|(\mathrm{Id} - \chi(\widetilde{D}^2_u))P_{[0,k],u} - (\mathrm{Id} - \chi(\widetilde{D}^2_u))\|_{0,u} < \tfrac{1}{2}, \tag{3.28}$$

where in the subscript of the left-hand side, “0” indicates the L^2-norm, while “u” indicates the parameter $u \in [0,1]$.

Rewrite (3.28) as

$$\|(\mathrm{Id} - P_{[0,k],u}) - \chi(\widetilde{D}^2_u)(\mathrm{Id} - P_{[0,k],u})\|_{0,u} < \tfrac{1}{2}. \tag{3.29}$$

Now we apply [36, Lemma 2.4] (recalled in [18, Lemma 5 in Section 2.4]), which can be thought of as a noncommutative generalization of [24, Lemma 1].

For any $u \in [0,1]$, we denote $P_{[0,k],u}(\chi)$ the orthogonal projection from $L^2_u(\Omega^*(\widetilde{M}, \widetilde{F}))$ onto $\mathrm{Im}(\chi(\widetilde{D}^2_u)(\mathrm{Id} - P_{[0,k],u}))$, which by Wu [36, Lemma 2.4] is closed. From [36, Lemma 2.4], one knows that $P_{[0,k],u}(\chi)$ is smooth with respect to $u \in [0,1]$.

For any $u \in [0,1]$, set

$$C^*(u) = (\mathrm{Im}(P_{[0,k],u}(\chi)))^{\perp} = (\mathrm{Im}(\chi(\widetilde{D}^2_u)(\mathrm{Id} - P_{[0,k],u})))^{\perp} \subset L^2_u(\Omega^*(\widetilde{M}, \widetilde{F})). \tag{3.30}$$

We first observe that all the operators appeared above are Γ-equivariant and preserve the obvious $\mathbf{Z}$-grading through out the context. Then $C^*(u)$ admits an obvious $\mathbf{Z}$-grading.

We now show that $d^{\widetilde{F}}$ preserves $C^*(u)$ for any $u \in [0,1]$.

Take any $x \in C^*(u)$, by definition, we know that for any $y \in L^2_u(\Omega^*(\widetilde{M}))$, one has

$$\langle \chi(\widetilde{D}^2_u)(\mathrm{Id} - P_{[0,k],u})y, x\rangle_u = 0. \tag{3.31}$$

Let $d_u^{\widetilde{F}*}$ denote the adjoint of $d^{\widetilde{F}}$ with respect to the inner product on $L^2_u(\Omega^*(\widetilde{M}, \widetilde{F}))$.

From the obvious identity

$$d^{\widetilde{F}} P_{[0,k],u} = P_{[0,k],u} d^{\widetilde{F}} P_{[0,k],u}, \tag{3.32}$$

one gets

$$P_{[0,k],u} d_u^{\widetilde{F}*} P_{[0,k],u} = P_{[0,k],u} d_u^{\widetilde{F}*}, \tag{3.33}$$

which implies

$$d_u^{\widetilde{F}*}(\mathrm{Id} - P_{[0,k],u}) = (\mathrm{Id} - P_{[0,k],u}) d_u^{\widetilde{F}*}(\mathrm{Id} - P_{[0,k],u}). \tag{3.34}$$

From (3.31) and (3.34) one deduces that for any $x \in C^*(u)$ and $y \in L^2_u(\Omega^*(\widetilde{M}))$, one has

$$\begin{aligned}\langle \chi(\widetilde{D}^2_u)(\mathrm{Id} - P_{[0,k],u})y, d^{\widetilde{F}}x\rangle_u &= \langle \chi(\widetilde{D}^2_u) d_u^{\widetilde{F}*}(\mathrm{Id} - P_{[0,k],u})y, x\rangle_u \\ &= \langle \chi(\widetilde{D}^2_u)(\mathrm{Id} - P_{[0,k],u}) d_u^{\widetilde{F}*}(\mathrm{Id} - P_{[0,k],u})y, x\rangle_u = 0,\end{aligned} \tag{3.35}$$

which implies that $d^{\widetilde{F}}x \in C^*(u)$.

Thus, for any $u \in [0, 1]$, $(C^*(u), d^{\widetilde{F}})$ is a cochain subcomplex of $L^2_u(\Omega^*(\widetilde{M}, \widetilde{F}))$. From (3.30), one sees it depends smoothly on $u \in [0, 1]$, which proves the second part of the proposition.

On the other hand, by the definition of χ and by (3.30), one gets (3.26) immediately.

It remains to show that as an $\mathcal{N}(\Gamma)$-Hilbert module, $C^*(u)$ is finitely generated. By Theorem 3.1, this follows from the following result.

Lemma 3.8. *There exists $K > k$ such that for any $u \in [0, 1]$, one has*

$$L^2_{u,[K,+\infty)}(\Omega^*(\widetilde{M}, \widetilde{F})) \subseteq \mathrm{Im}(P_{[0,k],u}(\chi)). \tag{3.36}$$

Proof. Let $P_{[K,+\infty)}(u)$ denote the orthogonal projection from $L^2_u(\Omega^*(\widetilde{M}, \widetilde{F}))$ onto $L^2_{u,[K,+\infty)}(\Omega^*(\widetilde{M}, \widetilde{F}))$.

Since $k > 10$, by the definition of χ, we need only to show that

$$\begin{aligned}L^2_{u,[K,+\infty)}(\Omega^*(\widetilde{M}, \widetilde{F})) &= \mathrm{Im}(P_{[K,+\infty)}(u) P_{[0,k],u}(\chi)) \\ &= \mathrm{Im}(P_{[K,+\infty)}(u)(\mathrm{Id} - P_{[0,k],u})).\end{aligned} \tag{3.37}$$

In order to prove (3.37), observe first that there exist constants $A > 0$, $B > 0$ and $C > 0$ such that for any $u \in [0, 1]$, one has that (cf. with (3.28) for subscript notation convention)

$$\frac{1}{C}\|\cdot\|_{0,u} \leqslant \|\cdot\|_0 \leqslant C\|\cdot\|_{0,u}, \tag{3.38}$$

and that for any $x \in \Omega^*(\widetilde{M}, \widetilde{F})$, one has

$$\|\widetilde{D}_u x\|_{0,u} \leqslant A(\|\widetilde{D}x\|_0 + B\|x\|_0). \tag{3.39}$$

Now assume that (3.37) does not hold for $K > k$ and some $u \in [0, 1]$, then there exists a nonzero element $x \in L^2_{u,[K,+\infty)}(\Omega^*(\widetilde{M}, \widetilde{F}))$ such that for any $y \in L^2_u(\Omega^*(\widetilde{M}, \widetilde{F}))$, one has

$$\langle P_{[K,+\infty)}(u)(\mathrm{Id} - P_{[0,k],u})y, x\rangle_u = \langle(\mathrm{Id} - P_{[0,k],u})y, x\rangle_u = 0. \tag{3.40}$$

From (3.40), one sees that $x \in \mathrm{Im}(P_{[0,k],u}) = L^2_{[0,k]}(\Omega^*(\widetilde{M}, \widetilde{F}))$. Thus, one has

$$\|\widetilde{D}x\|_0 \leqslant \sqrt{k}\|x\|_0. \tag{3.41}$$

From (3.38), (3.39) and (3.41), one gets

$$\|\widetilde{D}_u x\|_{0,u} \leqslant C(A\sqrt{k} + B)\|x\|_0. \tag{3.42}$$

On the other hand, since $x \in L^2_{u,[K,+\infty)}(\Omega^*(\widetilde{M}, \widetilde{F}))$, by (3.38) one has

$$\|\widetilde{D}_u x\|_{0,u} \geqslant \sqrt{K}\|x\|_{0,u} \geqslant \frac{\sqrt{K}}{C}\|x\|_0. \tag{3.43}$$

From (3.42), (3.43) and the assumption that $x \neq 0$, one finds

$$K \leqslant (C^2(A\sqrt{k} + B))^2. \tag{3.44}$$

Thus (3.36) holds when $K > k + (C^2(A\sqrt{k} + B))^2$.

The proof of Lemma 3.8 is completed. □

The proof of Proposition 3.7 is thus also completed. □

Remark 3.9. The method in the proof of Lemma 3.8 can also be used to give a direct analytic proof of (3.27).

We now come back to the proof of Theorem 3.4.

By (3.25) and Proposition 3.7, one gets that for any $0 \leqslant u \leqslant 1$,

$$T^{(2)}_{\mathrm{RS}}(\widetilde{M}, F, g^{TM}_u, g^F_u) = T_{(C^*(u),d^{\widetilde{F}})} \cdot T_{(C^*(u),d^{\widetilde{F}}),\perp}. \tag{3.45}$$

For any $s \in \mathbf{C}$ with $\mathrm{Re}(s) > n/2$ and $0 \leqslant u \leqslant 1$, set

$$\theta_u(s) = \sum_{i=0}^{n} (-1)^i i \zeta^i_{C^*(u),\perp}(s). \tag{3.46}$$

Let N denote the number operator on $\Omega^*(\widetilde{M}, \widetilde{F})$ acting by multiplication by i on $\Omega^i(\widetilde{M}, \widetilde{F})$. It extends to obvious actions on L^2-completions.

From (3.22) and (3.46), one can rewrite $\theta_u(s)$ as

$$\theta_u(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1}(\mathrm{Tr}_{\mathscr{N},s}[N \exp(-t\widetilde{D}^2_u)] - \mathrm{Tr}_{\mathscr{N},s}[N \exp(-tD^2_{C^*(u)})])\, \mathrm{d}t, \tag{3.47}$$

where $\mathrm{Tr}_{\mathscr{N},s}[\cdot] = \mathrm{Tr}_{\mathscr{N}}[(-1)^N \cdot]$ is the supertrace in the sense of Quillen [30], taking on bounded $\mathscr{N}(\Gamma)$-linear operators acting on $\Omega^*(\widetilde{M}, \widetilde{F})$ as well as their L^2-completions. In what follows we will also adopt the notation in [30] of supercommutators.

By proceeding as in [19, Lemma 8], one gets the following analogue of [19, (35)] in the current situation

$$\frac{\partial}{\partial u}\mathrm{Tr}_{\mathcal{N},s}[N\exp(-t\widetilde{D}_u^2)] = -t\mathrm{Tr}_{\mathcal{N},s}\left[N\frac{\partial \widetilde{D}_u^2}{\partial u}\exp(-t\widetilde{D}_u^2)\right]. \tag{3.48}$$

We now proceed similarly as in [31] and [2, Theorem 1.18].

Let $*_u^{\widetilde{F}}$ denote the Hodge star operator mapping from $\Omega^*(\widetilde{M},\widetilde{F})$ to $\Omega^*(\widetilde{M},\widetilde{F}\otimes o(T\widetilde{M}))$ with respect to g^{TM} and g^F, where $o(T\widetilde{M})$ is the orientation bundle of $o(T\widetilde{M})$. Let $*_u$ be the usual Hodge star operator associated to g^{TM} for the $F=\mathbf{C}$ case (cf. [37, Chapter 4]). Then one has

$$Q_u := (*_u^{\widetilde{F}})^{-1}\frac{\partial *_u^{\widetilde{F}}}{\partial u} = (*_u)^{-1}\frac{\partial *_u}{\partial u} + (g_u^{\widetilde{F}})^{-1}\frac{\partial g_u^{\widetilde{F}}}{\partial u}. \tag{3.49}$$

A direct verification shows that, when acting on $\Omega^i(\widetilde{M},\widetilde{F})$, one has

$$d_u^{\widetilde{F}*} = (-1)^i(*_u^{\widetilde{F}})^{-1}d^{\widetilde{F}\otimes o(T\widetilde{M})}*_u^{\widetilde{F}}. \tag{3.50}$$

From (3.49) and (3.50), one gets

$$\frac{\partial}{\partial u}d_u^{\widetilde{F}*} = [d_u^{\widetilde{F}*}, Q_u]. \tag{3.51}$$

On the other hand, by (3.3) one verifies directly that

$$[\widetilde{D}_u, N] = -d^{\widetilde{F}} + d_u^{\widetilde{F}*}. \tag{3.52}$$

From (3.3) one deduces that

$$\begin{aligned}\mathrm{Tr}_{\mathcal{N},s}\left[N\frac{\partial \widetilde{D}_u^2}{\partial u}\exp(-t\widetilde{D}_u^2)\right] &= \mathrm{Tr}_{\mathcal{N},s}\left[N\left[\widetilde{D}_u,\frac{\partial}{\partial u}d_u^{\widetilde{F}*}\right]\exp(-t\widetilde{D}_u^2)\right]\\ &= \mathrm{Tr}_{\mathcal{N},s}\left[N\left[\widetilde{D}_u,\frac{\partial d_u^{\widetilde{F}*}}{\partial u}\exp(-t\widetilde{D}_u^2)\right]\right]\\ &= \mathrm{Tr}_{\mathcal{N},s}\left[[N,\widetilde{D}_u]\frac{\partial d_u^{\widetilde{F}*}}{\partial u}\exp(-t\widetilde{D}_u^2)\right]\\ &\quad + \mathrm{Tr}_{\mathcal{N},s}\left[\widetilde{D}_u, N\frac{\partial d_u^{\widetilde{F}*}}{\partial u}\exp(-t\widetilde{D}_u^2)\right]. \end{aligned} \tag{3.53}$$

Clearly,

$$\begin{aligned}\mathrm{Tr}_{\mathcal{N},s}\left[\widetilde{D}_u, N\frac{\partial d_u^{\widetilde{F}*}}{\partial u}\exp(-t\widetilde{D}_u^2)\right] &= \mathrm{Tr}_{\mathcal{N},s}\left[\exp\left(-\frac{t}{2}\widetilde{D}_u^2\right)\widetilde{D}_u, N\frac{\partial d_u^{\widetilde{F}*}}{\partial u}\exp\left(-\frac{t}{2}\widetilde{D}_u^2\right)\right]\\ &= 0.\end{aligned} \tag{3.54}$$

Using the fact that $d_u^{\widetilde{F}*}$ commutes with $\widetilde{D}_u^2$, by (3.51) and (3.52), one deduces that

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{N},s}\left[[N,\widetilde{D}_u]\frac{\partial d_u^{\widetilde{F}*}}{\partial u}\exp(-t\widetilde{D}_u^2)\right]\\
&\quad=\mathrm{Tr}_{\mathcal{N},s}[(d^{\widetilde{F}}-d_u^{\widetilde{F}*})[d_u^{\widetilde{F}*},Q_u]\exp(-t\widetilde{D}_u^2)]\\
&\quad=\mathrm{Tr}_{\mathcal{N},s}\left[\exp\left(-\frac{t}{2}\widetilde{D}_u^2\right)(d^{\widetilde{F}}d_u^{\widetilde{F}*}Q_u-d^{\widetilde{F}}Q_ud_u^{\widetilde{F}*}+d_u^{\widetilde{F}*}Q_ud_u^{\widetilde{F}*})\exp\left(-\frac{t}{2}\widetilde{D}_u^2\right)\right]\\
&\quad=\mathrm{Tr}_{\mathcal{N},s}[Q_u(d^{\widetilde{F}}d_u^{\widetilde{F}*}+d_u^{\widetilde{F}*}d^{\widetilde{F}})\exp(-t\widetilde{D}_u^2)].
\end{aligned}\tag{3.55}$$

From (3.3), (3.48) and (3.53)–(3.55), one gets

$$\frac{\partial}{\partial u}\mathrm{Tr}_{\mathcal{N},s}[N\exp(-t\widetilde{D}_u^2)]=t\,\frac{\partial}{\partial t}\mathrm{Tr}_{\mathcal{N},s}[Q_u\exp(-t\widetilde{D}_u^2)]\tag{3.56}$$

(cf. [2, (1.113)]).

On the other hand, for any $0\leqslant u\leqslant 1$, let $P_{C^*(u)}$ denote the orthogonal projection from $L_u^2(\Omega^*(\widetilde{M},\widetilde{F}))$ onto $C^*(u)$. Then by Proposition 3.7, $P_{C^*(u)}$ depends smoothly on $u\in[0,1]$. Moreover, one has

$$d^{\widetilde{F}}P_{C^*(u)}=P_{C^*(u)}d^{\widetilde{F}}P_{C^*(u)}.\tag{3.57}$$

Let $d_{C^*(u)}^{\widetilde{F}*}:C^*(u)\to C^*(u)$ be the formal adjoint of

$$d_{C^*(u)}^{\widetilde{F}}=P_{C^*(u)}d_u^{\widetilde{F}}P_{C^*(u)}:C^*(u)\to C^*(u).\tag{3.58}$$

Then in view of (3.57), one has

$$d_{C^*(u)}^{\widetilde{F}*}=P_{C^*(u)}d_u^{\widetilde{F}*}P_{C^*(u)}=P_{C^*(u)}d_u^{\widetilde{F}*}.\tag{3.59}$$

Set

$$\widetilde{D}_{C^*(u)}=d_{C^*(u)}^{\widetilde{F}}+d_{C^*(u)}^{\widetilde{F}*}.\tag{3.60}$$

One has, similar as in (3.52), that

$$[\widetilde{D}_{C^*(u)},N]=-d_{C^*(u)}^{\widetilde{F}}+d_{C^*(u)}^{\widetilde{F}*}.\tag{3.61}$$

In order to have a formula for $(\partial/\partial u)d_{C^*(u)}^{\widetilde{F}*}$ similar to (3.51), by using (3.51) and (3.59), we compute

$$\begin{aligned}
\frac{\partial}{\partial u}d_{C^*(u)}^{\widetilde{F}*}&=\frac{\partial}{\partial u}(P_{C^*(u)}d_u^{\widetilde{F}*})=\left(\frac{\partial}{\partial u}P_{C^*(u)}\right)d_u^{\widetilde{F}*}+P_{C^*(u)}\frac{\partial}{\partial u}d_u^{\widetilde{F}*}\\
&=\left(\frac{\partial}{\partial u}P_{C^*(u)}\right)d_u^{\widetilde{F}*}+P_{C^*(u)}[d_u^{\widetilde{F}*},Q_u]\\
&=\left(\frac{\partial}{\partial u}P_{C^*(u)}\right)d_u^{\widetilde{F}*}+P_{C^*(u)}d_u^{\widetilde{F}*}Q_u-P_{C^*(u)}Q_ud_u^{\widetilde{F}*}\\
&=[d_{C^*(u)}^{\widetilde{F}*},Q_u]+\left(\frac{\partial}{\partial u}P_{C^*(u)}\right)d_u^{\widetilde{F}*}+Q_uP_{C^*(u)}d_u^{\widetilde{F}*}-P_{C^*(u)}Q_ud_u^{\widetilde{F}*}.
\end{aligned}\tag{3.62}$$

Since $C^*(u)$, $0\leqslant u\leqslant 1$, are finitely generated Hilbert modules, one sees easily that an analogue of (3.48) holds for $\widetilde{D}^2_{C^*(u)}$. Thus, by using (3.61), (3.62) and proceeding as in (3.53)–(3.56), one deduces

$$\begin{aligned}\frac{\partial}{\partial u}\mathrm{Tr}_{\mathcal{N},s}[N\exp(-t\widetilde{D}^2_{C^*(u)})] &= -t\mathrm{Tr}_{\mathcal{N},s}\left[N\,\frac{\partial \widetilde{D}^2_{C^*(u)}}{\partial u}\,\exp(-t\widetilde{D}^2_{C^*(u)})\right]\\ &= -t\mathrm{Tr}_{\mathcal{N},s}\left[[N,\widetilde{D}_{C^*(u)}]\frac{\partial d^{\widetilde{F}*}_{C^*(u)}}{\partial u}\,\exp(-t\widetilde{D}^2_{C^*(u)})\right]\\ &= t\frac{\partial}{\partial t}\,\mathrm{Tr}_{\mathcal{N},s}[Q_u\exp(-t\widetilde{D}^2_{C^*(u)})]\\ &\quad -t\mathrm{Tr}_{\mathcal{N},s}\left[(d^{\widetilde{F}}_{C^*(u)}-d^{\widetilde{F}*}_{C^*(u)})\left(\frac{\partial P_{C^*(u)}}{\partial u}\,d^{\widetilde{F}*}_u+Q_uP_{C^*(u)}d^{\widetilde{F}*}_u-P_{C^*(u)}Q_ud^{\widetilde{F}*}_u\right)\exp(-t\widetilde{D}^2_{C^*(u)})\right]\\ &= t\frac{\partial}{\partial t}\,\mathrm{Tr}_{\mathcal{N},s}[Q_u\exp(-t\widetilde{D}^2_{C^*(u)})]\\ &\quad -t\mathrm{Tr}_{\mathcal{N},s}\left[(d^{\widetilde{F}}_{C^*(u)}-d^{\widetilde{F}*}_{C^*(u)})\left(P_{C^*(u)}\frac{\partial P_{C^*(u)}}{\partial u}\,d^{\widetilde{F}*}_uP_{C^*(u)}+Q_u[P_{C^*(u)},d^{\widetilde{F}*}_u]\right)\exp(-t\widetilde{D}^2_{C^*(u)})\right].\end{aligned} \tag{3.63}$$

Denote for $0\leqslant u\leqslant 1$ that

$$f(u)=(d^{\widetilde{F}}_{C^*(u)}-d^{\widetilde{F}*}_{C^*(u)})\left(P_{C^*(u)}\frac{\partial P_{C^*(u)}}{\partial u}\,d^{\widetilde{F}*}_uP_{C^*(u)}+Q_u[P_{C^*(u)},d^{\widetilde{F}*}_u]\right). \tag{3.64}$$

Since $C^*(u)$ contains $L^2_{u,[0,1]}(\Omega^*(\widetilde{M},\widetilde{F}))$ for $0\leqslant u\leqslant 1$ (cf. (3.26)), one sees that when $t\to+\infty$,

$$\mathrm{Tr}_{\mathcal{N},s}[Q_u\exp(-t\widetilde{D}^2_u)]-\mathrm{Tr}_{\mathcal{N},s}[Q_u\exp(-t\widetilde{D}^2_{C^*(u)})]$$

is of exponential decay.

On the other hand, since, when restricted to the subcomplex $(L^2_{u,[0,1]}(\Omega^*(\widetilde{M},\widetilde{F})),d^{\widetilde{F}})$ of $(C^*(u),d^{\widetilde{F}})$, $d^{\widetilde{F}*}_u$ commutes with $P_{C^*(u)}$, while

$$P_{C^*(u)}\frac{\partial P_{C^*(u)}}{\partial u}\,P_{C^*(u)}=0, \tag{3.65}$$

from (3.64), (3.65) one gets

$$f(u)|_{L^2_{u,[0,1]}(\Omega^*(\widetilde{M},\widetilde{F}))}=0. \tag{3.66}$$

From (3.26) and (3.66), one sees that as $t\to+\infty$,

$$\mathrm{Tr}_{\mathcal{N},s}[f(u)\exp(-t\widetilde{D}^2_{C^*(u)})]$$

is of exponential decay.

By (3.47), (3.56), (3.63), (3.64) and (3.66), we have for $\mathrm{Re}(s)$ large enough that

$$\begin{aligned}\frac{\partial\theta_u(s)}{\partial u} &= \frac{1}{\Gamma(s)}\int_0^{+\infty} t^s\frac{\partial}{\partial t}(\mathrm{Tr}_{\mathcal{N},s}[Q_u\exp(-t\widetilde{D}_u^2)]-\mathrm{Tr}_{\mathcal{N},s}[Q_u\exp(-t\widetilde{D}^2_{C^*(u)})])\,\mathrm{d}t\\ &\quad -\frac{1}{\Gamma(s)}\int_0^{+\infty} t^s\mathrm{Tr}_{\mathcal{N},s}[f(u)\exp(-t\widetilde{D}^2_{C^*(u)})]\,\mathrm{d}t\\ &= \frac{-s}{\Gamma(s)}\int_0^{+\infty} t^{s-1}(\mathrm{Tr}_{\mathcal{N},s}[Q_u\exp(-t\widetilde{D}_u^2)]-\mathrm{Tr}_{\mathcal{N},s}[Q_u\exp(-t\widetilde{D}^2_{C^*(u)})])\,\mathrm{d}t\\ &\quad -\frac{1}{\Gamma(s)}\int_0^{+\infty} t^s\mathrm{Tr}_{\mathcal{N},s}[f(u)\exp(-t\widetilde{D}^2_{C^*(u)})]\,\mathrm{d}t. \end{aligned}\tag{3.67}$$

Now by proceeding as in [19, Lemma 4], and using the standard heat kernel asymptotic expansion on the closed manifold M, one sees that as $t\to 0^+$, for any positive integer l one has an asymptotic expansion

$$\mathrm{Tr}_{\mathcal{N},s}[Q_u\exp(-t\widetilde{D}_u^2)] = \sum_{j=-n/2}^{l} M_{j,u}t^j + o(t^l). \tag{3.68}$$

From (3.67) and (3.68), one finds that for any $0\leqslant u\leqslant 1$, one has

$$\begin{aligned}\frac{\partial}{\partial u}\left(\left.\frac{\partial\theta_u(s)}{\partial s}\right|_{s=0}\right) &= -M_{0,u}+\mathrm{Tr}_{\mathcal{N},s}[Q_uP_{C^*(u)}]\\ &\quad -\int_0^{+\infty}\mathrm{Tr}_{\mathcal{N},s}[f(u)\exp(-t\widetilde{D}^2_{C^*(u)})]\,\mathrm{d}t.\end{aligned}\tag{3.69}$$

Now observe that at $u=0$, by (3.30), one has

$$(C^*(0), d^{\widetilde{F}}) = (L^2_{[0,k]}(\Omega^*(\widetilde{M},\widetilde{F})), d^{\widetilde{F}}). \tag{3.70}$$

Thus one again has the fact that $d_u^{\widetilde{F}*}$ commutes with $P_{C^*(u)}$, which, together with (3.65), implies that

$$f(0)=0. \tag{3.71}$$

From (3.24), (3.46), (3.69) and (3.71), one finds

$$\left.\frac{\partial\log T_{(C^*(u),d^{\widetilde{F}}),\perp}}{\partial u}\right|_{u=0} = -\frac{M_{0,0}}{2}+\frac{1}{2}\mathrm{Tr}_{\mathcal{N},s}[Q_0P_{C^*(0)}]. \tag{3.72}$$

Now let us consider the variation of $T_{(C^*(u),d^{\widetilde{F}})}$ near $u=0$.

Observe that for any $\omega,\omega'\in C^*(0)=L^2_{[0,k]}(\Omega^*(\widetilde{M},\widetilde{F}))$, the induced inner product of them in $C^*(u)$ is given by

$$\begin{aligned}\langle P_{C^*(u)}\omega, P_{C^*(u)}\omega'\rangle_u &= \langle\omega, P_{C^*(u)}\omega'\rangle_u = \int_{\widetilde{M}}\langle\omega\wedge *_u^{\widetilde{F}}P_{C^*(u)}\omega'\rangle_{\widetilde{F}}\\ &= \langle\omega, (*^{\widetilde{F}})^{-1} *_u^{\widetilde{F}} P_{C^*(u)}\omega'\rangle.\end{aligned}\tag{3.73}$$

Set for $0\leqslant u\leqslant 1$ that

$$A_u = P_{C^*(0)}(*^{\widetilde{F}})^{-1} *_u^{\widetilde{F}} P_{C^*(u)}P_{C^*(0)} : C^*(0) \to C^*(0). \tag{3.74}$$

From (2.25)–(2.27), (3.21), (3.73) and (3.74), one finds,

$$\log \frac{T_{(C^*(u),d^{\widetilde{F}})}}{T_{(C^*(0),d^{\widetilde{F}})}} = -\frac{1}{2}\sum_{i=0}^{n}(-1)^i \log \operatorname{Det}_{\tau_{\mathcal{N}(\Gamma)}}(A_u|_{C^i(0)}). \tag{3.75}$$

From (2.16) and (3.75), one deduces

$$\frac{\partial}{\partial u}\log \frac{T_{(C^*(u),d^{\widetilde{F}})}}{T_{(C^*(0),d^{\widetilde{F}})}} = -\frac{1}{2}\operatorname{Tr}_{\mathcal{N},s}\left[A_u^{-1}\frac{\partial A_u}{\partial u}\right]. \tag{3.76}$$

By (3.74), one sees directly that

$$A_u|_{u=0} = \mathrm{Id}|_{C^*(0)}. \tag{3.77}$$

From (3.49), (3.65), (3.74), (3.76) and (3.77), one finds

$$\begin{aligned}\frac{\partial}{\partial u}\bigg|_{u=0}\log \frac{T_{(C^*(u),d^{\widetilde{F}})}}{T_{(C^*(0),d^{\widetilde{F}})}} &= -\frac{1}{2}\operatorname{Tr}_{\mathcal{N},s}\left[P_{C^*(0)}(*^{\widetilde{F}})^{-1}\frac{\partial *_u^{\widetilde{F}}}{\partial u}\bigg|_{u=0}P_{C^*(0)}\right]\\ &= -\frac{1}{2}\operatorname{Tr}_{\mathcal{N},s}[Q_0P_{C^*(0)}].\end{aligned} \tag{3.78}$$

From (3.45), (3.72) and (3.78), one gets

$$\frac{\partial}{\partial u}\bigg|_{u=0}\log \frac{T_{\mathrm{RS}}^{(2)}(\widetilde{M},F,g_u^{TM},g_u^F)}{T_{\mathrm{RS}}^{(2)}(\widetilde{M},F,g^{TM},g^F)} = -\frac{M_{0,0}}{2}. \tag{3.79}$$

Since (3.79) holds for arbitrary (g^{TM},g^F), one gets indeed that for any $0\leqslant u\leqslant 1$,

$$\frac{\partial}{\partial u}\log \frac{T_{\mathrm{RS}}^{(2)}(\widetilde{M},F,g_u^{TM},g_u^F)}{T_{\mathrm{RS}}^{(2)}(\widetilde{M},F,g^{TM},g^F)} = -\frac{M_{0,u}}{2}. \tag{3.80}$$

Now by using [19, Lemma 4] again, one sees that for any $0\leqslant u\leqslant 1$, $M_{0,u}$ is exactly the same quantity appears in [4, Theorem 4.14], where a similar result is proved for the usual Ray–Singer metrics.

Formula (3.15) then follows from the evaluation of this $M_{0,u}$, $0\leqslant u\leqslant 1$, in [4, Theorem 4.20], and an integration from 0 to 1 of the obtained result.

The proof of Theorem 3.4 is completed. □

Remark 3.10. As was mentioned in Remark 2.3, the "torsion element" dealt with here is dual to the Ray–Singer metric discussed in [4], at least in the $\Gamma=\{e\}$ case. This explains that the right-hand side of (3.15) differs from that of [4, (4.13)] by a factor of $-\frac{1}{2}$.

Remark 3.11. If for any $u\in[0,1]$, $\mathrm{Spec}(\widetilde{D}_u^2)$ contains a nonempty gap, then the proof of Theorem 3.4 can be simplified a lot. Here we did not make this assumption as usually $\mathrm{Spec}(\widetilde{D}_u^2)$, $u\in[0,1]$, may not be discrete when Γ is an infinite group.

4. Infinite covering spaces and a formula relating L^2-Milnor torsion element to L^2-Ray–Singer torsion element

In this section, we state the main result of this paper, which is an extension of [4, Theorem 0.2] in the infinite covering space case, and prove it modulus two intermediate results.

This section is organized as follows. In Section 4.1, we recall Shubin's de Rham theorem for extended cohomologies. In Section 4.2, we state the above mentioned main result of this paper as Theorem 4.2. In Section 4.3, we state two intermediate results and prove Theorem 4.2.

4.1. An extended de Rham theorem

We assume that we are in the same situation as in Section 2.4.

By a simple argument of Helffer–Sjöstrand [16, Proposition 5.1] (cf. [4, Section 7b]), we may and we well assume that g^{TM} there satisfies the following property without altering the L^2-Thom–Smale cochain complex $(C^*(W^{\mathrm{u}}, \widetilde{F}), \widetilde{\partial})$,

$(*)$ For any $x \in B$, there is a system of coordinates $y = (y^1, \ldots, y^n)$ centered at x such that near x,

$$g^{TM} = \sum_{i=1}^{n} |dy^i|^2, \quad f(y) = f(x) - \frac{1}{2} \sum_{i=1}^{\mathrm{ind}(x)} |y^i|^2 + \frac{1}{2} \sum_{i=\mathrm{ind}(x)+1}^{n} |y^i|^2. \tag{4.1}$$

By a result of Laudenbach [17], $\{W^{\mathrm{u}}(x) : x \in B\}$ form a CW decomposition of M. As a consequence, $\{W^{\mathrm{u}}(\widetilde{x}) : \widetilde{x} \in \widetilde{B}\}$ form a (Γ-equivariant) CW decomposition of $\widetilde{M}$.

For any $\widetilde{x} \in \widetilde{B}$, $\widetilde{F}$ is canonically trivialized over each cell $W^{\mathrm{u}}(\widetilde{x})$.

Let $\widetilde{P}_\infty$ be the de Rham map defined by

$$\alpha \in \Omega^*(\widetilde{M}, \widetilde{F}) \cap L^2(\Omega^*(\widetilde{M}, \widetilde{F})) \to \widetilde{P}_\infty \alpha = \sum_{\widetilde{x} \in \widetilde{B}} W^{\mathrm{u}}(\widetilde{x})^* \int_{W^{\mathrm{u}}(\widetilde{x})} \alpha \in C^*(W^{\mathrm{u}}, \widetilde{F}). \tag{4.2}$$

Let $\mathbf{H}^1(\widetilde{M}, \widetilde{F})$ denote the first Sobolev space with respect to a (fixed, Γ-invariant) first Sobolev norm on $\Omega^*(\widetilde{M}, \widetilde{F})$. By Stokes theorem, one verifies that when acting on $\Omega^*(\widetilde{M}, \widetilde{F}) \cap \mathbf{H}^1(\Omega^*(\widetilde{M}, \widetilde{F}))$, one has

$$\widetilde{\partial} \widetilde{P}_\infty = \widetilde{P}_\infty d^{\widetilde{F}}. \tag{4.3}$$

From (4.3), one deduces easily that $\widetilde{P}_\infty$ induces a $\mathbf{Z}$-grading preserving homomorphism $\widetilde{P}_\infty^{\mathscr{H}}$ between the extended cohomologies (cf. [6,13,33]),

$$\widetilde{P}_\infty^{\mathscr{H}} : \mathscr{H}^{(2)}_{\mathrm{dR}}(\Omega^*(\widetilde{M}, \widetilde{F}), d^{\widetilde{F}}) \to \mathscr{H}^*(C^*(W^{\mathrm{u}}, \widetilde{F}), \widetilde{\partial}). \tag{4.4}$$

The following theorem has been proved by Shubin ([33, Theorem 3.1]).

Theorem 4.1. *The canonical homomorphism* $\widetilde{P}_\infty^{\mathscr{H}}$ *in* (4.4) *is an isomorphism.*

By Theorem 4.1, the isomorphism $\widetilde{P}_\infty^{\mathscr{H}}$ in (4.4) induces a natural isomorphism between the determinant lines,

$$\widetilde{P}_\infty^{\det \mathscr{H}} : \det \mathscr{H}^{(2)}_{\mathrm{dR}}(\Omega^*(\widetilde{M}, \widetilde{F}), d^{\widetilde{F}}) \to \det \mathscr{H}^*(C^*(W^{\mathrm{u}}, \widetilde{F}), \widetilde{\partial}). \tag{4.5}$$

4.2. An extended Cheeger–Müller theorem

Let h^{TM} be an arbitrary smooth metric on *TM*. Then by Definition 3.3, one has an associated L^2-Ray–Singer torsion element

$$T_{\mathrm{RS}}^{(2)}(\widetilde{M}, F, h^{TM}, g^F) \in \det \mathscr{H}_{\mathrm{dR}}^{(2)}(\Omega^*(\widetilde{M}, \widetilde{F}), d^{\widetilde{F}}). \tag{4.6}$$

From (4.5) and (4.6), one gets a well-defined element

$$\widetilde{P}_\infty^{\det \mathscr{H}}(T_{\mathrm{RS}}^{(2)}(\widetilde{M}, F, h^{TM}, g^F)) \in \det \mathscr{H}^*(C^*(W^{\mathrm{u}}, \widetilde{F}), \widetilde{\partial}). \tag{4.7}$$

On the other hand, by Definition 2.9, one has a well-defined L^2-Milnor torsion element

$$\rho_{(\widetilde{M}, F, g^F, -X)} \in \det \mathscr{H}^*(C^*(W^{\mathrm{u}}, \widetilde{F}), \widetilde{\partial}), \tag{4.8}$$

where $X = \nabla f$ is the gradient vector field of *f* associated to g^{TM}.

Let $\psi(TM, \nabla^{TM})$ be the Mathai–Quillen current [23] over *TM*, associated to h^{TM}, defined in [4, Definition 3.6]. As indicated in [4, Remark 3.8], the pull-back current $X^*\psi(TM, \nabla^{TM})$ is well-defined over *M*.

The main result of this paper, which extends [4, Theorem 0.2] to the infinite covering spaces case, can be stated as follows.

Theorem 4.2. *The following identity in* **R** *holds*:

$$\log \frac{\widetilde{P}_\infty^{\det \mathscr{H}}(T_{\mathrm{RS}}^{(2)}(\widetilde{M}, F, h^{TM}, g^F))}{\rho_{(\widetilde{M}, F, g^F, -X)}} = \frac{1}{2} \int_M \theta(F, h^F) X^*\psi(TM, \nabla^{TM}). \tag{4.9}$$

Remark 4.3. If $\Gamma = \{e\}$ is trivial, then (4.9) reduces to [4, Theorem 0.2], which generalizes the Cheeger–Müller theorem (cf. [12,26,27]) to the case of general flat vector bundles. It is interesting to observe that the right-hand side of (4.9) does not depend on Γ.

Remark 4.4. When Γ is infinite, $n = \dim M$ is odd and ∇^F preserves the volume form determined by g^F on $\det(F)$, Theorem 4.2 was proved in [6, Theorem 13.8]. When Γ is infinite and $(\widetilde{M}, F)$ is of determinant class (cf. [8]), Theorem 4.2 was proved in [7] (cf. [7, Remark to Theorem 1.1 in Section 6.1]) as an extension of the main result of [8] to the nonunitary case. Both proofs in [7,6] use essentially the concept of "relative torsion" introduced in [11]. In what follows, we will give a direct heat kernel proof of Theorem 4.2 in the spirit of [4].

4.3. Witten deformation and a proof of Theorem 4.2

First of all, in view of Proposition 2.10 and Theorem 3.4, by proceeding as in [4, Section 7b], in order to prove Theorem 4.2, we need only to prove it in the case where $h^{TM} = g^{TM}$. Moreover, we may well assume that g^F is flat near *B*. From now on, we will make these assumptions.

Let $T_{\mathrm{RS}}(M,F,g^{TM},g^F)$ (resp. $\rho_{(M,F,g^F,-X)}$) be the Ray–Singer (resp. Milnor) torsion element corresponding to the $\Gamma=\{e\}$ case. Then [4, Theorem 0.2], in the above choice of metrics, takes the form

$$\log\frac{P_\infty^{\det H}(T_{\mathrm{RS}}(M,F,g^{TM},g^F))}{\rho_{(M,F,g^F,-X)}}=\frac{1}{2}\int_M\theta(F,h^F)X^*\psi(TM,\nabla^{TM}). \tag{4.10}$$

Thus, in order to prove Theorem 4.2, we need only to prove the following identity in $\mathbf{R}^+$:

$$\frac{\widetilde{P}_\infty^{\det \mathscr{H}}(T_{\mathrm{RS}}^{(2)}(\widetilde{M},F,g^{TM},g^F))}{\rho_{(\widetilde{M},F,g^F,-X)}}=\frac{P_\infty^{\det H}(T_{\mathrm{RS}}(M,F,g^{TM},g^F))}{\rho_{(M,F,g^F,-X)}}. \tag{4.11}$$

In what follows, as above, whenever we do something on $\widetilde{M}$, we will assume the same thing has been done on M also, and the results on M will be formulated through the corresponding results on $\widetilde{M}$ by simply withdraw the "$\sim$" notation. Also, while on $\widetilde{M}$ we use "$\mathscr{H}$" to denote the extended cohomology, we use "H" to denote the cohomology on M.

As in [4,8,7], we will use the deformation associated to the Morse function f introduced by Witten [35] to prove (4.11).

Recall from [4, Section 5b] that the Witten deformation is equivalent to a deformation of the metric on the flat vector bundle F. Thus, following [4, Definition 5.1], for any $T\geqslant 0$, let g_T^F be the smooth metric on F given by

$$g_T^F=\mathrm{e}^{-2Tf}g^F. \tag{4.12}$$

Let $L_T^2(\Omega^*(\widetilde{M},\widetilde{F}))$ be the associated Hilbert space. Let $d_T^{\widetilde{F}*}$ be the corresponding formal adjoint of $d^{\widetilde{F}}$. Recall that $\widetilde{f}$ denotes the lifting of f on $\widetilde{M}$. Then one has

$$d_T^{\widetilde{F}*}=\mathrm{e}^{2T\widetilde{f}}d^{\widetilde{F}*}\mathrm{e}^{-2T\widetilde{f}}:\Omega^*(\widetilde{M},\widetilde{F})\to\Omega^*(\widetilde{M},\widetilde{F}). \tag{4.13}$$

Set

$$\widetilde{D}_T=d^{\widetilde{F}}+d_T^{\widetilde{F}*},\quad \widetilde{D}_T^2=(d^{\widetilde{F}}+d_T^{\widetilde{F}*})^2=d_T^{\widetilde{F}*}d^{\widetilde{F}}+d^{\widetilde{F}}d_T^{\widetilde{F}*}. \tag{4.14}$$

Then $\widetilde{D}_T^2$ preserves the $\mathbf{Z}$-grading of $\Omega^*(\widetilde{M},\widetilde{F})$. Let

$$\begin{aligned}(L_T^2(\Omega^*(\widetilde{M},\widetilde{F})),d^{\widetilde{F}}):0\to L_T^2(\Omega^0(\widetilde{M},\widetilde{F}))&\stackrel{d^{\widetilde{F}}}{\to}L_T^2(\Omega^1(\widetilde{M},\widetilde{F}))\\ \to\cdots&\stackrel{d^{\widetilde{F}}}{\to}L_T^2(\Omega^n(\widetilde{M},\widetilde{F}))\to 0\end{aligned} \tag{4.15}$$

denote the corresponding deformed complex of $(L^2(\Omega^*(\widetilde{M},\widetilde{F})),d^{\widetilde{F}})$ in (3.2).

By [8, Proposition 5.2] and [32], one knows that there exist $C'>0$, $C''>0$ and $T_0>0$ such that whenever $T\geqslant T_0$, one has

$$\mathrm{Spec}(\widetilde{D}_T^2)\cap(\mathrm{e}^{-C'T},C''T)=\emptyset. \tag{4.16}$$

We may well assume that the T_0 above can be chosen so that $1\notin(\mathrm{e}^{-C'T},C''T)$ for $T\geqslant T_0$. Then (4.16) allows us to split the deformed complex $(L_T^2(\Omega^*(\widetilde{M},\widetilde{F})),d^{\widetilde{F}})$ into an orthogonal direct sum of two subcomplexes,

$$(L_T^2(\Omega^*(\widetilde{M},\widetilde{F})),d^{\widetilde{F}})=(L_{T,[0,1]}^2(\Omega^*(\widetilde{M},\widetilde{F})),d^{\widetilde{F}})\oplus(L_{T,[1,+\infty)}^2(\Omega^*(\widetilde{M},\widetilde{F})),d^{\widetilde{F}}), \tag{4.17}$$

where $L^2_{T,[0,1]}(\Omega^*(\widetilde{M},\widetilde{F}))$ (resp. $L^2_{T,[1,+\infty)}(\Omega^*(\widetilde{M},\widetilde{F}))$) is the image of the spectral projection of $\widetilde{D}_T^2$ corresponding to the spectral interval [0, 1] (resp. $[1,+\infty)$).

It is clear that splitting (4.17) is Γ-equivariant.

For any $T \geqslant T_0$, let $\widetilde{P}_{\infty,T}$ denote the restriction of $\widetilde{P}_\infty$ on $L^2_{T,[0,1]}(\Omega^*(\widetilde{M},\widetilde{F}))$,

$$\widetilde{P}_{\infty,T} = \widetilde{P}_\infty : L^2_{T,[0,1]}(\Omega^*(\widetilde{M},\widetilde{F})) \to C^*(W^{\mathrm{u}},\widetilde{F}). \tag{4.18}$$

By (4.3), one has

$$\widetilde{\partial}\widetilde{P}_{\infty,T} = \widetilde{P}_{\infty,T} d^{\widetilde{F}}. \tag{4.19}$$

By Theorem 4.1, $\widetilde{P}_{\infty,T}$ induces an isomorphism between the extended de Rham cohomology and $\mathscr{H}^*(C^*(W^{\mathrm{u}},\widetilde{F}),\widetilde{\partial})$. Thus it induces an isomorphism between the determinant lines

$$\begin{aligned}\widetilde{P}^{\det \mathscr{H}}_{\infty,T} : \det \mathscr{H}^{(2)}_{\mathrm{dR}}(\Omega^*(\widetilde{M},\widetilde{F}), d^{\widetilde{F}}) &\simeq \det \mathscr{H}^*(L^2_{T,[0,1]}(\Omega^*(\widetilde{M},\widetilde{F})), d^{\widetilde{F}}) \\ &\to \det \mathscr{H}^*(C^*(W^{\mathrm{u}},\widetilde{F}),\widetilde{\partial}).\end{aligned} \tag{4.20}$$

Clearly, by choosing T_0 sufficiently large, all these discussions also hold on M.

By Theorem 3.4 [4, Theorem 0.1] and using the notation in Definition 3.3, one sees that (4.11) is equivalent to

$$\begin{aligned}&\frac{\widetilde{P}^{\det \mathscr{H}}_{\infty,T}(\mathscr{T}_{[0,1]}(\widetilde{M},F,g^{TM},g^F_T))}{\rho_{(\widetilde{M},F,g^F,-X)}} \cdot \mathscr{T}_{[1,+\infty)}(\widetilde{M},F,g^{TM},g^F_T) \\ &\quad = \frac{P^{\det H}_{\infty,T}(\mathscr{T}_{[0,1]}(M,F,g^{TM},g^F_T))}{\rho_{(M,F,g^F,-X)}} \cdot \mathscr{T}_{[1,+\infty)}(M,F,g^{TM},g^F_T),\end{aligned} \tag{4.21}$$

where we use the notation $\mathscr{T}$ for torsion elements, in order to avoid possible confusions with the notation T for the deformed parameter.

We now state two intermediate results which will be proved in the next two sections.

Theorem 4.5. *The following identity holds*:

$$\lim_{T\to+\infty}\left(\log\frac{\widetilde{P}^{\det \mathscr{H}}_{\infty,T}(\mathscr{T}_{[0,1]}(\widetilde{M},F,g^{TM},g^F_T))}{\rho_{(\widetilde{M},F,g^F,-X)}} - \log\frac{P^{\det H}_{\infty,T}(\mathscr{T}_{[0,1]}(M,F,g^{TM},g^F_T))}{\rho_{(M,F,g^F,-X)}}\right) = 0. \tag{4.22}$$

Theorem 4.6. *The following identity holds*:

$$\lim_{T\to+\infty}(\log \mathscr{T}_{[1,+\infty)}(\widetilde{M},F,g^{TM},g^F_T) - \log \mathscr{T}_{[1,+\infty)}(M,F,g^{TM},g^F_T)) = 0. \tag{4.23}$$

Proof of Theorem 4.2. By Theorem 3.4 and [4, Theorem 0.1], one knows that

$$\frac{\dfrac{\widetilde{P}^{\det \mathscr{H}}_{\infty,T}(\mathscr{T}_{[0,1]}(\widetilde{M},F,g^{TM},g^F_T))}{\rho_{(\widetilde{M},F,g^F,-X)}}}{\dfrac{P^{\det H}_{\infty,T}(\mathscr{T}_{[0,1]}(M,F,g^{TM},g^F_T))}{\rho_{(M,F,g^F,-X)}}} \cdot \frac{\mathscr{T}_{[1,+\infty)}(\widetilde{M},F,g^{TM},g^F_T)}{\mathscr{T}_{[1,+\infty)}(M,F,g^{TM},g^F_T)} \tag{4.24}$$

does not depend on $T \geqslant 0$. By Theorems 4.5 and 4.6, one sees that it equals to 1. Thus, one gets (4.21) which implies (4.11).

From (4.10) and (4.11), one gets (4.9).

The proof of Theorem 4.2 is completed. $\square$

Remark 4.7. It is clear that the strategy of the above proof is similar to those in [4,7,8], where one uses the Witten deformation and studies the asymptotic properties of the small and large eigen-complexes. A notable point here is that by Theorem 4.6, one is able to avoid the repeat of the local index computations in [4, Sections 12–15]. Moreover, as one compares directly with the usual Ray–Singer torsion element, one is able to avoid the comparison arguments in [7,8], and thus able to prove Theorem 4.2 directly.

5. A proof of Theorem 4.5

In this section, we prove Theorem 4.5. Recall that an asymptotic formula for

$$\log \frac{P^{\det H}_{\infty,T}(\mathscr{T}_{[0,1]}(M, F, g^{TM}, g^F_T))}{\rho_{(M,F,g^F,-X)}}, \quad \text{as } T \to +\infty,$$

has been established in [4, Theorem 7.6]. Thus, we need only to show that a similar asymptotic formula holds for

$$\log \frac{\widetilde{P}^{\det \mathscr{H}}_{\infty,T}(\mathscr{T}_{[0,1]}(\widetilde{M}, F, g^{TM}, g^F_T))}{\rho_{(\widetilde{M},F,g^F,-X)}}.$$

While such an asymptotic formula can be proved by using the L^2-Helffer–Sjöstrand–Witten theory developed in [8], we will use an L^2-generalization of the arguments in [5, Section 6] to prove it (the idea of using an extension of the arguments in [5, Section 6] to prove an L^2-Cheeger–Müller theorem first appeared in [15], where Gong dealt with the case where ∇^F preserves g^F and the Novikov–Shubin invariants associated to $(\widetilde{M}, F)$ are all positive).

We continue the discussion in Section 4.3.

For any $T \geqslant 0$, following Witten [35], set

$$d^{\widetilde{F}}_T = \mathrm{e}^{-T\widetilde{f}} d^{\widetilde{F}} \mathrm{e}^{T\widetilde{f}}, \quad \delta^{\widetilde{F}*}_T = \mathrm{e}^{T\widetilde{f}} d^{\widetilde{F}*} \mathrm{e}^{-T\widetilde{f}} : \Omega^*(\widetilde{M}, \widetilde{F}) \to \Omega^*(\widetilde{M}, \widetilde{F}). \tag{5.1}$$

Then $\delta^{\widetilde{F}*}_T$ is the formal adjoint of $d^{\widetilde{F}}_T$ with respect to the usual inner product (associated to (g^{TM}, g^F)) on $L^2(\Omega^*(\widetilde{M}, \widetilde{F}))$. Set

$$\widetilde{D'}_T = d^{\widetilde{F}}_T + \delta^{\widetilde{F}*}_T, \quad \widetilde{D'}^2_T = (d^{\widetilde{F}}_T + \delta^{\widetilde{F}*}_T)^2 = \delta^{\widetilde{F}*}_T d^{\widetilde{F}} + d^{\widetilde{F}} \delta^{\widetilde{F}*}_T. \tag{5.2}$$

Then $\widetilde{D'}^2_T$ preserves the **Z**-grading of $\Omega^*(\widetilde{M}, \widetilde{F})$.

The following formula is clear (cf. [4, Proposition 5.4]),

$$\widetilde{D'}_T = \mathrm{e}^{-T\widetilde{f}} \widetilde{D}_T \mathrm{e}^{T\widetilde{f}}, \quad \widetilde{D'}^2_T = \mathrm{e}^{-T\widetilde{f}} \widetilde{D}^2_T \mathrm{e}^{T\widetilde{f}}. \tag{5.3}$$

Let

$$(L^2(\Omega^*(\widetilde{M}, \widetilde{F})), d_T^{\widetilde{F}}) : 0 \to L^2(\Omega^0(\widetilde{M}, \widetilde{F})) \xrightarrow{d_T^{\widetilde{F}}} L^2(\Omega^1(\widetilde{M}, \widetilde{F})) \to \cdots \xrightarrow{d_T^{\widetilde{F}}} L^2(\Omega^n(\widetilde{M}, \widetilde{F})) \to 0 \tag{5.4}$$

denote the corresponding deformed complex of $(L^2(\Omega^*(\widetilde{M}, \widetilde{F})), d^{\widetilde{F}})$ in (3.2).

From (4.16) and (5.3), one sees that as $T > 0$ is sufficiently large, $(L^2(\Omega^*(\widetilde{M}, \widetilde{F})), d_T^{\widetilde{F}})$ decomposes to an orthogonal direct sum of two subcomplexes,

$$(L^2(\Omega^*(\widetilde{M}, \widetilde{F})), d_T^{\widetilde{F}}) = (L^2_{[0,1],T}(\Omega^*(\widetilde{M}, \widetilde{F})), d_T^{\widetilde{F}}) \oplus (L^2_{[1,+\infty),T}(\Omega^*(\widetilde{M}, \widetilde{F})), d_T^{\widetilde{F}}), \tag{5.5}$$

where $L^2_{[0,1],T}(\Omega^*(\widetilde{M}, \widetilde{F}))$ (resp. $L^2_{[1,+\infty),T}(\Omega^*(\widetilde{M}, \widetilde{F}))$) is the image of the spectral projection of $\widetilde{D'}^2_T$ corresponding to the spectral interval [0, 1] (resp. $[1, +\infty)$).

Let $\widetilde{P}_{[0,1],T}$ (resp. $\widetilde{P}'_{[0,1],T}$) denote the orthogonal projection from $L^2_T(\Omega^*(\widetilde{M}, \widetilde{F}))$ (resp. $L^2(\Omega^*(\widetilde{M}, \widetilde{F}))$) onto $L^2_{T,[0,1]}(\Omega^*(\widetilde{M}, \widetilde{F}))$ (resp. $L^2_{[0,1],T}(\Omega^*(\widetilde{M}, \widetilde{F}))$). Then by (5.3), one has

$$\widetilde{P}'_{[0,1],T} = e^{-T\widetilde{f}}\widetilde{P}_{[0,1],T}e^{T\widetilde{f}}. \tag{5.6}$$

Thus, the linear map $s \to e^{-T\widetilde{f}}s$ identifies $L^2_{T,[0,1]}(\Omega^*(\widetilde{M}, \widetilde{F}))$ isometrically with $L^2_{[0,1],T}(\Omega^*(\widetilde{M}, \widetilde{F}))$.

Let $\varepsilon > 0$ be such that for any $x \in B$, (4.1) holds on

$$B_M(x, \varepsilon) = \{y \in M : d^{g^{TM}}(x, y) \leqslant \varepsilon\},$$

and that g^F is flat on $B_M(x, \varepsilon)$. Such an $\varepsilon > 0$ clearly exists.

For any $\widetilde{x} \in \widetilde{B}$, we use the local coordinate system $(\widetilde{y}^1, \ldots, \widetilde{y}^n)$ lifted from (4.1) near $x = \pi(\widetilde{x}) \in B$. In particular, $\mathbf{R}^{\mathrm{ind}(\widetilde{x})} \simeq T_{\widetilde{x}}W^{\mathrm{u}}(\widetilde{x})$ inherits an orientation from the orientation of $W^{\mathrm{u}}(\widetilde{x})$. Let $\rho_{\widetilde{x}}$ be the volume form on the oriented vector space $\mathbf{R}^{\mathrm{ind}(\widetilde{x})}$. One can assume that $\widetilde{y}^1, \ldots, \widetilde{y}^{\mathrm{ind}(x)}$ are such that

$$\rho_{\widetilde{x}} = d\widetilde{y}^1 \wedge \cdots \wedge d\widetilde{y}^{\mathrm{ind}(x)}. \tag{5.7}$$

Let $\gamma : \mathbf{R} \to [0, 1]$ be a smooth function such that

$$\gamma(a) = 1 \quad \text{if } a < \tfrac{1}{2}, \text{ while } \gamma(a) = 0 \text{ if } a > 1. \tag{5.8}$$

If $\widetilde{y} \in \mathbf{R}^n$, set

$$\mu(\widetilde{y}) = \gamma\left(\frac{|\widetilde{y}|}{\varepsilon}\right). \tag{5.9}$$

We can consider μ as a smooth function defined on $\widetilde{M}$ with nonnegative values, which vanishes on $\widetilde{M} \backslash B_{\widetilde{M}}(\widetilde{x}, \varepsilon)$.

For any $T > 0$, set

$$\alpha_T = \int_{\mathbf{R}^n} \mu^2(\widetilde{y}) \exp(-T|\widetilde{y}|^2)\, d\widetilde{y}. \tag{5.10}$$

Clearly, there is $c > 0$ such that as $T \to +\infty$,

$$\alpha_T = \left(\frac{\pi}{T}\right)^{n/2} + O(\mathrm{e}^{-cT}). \tag{5.11}$$

Following [4, Definition 8.7; 5, Definition 6.5], for any $T > 0$, let $\widetilde{J}_T$ be the linear map from $C^*(W^{\mathrm{u}}, \widetilde{F})$ into $L^2(\Omega^*(\widetilde{M}, \widetilde{F}))$ such that if $\widetilde{x} \in \widetilde{B}$, $h \in \widetilde{F}_{\widetilde{x}}$, $\widetilde{y} \in B_{\widetilde{M}}(\widetilde{x}, \varepsilon)$,

$$\widetilde{J}_T(W^{\mathrm{u}}(\widetilde{x})^* \otimes h)(\widetilde{y}) = \frac{\mu(\widetilde{y})}{(\alpha_T)^{1/2}} \exp\left(\frac{-T|\widetilde{y}|^2}{2}\right) \rho_{\widetilde{x}} \otimes h. \tag{5.12}$$

Clearly, $\widetilde{J}_T$ is an isometry from $C^*(W^{\mathrm{u}}, \widetilde{F})$ into $L^2(\Omega^*(\widetilde{M}, \widetilde{F}))$, which preserves the $\mathbf{Z}$-grading.

Let $\widetilde{e}'_T : C^*(W^{\mathrm{u}}, \widetilde{F}) \to L^2_{[0,1],T}(\Omega^*(\widetilde{M}, \widetilde{F}))$ be defined by

$$\widetilde{e}'_T = \widetilde{P}'_{[0,1],T} \widetilde{J}_T. \tag{5.13}$$

The following L^2-extension of [4, Theorem 8.8; 5, Theorem 6.7] can be proved by an easy adaptation of the arguments in [4, Theorem 8.8; 5, Theorem 6.7] to the L^2-setting (cf. [8, Proposition 5.4; 15, Lemma 3.6]).

Proposition 5.1. *There exists $c > 0$ such that as $T \to +\infty$, for any $s \in C^*(W^{\mathrm{u}}, \widetilde{F})$,*

$$(\widetilde{e}'_T - \widetilde{J}_T)s = O(\mathrm{e}^{-cT})s \quad \textit{uniformly on } \widetilde{M}. \tag{5.14}$$

Let $\widetilde{e}_T : C^*(W^{\mathrm{u}}, \widetilde{F}) \to L^2_{T,[0,1]}(\Omega^*(\widetilde{M}, \widetilde{F}))$ be defined by

$$\widetilde{e}_T = \mathrm{e}^{T\widetilde{f}} \widetilde{e}'_T. \tag{5.15}$$

Using Proposition 5.1, the following L^2-extension of [5, Theorem 6.9] can be proved by an easy adaptation of the arguments in [5, Theorem 6.9] to the L^2-setting (cf. [15, Lemma 3.6]).

Proposition 5.2. *There exists $c > 0$ such that as $T \to +\infty$,*

$$\widetilde{e}^*_T \widetilde{e}_T = 1 + O(\mathrm{e}^{-cT}). \tag{5.16}$$

Moreover, for $T > 0$ large enough, $\widetilde{e}_T : C^*(W^{\mathrm{u}}, \widetilde{F}) \to L^2_{T,[0,1]}(\Omega^*(\widetilde{M}, \widetilde{F}))$ is an $\mathcal{N}(\Gamma)$-linear isomorphism between the $\mathbf{Z}$-graded $\mathcal{N}(\Gamma)$-Hilbert modules.

Proof. Formula (5.16) can be proved in the same way as in [5, Theorem 6.9] (cf. [15, Lemma 3.6]). Moreover, an easy argument (cf. [15, pp. 75–76]) shows that

$$\mathrm{Im}(\widetilde{e}_T) = L^2_{T,[0,1]}(\Omega^*(\widetilde{M}, \widetilde{F})). \tag{5.17}$$

The proof of Proposition 5.2 is completed. □

Recall that the map $\widetilde{P}_{\infty,T} : L^2_{T,[0,1]}(\Omega^*(\widetilde{M}, \widetilde{F})) \to C^*(W^{\mathrm{u}}, \widetilde{F})$ has been defined in (4.18).

Let $\widetilde{\mathscr{F}} : C^*(W^{\mathrm{u}}, \widetilde{F}) \to C^*(W^{\mathrm{u}}, \widetilde{F})$ be acting on $W^{\mathrm{u}}(\widetilde{x}) \otimes \widetilde{F}_{\widetilde{x}}$, $\widetilde{x} \in \widetilde{B}$, by multiplication by $\widetilde{f}(\widetilde{x})$. Also, we still denote by $N : C^*(W^{\mathrm{u}}, \widetilde{F}) \to C^*(W^{\mathrm{u}}, \widetilde{F})$ the operator acting on $C^i(W^{\mathrm{u}}, \widetilde{F})$ by multiplication by i.

The following L^2-extension of [5, Theorem 6.11] can be proved by exactly the same arguments as in [5, Theorem 6.11] (cf. [15, Lemma 4.2]).

Proposition 5.3. *There exists $c>0$ such that as $T\to+\infty$,*

$$\widetilde{P}_{\infty,T}\widetilde{e}_T=\mathrm{e}^{T\widetilde{\mathscr{F}}}\left(\frac{\pi}{T}\right)^{N/2-n/4}(1+O(\mathrm{e}^{-cT})). \tag{5.18}$$

*In particular, for $T>0$ large enough, $\widetilde{P}_{\infty,T}\widetilde{e}_T : C^*(W^{\mathrm{u}},\widetilde{F})\to C^*(W^{\mathrm{u}},\widetilde{F})$ is an $\mathscr{N}(\Gamma)$-linear isomorphism between **Z**-graded $\mathscr{N}(\Gamma)$-Hilbert modules.*

From (4.19) and Propositions 5.2, 5.3, one sees that when $T>0$ is large enough,

$$\widetilde{P}_{\infty,T} : L^2_{T,[0,1]}(\Omega^*(\widetilde{M},\widetilde{F}))\to C^*(W^{\mathrm{u}},\widetilde{F}) \tag{5.19}$$

is a cochain isomorphism between finite length cochain complexes of $\mathscr{N}(\Gamma)$-Hilbert modules.

From Proposition 2.7, one deduces easily that

$$\frac{\widetilde{P}^{\det\mathscr{H}}_{\infty,T}(\mathscr{T}_{[0,1]}(\widetilde{M},F,g^{TM},g^F_T))}{\rho_{(\widetilde{M},F,g^F,-X)}}=\prod_{i=0}^{n}\mathrm{Det}_{\tau_{\mathscr{N}(\Gamma)}}(\widetilde{P}^*_{\infty,T}\widetilde{P}_{\infty,T}|_{L^2_{T,[0,1]}(\Omega^i(\widetilde{M},\widetilde{F}))})^{(-1)^i/2}, \tag{5.20}$$

which can be thought of as an L^2-extension of [5, Theorem 6.17].

Now for any $0\leqslant i\leqslant n$, by Propositions 2.1, 5.2 and 5.3, one computes that when $T>0$ is large enough,

$$\begin{aligned}
&\mathrm{Det}_{\tau_{\mathscr{N}(\Gamma)}}(\widetilde{P}^*_{\infty,T}\widetilde{P}_{\infty,T}|_{L^2_{T,[0,1]}(\Omega^i(\widetilde{M},\widetilde{F}))})\\
&\quad=\mathrm{Det}_{\tau_{\mathscr{N}(\Gamma)}}(\widetilde{e}_T\widetilde{e}^*_T\widetilde{P}^*_{\infty,T}\widetilde{P}_{\infty,T}|_{L^2_{T,[0,1]}(\Omega^i(\widetilde{M},\widetilde{F}))})\\
&\qquad\cdot\mathrm{Det}^{-1}_{\tau_{\mathscr{N}(\Gamma)}}(\widetilde{e}_T\widetilde{e}^*_T|_{L^2_{T,[0,1]}(\Omega^i(\widetilde{M},\widetilde{F}))})\\
&\quad=\mathrm{Det}_{\tau_{\mathscr{N}(\Gamma)}}((\widetilde{P}_{\infty,T}\widetilde{e}_T)^*\widetilde{P}_{\infty,T}\widetilde{e}_T|_{C^i(W^{\mathrm{u}},\widetilde{F})})\cdot\mathrm{Det}^{-1}_{\tau_{\mathscr{N}(\Gamma)}}(\widetilde{e}^*_T\widetilde{e}_T|_{C^i(W^{\mathrm{u}},\widetilde{F})})\\
&\quad=\mathrm{Det}_{\tau_{\mathscr{N}(\Gamma)}}\left((1+O(\mathrm{e}^{-cT}))^*\left(\frac{\pi}{T}\right)^{N-n/2}\mathrm{e}^{2T\widetilde{\mathscr{F}}}(1+O(\mathrm{e}^{-cT}))|_{C^i(W^{\mathrm{u}},\widetilde{F})}\right)\\
&\qquad\cdot\mathrm{Det}^{-1}_{\tau_{\mathscr{N}(\Gamma)}}((1+O(\mathrm{e}^{-cT}))|_{C^i(W^{\mathrm{u}},\widetilde{F})})\\
&\quad=\mathrm{Det}_{\tau_{\mathscr{N}(\Gamma)}}\left(\left(\frac{\pi}{T}\right)^{N-n/2}\mathrm{e}^{2T\widetilde{\mathscr{F}}}(1+O(\mathrm{e}^{-cT}))(1+O(\mathrm{e}^{-cT}))^*\Big|_{C^i(W^{\mathrm{u}},\widetilde{F})}\right)\\
&\qquad\cdot\mathrm{Det}^{-1}_{\tau_{\mathscr{N}(\Gamma)}}((1+O(\mathrm{e}^{-cT}))|_{C^i(W^{\mathrm{u}},\widetilde{F})}).
\end{aligned} \tag{5.21}$$

From Proposition 2.1 and (5.21), one deduces that as $T\to+\infty$,

$$\begin{aligned}
\log\,\mathrm{Det}_{\tau_{\mathscr{N}(\Gamma)}}(\widetilde{P}^*_{\infty,T}\widetilde{P}_{\infty,T}|_{L^2_{T,[0,1]}(\Omega^i(\widetilde{M},\widetilde{F}))})&=\mathrm{rk}(F)\left(i-\frac{n}{2}\right)\log\left(\frac{\pi}{T}\right)\\
&\quad+2T\,\mathrm{rk}(F)\sum_{x\in B,\mathrm{ind}(x)=i}f(x)+o(1).
\end{aligned} \tag{5.22}$$

Following [4, (7.13)–(7.15)], we introduce the notations

$$\chi(F)=\sum_{i=0}^{\dim M}(-1)^i\dim H^i(M,F)=\mathrm{rk}(F)\sum_{x\in B}(-1)^{\mathrm{ind}(x)},$$

$$\widetilde{\chi}'(F)=\mathrm{rk}(F)\sum_{x\in B}(-1)^{\mathrm{ind}(x)}\mathrm{ind}(x),$$

$$\mathrm{Tr}_s^B[f]=\sum_{x\in B}(-1)^{\mathrm{ind}(x)}f(x). \tag{5.23}$$

From (5.20), (5.22) and (5.23), one gets that as $T\to+\infty$,

$$\log\frac{\widetilde{P}^{\det\mathscr{H}}_{\infty,T}(\mathscr{T}_{[0,1]}(\widetilde{M},F,g^{TM},g_T^F))}{\rho_{(\widetilde{M},F,g^F,-X)}}-T\mathrm{rk}(F)\mathrm{Tr}_s^B[f]-\left(\frac{n}{4}\chi(F)-\frac{\widetilde{\chi}'(F)}{2}\right)\log\left(\frac{T}{\pi}\right)\to 0. \tag{5.24}$$

On the other hand, in using our notation, one sees that [4, Theorem 7.6] is equivalent to the following formula:

$$\log\frac{P^{\det H}_{\infty,T}(\mathscr{T}_{[0,1]}(M,F,g^{TM},g_T^F))}{\rho_{(M,F,g^F,-X)}}-T\mathrm{rk}(F)\mathrm{Tr}_s^B[f]-\left(\frac{n}{4}\chi(F)-\frac{\widetilde{\chi}'(F)}{2}\right)\log\left(\frac{T}{\pi}\right)\to 0. \tag{5.25}$$

From (5.24) and (5.25), one gets (4.22).

The proof of Theorem 4.5 is completed. □

6. A proof of Theorem 4.6

In this section, we prove Theorem 4.6. The method of finite propagation speed will play an essential role in the proof.

This section is organized as follows. In Section 6.1, we give an explicit expression of $\mathscr{T}_{[1,+\infty)}(\widetilde{M},F,g^{TM},g_T^F)$ and decompose it into two parts: a part involving the integration from 0 to 1 and the other from 1 to $+\infty$. In Section 6.2, we show that the part involving the integration from 1 to $+\infty$ tends to zero as $T\to+\infty$. In Section 6.3, we deal with the part involving the integration from 0 to 1. We show that this part can be further decomposed into two parts and one of them tends to zero as $T\to+\infty$. In Section 6.4, we complete the proof of Theorem 4.6.

6.1. An expression of $\mathscr{T}_{[1,+\infty)}(\widetilde{M},F,g^{TM},g_T^F)$

We continue the discussion in Section 5.

Recall that when $T>0$ is large enough, $1\notin\mathrm{Spec}(\widetilde{D}_T^2)$ and $\mathscr{T}_{[1,+\infty)}(\widetilde{M},F,g^{TM},g_T^F)$ can be defined as in (3.7) by setting $\varepsilon=1$ and by replacing g^F there by g_T^F.

More precisely, for $T>0$ large enough and $s\in\mathbf{C}$ with $\mathrm{Re}(s)>n/2$, set (cf. (3.6) and (3.47)),

$$\theta_T(s)=\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\mathrm{Tr}_{\mathscr{N},s}[N\ \exp(-t\widetilde{D}_T^2|_{L^2_{T,[1,+\infty)}(\Omega^*(\widetilde{M},\widetilde{F}))})]\,\mathrm{d}t. \tag{6.1}$$

Then $\theta_T(s)$ can be extended as a meromorphic function on $\mathbf{C}$ which is holomorphic at $s=0$, and

$$\log\mathscr{T}_{[1,+\infty)}(\widetilde{M},F,g^{TM},g_T^F)=\frac{1}{2}\left.\frac{\partial\theta_T(s)}{\partial s}\right|_{s=0}. \tag{6.2}$$

Now by Lott [19, Lemma 4] and Bismut and Zhang [4, Theorem 7.10], one sees that for any $T\geqslant 0$, when $t\to 0^+$, one has the asymptotic expansion

$$\mathrm{Tr}_{\mathscr{N},s}[N\exp(-t\widetilde{D}_T^2)]=\frac{a_{-1}}{\sqrt{t}}+a_0+O(\sqrt{t}), \tag{6.3}$$

where a_{-1}, a_0 are defined in [4, (7.55)] and do not depend on T as well as Γ.

Also, since $L^2_{T,[0,1)}(\Omega^*(\widetilde{M},\widetilde{F}))$ is a finitely generated $\mathscr{N}(\Gamma)$-Hilbert module and is $\mathscr{N}(\Gamma)$-isomorphic to $C^*(W^{\mathrm{u}},\widetilde{F})$ when $T>0$ is large enough, one sees that when $T>0$ is large enough, one has the asymptotic expansion

$$\mathrm{Tr}_{\mathscr{N},s}\left[N\exp\left(-t\widetilde{D}_T^2|_{L^2_{T,[0,1)}(\Omega^*(\widetilde{M},\widetilde{F}))}\right)\right]=\widetilde{\chi}'(F)+O(\sqrt{t}). \tag{6.4}$$

From (6.1), (6.3) and (6.4), one can rewrite (6.1) as

$$\begin{aligned}\theta_T(s)=&\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left(\mathrm{Tr}_{\mathscr{N},s}[N\exp(-t\widetilde{D}_T^2|_{L^2_{T,[1,+\infty)}(\Omega^*(\widetilde{M},\widetilde{F}))})]-\frac{a_{-1}}{\sqrt{t}}-a_0+\widetilde{\chi}'(F)\right)\mathrm{d}t\\&+\frac{1}{\Gamma(s)}\int_1^{+\infty}t^{s-1}\mathrm{Tr}_{\mathscr{N},s}[N\exp(-t\widetilde{D}_T^2|_{L^2_{T,[1,+\infty)}(\Omega^*(\widetilde{M},\widetilde{F}))})]\,\mathrm{d}t\\&+\frac{a_{-1}}{\Gamma(s)(s-\frac{1}{2})}+\frac{a_0-\widetilde{\chi}'(F)}{\Gamma(s+1)}.\end{aligned} \tag{6.5}$$

From (6.3) to (6.5), we get

$$\begin{aligned}\left.\frac{\partial\theta_T(s)}{\partial s}\right|_{s=0}=&\int_0^1\left(\mathrm{Tr}_{\mathscr{N},s}[N\exp(-t\widetilde{D}_T^2|_{L^2_{T,[1,+\infty)}(\Omega^*(\widetilde{M},\widetilde{F}))})]-\frac{a_{-1}}{\sqrt{t}}-a_0+\widetilde{\chi}'(F)\right)\frac{\mathrm{d}t}{t}\\&+\int_1^{+\infty}\mathrm{Tr}_{\mathscr{N},s}[N\exp(-t\widetilde{D}_T^2|_{L^2_{T,[1,+\infty)}(\Omega^*(\widetilde{M},\widetilde{F}))})]\frac{\mathrm{d}t}{t}\\&-2a_{-1}-\Gamma'(1)(a_0-\widetilde{\chi}'(F)).\end{aligned} \tag{6.6}$$

Set

$$\widetilde{\theta}_1(T)=\int_0^1\left(\mathrm{Tr}_{\mathscr{N},s}[N\exp(-t\widetilde{D}_T^2|_{L^2_{T,[1,+\infty)}(\Omega^*(\widetilde{M},\widetilde{F}))})]-\frac{a_{-1}}{\sqrt{t}}-a_0+\widetilde{\chi}'(F)\right)\frac{\mathrm{d}t}{t}, \tag{6.7}$$

$$\widetilde{\theta}_2(T)=\int_1^{+\infty}\mathrm{Tr}_{\mathscr{N},s}[N\exp(-t\widetilde{D}_T^2|_{L^2_{T,[1,+\infty)}(\Omega^*(\widetilde{M},\widetilde{F}))})]\frac{\mathrm{d}t}{t}. \tag{6.8}$$

In the next subsections, we will study the behavior as $T\to+\infty$ of $\widetilde{\theta}_1(T)$ and $\widetilde{\theta}_2(T)$ separately.

6.2. The behavior of $\widetilde{\theta}_2(T)$ as $T \to +\infty$

In this subsection, we prove the following result.

Proposition 6.1. *The following identity holds*:

$$\lim_{T\to+\infty} \widetilde{\theta}_2(T) = 0. \tag{6.9}$$

Proof. We first prove the following L^2-analogue of [4, Theorem 7.8].

Lemma 6.2. *For any* $t > 0$,

$$\lim_{T\to+\infty} \mathrm{Tr}_{\mathcal{N},s}[N \exp(-t\widetilde{D}_T^2|_{L^2_{T,[1,+\infty)}(\Omega^*(\widetilde{M},\widetilde{F}))})] = 0. \tag{6.10}$$

Moreover, there exist $C > 0$, $c > 0$ *and* $T_1 > 0$ *such that for* $t \geqslant 1$, $T \geqslant T_1$, *one has*

$$|\mathrm{Tr}_{\mathcal{N},s}[N \exp(-t\widetilde{D}_T^2|_{L^2_{T,[1,+\infty)}(\Omega^*(\widetilde{M},\widetilde{F}))})]| \leqslant c \exp(-Ct). \tag{6.11}$$

Proof. Let $\widetilde{J}_T : C^*(W^{\mathrm{u}}, \widetilde{F}) \to L^2(\Omega^*(\widetilde{M}, \widetilde{F}))$ be the map defined in (5.12). Let

$$\widetilde{E}_T = \widetilde{J}_T(C^*(W^{\mathrm{u}}, \widetilde{F})) \tag{6.12}$$

be the image of $\widetilde{J}_T$. Since $\widetilde{J}_T$ is an isometry, $\widetilde{E}_T \subseteq L^2(\Omega^*(\widetilde{M}, \widetilde{F}))$ is closed.

Let $\widetilde{E}_T^{\perp}$ denote the orthogonal complement of $\widetilde{E}_T$ in $L^2(\Omega^*(\widetilde{M}, \widetilde{F}))$, that is,

$$L^2(\Omega^*(\widetilde{M}, \widetilde{F})) = \widetilde{E}_T \oplus \widetilde{E}_T^{\perp}. \tag{6.13}$$

Let $\widetilde{P}_T$ (resp. $\widetilde{P}_T^{\perp}$) denote the orthogonal projection from $L^2(\Omega^*(\widetilde{M}, \widetilde{F}))$ onto $\widetilde{E}_T$ (resp. $\widetilde{E}_T^{\perp}$).

Recall that $\widetilde{D}'_T$ has been defined in (5.3).

Following Bismut–Lebeau [3, Section 9], we define

$$\begin{aligned} &\widetilde{D}'_{T,1} = \widetilde{P}_T \widetilde{D}'_T \widetilde{P}_T, \quad \widetilde{D}'_{T,2} = \widetilde{P}_T \widetilde{D}'_T \widetilde{P}_T^{\perp}, \\ &\widetilde{D}'_{T,3} = \widetilde{P}_T^{\perp} \widetilde{D}'_T \widetilde{P}_T, \quad \widetilde{D}'_{T,4} = \widetilde{P}_T^{\perp} \widetilde{D}'_T \widetilde{P}_T^{\perp}. \end{aligned} \tag{6.14}$$

Recall that $\mathbf{H}^1(\widetilde{M}, \widetilde{F})$ denotes the first Sobolev space with respect to a (fixed, Γ-invariant) first Sobolev norm on $\Omega^*(\widetilde{M}, \widetilde{F})$.

By proceeding as in [3, Section 9b], which can be made much simpler in the current situation (cf. [37, Proposition 5.2]), one deduces that

(i) The following identity holds:

$$\widetilde{D}'_{T,1} = 0. \tag{6.15}$$

(ii) There exists $T_2 > 0$ such that for any $s \in \widetilde{E}_T^{\perp} \cap \mathbf{H}^1(\widetilde{M}, \widetilde{F})$, $s' \in \widetilde{E}_T \cap \mathbf{H}^1(\widetilde{M}, \widetilde{F})$ and $T \geqslant T_2$, one has

$$\|\widetilde{D}'_{T,2}s\|_0 \leqslant \frac{\|s\|_0}{T} \quad \text{and} \quad \|\widetilde{D}'_{T,3}s'\|_0 \leqslant \frac{\|s'\|_0}{T}. \tag{6.16}$$

(iii) There exist $T_3>0$ and $C'>0$ such that for any $s\in\widetilde{E}_T^{\perp}\cap\mathbf{H}^1(\widetilde{M},\widetilde{F})$ and $T\geqslant T_3$, one has

$$\|\widetilde{D}'_{T,4}s\|_0\geqslant C'\sqrt{T}\|s\|_0. \tag{6.17}$$

Following [3, (9.113)], for $T\geqslant 1$, set

$$U_T=\left\{\lambda\in\mathbf{C}:1\leqslant|\lambda|\leqslant\frac{C'\sqrt{T}}{4}\right\}. \tag{6.18}$$

From (6.15) to (6.18), one can proceed as in [3, Section 9e] to show that there exists $T_4\geqslant 1$ such that for any $T\geqslant T_4$, $\lambda\in U_T$, $\lambda-\widetilde{D}'_T$ is invertible. For any positive integer $p\geqslant n+2$, there exists $C''>0$ such that if $T\geqslant T_4$, $\lambda\in U_T$, the following L^2-analogue of [3, (9.142)] in our current situation holds,

$$|\mathrm{Tr}_{\mathscr{N},s}[N(\lambda-\widetilde{D}'_T)^{-p}]-\lambda^{-p}\widetilde{\chi}'(F)|\leqslant\frac{C''}{\sqrt{T}}(1+|\lambda|)^{p+1}. \tag{6.19}$$

From (6.19), one can then proceed as in [4, Section 10c], with an obvious L^2-modification, to complete the proof of Lemma 6.2. □

From (5.3), (6.8), Lemma 6.2 and the dominate convergence theorem, one gets (6.9).

The proof of Proposition 6.1 is completed. □

Remark 6.3. Alternatively, one can proceed as in [8, Corollary 6.9], which works equally in the nonunitary case, to get the following analogue of [8, (6.75)]: there exist $T_5>0$, $c'>0$, $c''>0$ such that if $t\geqslant 1$, $T\geqslant T_5$, then

$$|\mathrm{Tr}_{\mathscr{N},s}[N\exp(-t\widetilde{D}'^2_T|_{L^2_{[1,+\infty),T}(\Omega^*(\widetilde{M},\widetilde{F}))})]|\leqslant c'\exp(-c''tT). \tag{6.20}$$

From (5.3) and (6.20), one also gets Lemma 6.2.

6.3. *A behavior of* $\widetilde{\theta}_1(T)$ *as* $T\to+\infty$

From (5.3) and (6.7), one can rewrite $\widetilde{\theta}_1(T)$ as

$$\begin{aligned}\widetilde{\theta}_1(T)&=\int_0^1\left(\mathrm{Tr}_{\mathscr{N},s}[N\exp(-t\widetilde{D}'^2_T|_{L^2_{[1,+\infty),T}(\Omega^*(\widetilde{M},\widetilde{F}))})]-\frac{a_{-1}}{\sqrt{t}}-a_0+\widetilde{\chi}'(F)\right)\frac{\mathrm{d}t}{t}\\&=2\int_0^1\left(\mathrm{Tr}_{\mathscr{N},s}[N\exp(-t^2\widetilde{D}'^2_T|_{L^2_{[1,+\infty),T}(\Omega^*(\widetilde{M},\widetilde{F}))})]-\frac{a_{-1}}{t}-a_0+\widetilde{\chi}'(F)\right)\frac{\mathrm{d}t}{t}.\end{aligned} \tag{6.21}$$

We now proceed as in [3, Section 13b].

Let $\alpha>0$ be a positive constant. Let $h:\mathbf{R}\to[0,1]$ be a smooth function such that

$$h(t)=1\quad\text{if }|t|\leqslant\frac{\alpha}{2},\quad h(t)=0\text{ if }|t|\geqslant\alpha. \tag{6.22}$$

Set

$$g(t)=1-h(t). \tag{6.23}$$

Following [3, (13.9)], if $t \in (0,1]$, $a \in \mathbf{C}$, set

$$H_t(a) = \int_{-\infty}^{+\infty} \exp(\mathrm{i}u\sqrt{2}a) \exp\left(-\frac{u^2}{2}\right) h(ut) \frac{\mathrm{d}u}{\sqrt{2\pi}}.$$

$$G_t(a) = \int_{-\infty}^{+\infty} \exp(\mathrm{i}u\sqrt{2}a) \exp\left(-\frac{u^2}{2}\right) g(ut) \frac{\mathrm{d}u}{\sqrt{2\pi}}. \tag{6.24}$$

Then

$$\exp(-a^2) = H_t(a) + G_t(a). \tag{6.25}$$

Since h is even, H_t and G_t are even functions, which takes values in $\mathbf{R}$. Moreover, H_t and G_t lie in the Schwartz space $S(\mathbf{R})$.

For simplicity, denote

$$\widetilde{D}''_T = \widetilde{D}'_T|_{L^2_{[1,+\infty),T}(\Omega^*(\widetilde{M},\widetilde{F}))}. \tag{6.26}$$

From (6.25), (6.26), one deduces that

$$\exp(-t^2\widetilde{D}''^2_T) = H_t(t\widetilde{D}''_T) + G_t(t\widetilde{D}''_T). \tag{6.27}$$

Since $H_t, G_t \in S(\mathbf{R})$ and since $t\widetilde{D}'_T$ verifies the elliptic estimate, one verifies easily that $H_t(t\widetilde{D}''_T)$ and $G_t(t\widetilde{D}''_T)$ are given by smooth kernels, and so are of trace class with respect to $\mathrm{Tr}_{\mathscr{N}}$ (cf. [1]).

The main result of this section can be stated as follows. It can be thought of as an L^2-analogue of [3, Theorem 13.4] in our situation.

Proposition 6.4. *There exist $c>0$, $C>0$, $T_0>0$ such that for any $t \in (0,1]$, $T \geqslant T_0$,*

$$|\mathrm{Tr}_{\mathscr{N},s}[NG_t(t\widetilde{D}''_T)]| \leqslant \frac{c}{\sqrt{T}} \exp\left(-\frac{C}{t^2}\right). \tag{6.28}$$

Proof. Set

$$I_t(a) = \int_{-\infty}^{+\infty} \exp(\mathrm{i}u\sqrt{2}a) \exp\left(-\frac{u^2}{2t^2}\right) g(u) \frac{\mathrm{d}u}{t\sqrt{2\pi}}. \tag{6.29}$$

Then

$$G_t(a) = I_t\left(\frac{a}{t}\right). \tag{6.30}$$

Observe that $g(t)=0$ near $t=0$. For $p \in \mathbf{N}$, set as in [3, (13.15)]

$$I_{t,p}(a) = (p-1)! \int_{-\infty}^{+\infty} \exp(\mathrm{i}u\sqrt{2}a) \exp\left(-\frac{u^2}{2t^2}\right) \frac{g(u)}{(\mathrm{i}u\sqrt{2})^{p-1}} \frac{\mathrm{d}u}{t\sqrt{2\pi}}. \tag{6.31}$$

Clearly,

$$\frac{I^{(p-1)}_{t,p}(a)}{(p-1)!} = I_t(a). \tag{6.32}$$

Then $a \in \mathbf{C} \to I_{t,p}(a)$ is holomorphic. Moreover, for any $c>0$, if $|\mathrm{Im}(a)| \leqslant c$, as $|a| \to +\infty$, $I_{t,p}(a)$ decay faster than any $|a|^{-m}$.

Let $\Delta = \Delta_+ \cup \Delta_- \subset \mathbf{C}$ be the contour considered in [4, Section 10c]. That is

$$\Delta_+ = \{x+\mathrm{i}y : x=1, -1 \leqslant y \leqslant 1\} \cup \{x+\mathrm{i}y : x \geqslant 1, y=1\} \cup \{x+\mathrm{i}y : x \geqslant 1, y=-1\},$$

$$\Delta_- = \{x+\mathrm{i}y : x=-1, -1 \leqslant y \leqslant 1\} \cup \{x+\mathrm{i}y : x \leqslant -1, y=1\} \cup \{x+\mathrm{i}y : x \leqslant -1, y=-1\}.$$

We orient $\Delta_\pm$ in counter clockwise manner. Then for any $a \in \mathbf{R}$ with $|a|>1$,

$$I_t(a) = \frac{1}{2\pi \mathrm{i}} \int_\Delta I_t(\lambda)(\lambda - a)^{-1}\,\mathrm{d}\lambda. \tag{6.33}$$

Equivalently,

$$I_t(a) = \frac{1}{2\pi \mathrm{i}} \int_\Delta I_{t,p}(\lambda)(\lambda - a)^{-p}\,\mathrm{d}\lambda \tag{6.34}$$

for any $p \in \mathbf{N}$.

From (6.30), one has

$$G_t(t\widetilde{D}''_T) = I_t(\widetilde{D}''_T). \tag{6.35}$$

We now take $p \geqslant \dim M + 2$. From (6.34) and (6.35), one gets

$$G_t(t\widetilde{D}''_T) = \frac{1}{2\pi \mathrm{i}} \int_\Delta I_{t,p}(\lambda)(\lambda - \widetilde{D}''_T)^{-p}\,\mathrm{d}\lambda. \tag{6.36}$$

For $T>0$, let $U_T \subset \mathbf{C}$ be defined as in (6.18). Recall that if $T>0$ is large enough, if $\lambda \in U_T$, then (6.19) holds.

Also, since $g(t)$ vanishes near $t=0$, we deduce from (6.31) that for any $m \in \mathbf{N}$, there exist $c_m, C_m > 0$ such that if $\lambda \in \Delta$,

$$|\lambda^m I_{t,p}(\lambda)| \leqslant c_m \exp\left(-\frac{C_m}{t^2}\right) \tag{6.37}$$

(cf. [3, (13.23)]).

From (6.19), (6.37), it is clearly that when $T>0$ is large enough,

$$\left|\mathrm{Tr}_{\mathcal{N},s}\left[N\frac{1}{2\pi \mathrm{i}}\int_{\Delta\cap U_T} I_{t,p}(\lambda)(\lambda - \widetilde{D}'_T)^{-p}\right]\mathrm{d}\lambda - \widetilde{\chi}'(F)\frac{1}{2\pi \mathrm{i}}\int_{\Delta\cap U_T} I_{t,p}(\lambda)\lambda^{-p}\,\mathrm{d}\lambda\right| \leqslant \frac{c}{\sqrt{T}}\exp\left(-\frac{C}{t^2}\right) \tag{6.38}$$

for some positive constants $c, C>0$.

On the other hand, for $T>0$ large enough, if $\lambda \in \Delta \cap^c U_T$, by proceeding as in [3, Section 9e], one deduces that the following L^2-analogue of [3, (9.170)] holds,

$$|\mathrm{Tr}_{\mathcal{N},s}[N(\lambda - \widetilde{D}'_T)^{-p}]| \leqslant c'(1+|\lambda|)^p \tag{6.39}$$

for some constant $c'>0$.

From (6.37) and (6.39), one finds that for any $m \in \mathbf{N}$, if $T > 0$ is large enough, then

$$\left|\mathrm{Tr}_{\mathcal{N},s}\left[N\frac{1}{2\pi i}\int_{\Delta\cap^c U_T} I_{t,p}(\lambda)(\lambda-\widetilde{D}'_T)^{-p}\right]\mathrm{d}\lambda\right| \leqslant c'_m\left(\frac{1}{T}\right)^m \exp\left(-\frac{C'_m}{t^2}\right), \tag{6.40}$$

for some positive constants c'_m, $C'_m > 0$. Also, by (6.37), one has

$$\left|\frac{1}{2\pi i}\int_{\Delta\cap^c U_T} I_{t,p}(\lambda)\lambda^{-p}\,\mathrm{d}\lambda\right| \leqslant c''_m\left(\frac{1}{T}\right)^m \exp\left(-\frac{C''_m}{t^2}\right) \tag{6.41}$$

for some positive constants c''_m, $C''_m > 0$.

From (6.36), (6.38), (6.40), (6.41) and the obvious identity

$$\frac{1}{2\pi\mathrm{i}}\int_{\Delta} I_{t,p}(\lambda)\lambda^{-p}\,\mathrm{d}\lambda = 0, \tag{6.42}$$

one gets (6.28).

The proof of Proposition 6.4 is completed. □

6.4. A Proof of Theorem 4.6

It is clear that all the above analysis works equally well for the $\Gamma = \{e\}$ case. As was indicated earlier, we will use notations without "$\sim$" to denote the corresponding quantities for the $\Gamma = \{e\}$ case.

We now will compare $\widetilde{\theta}_1(T)$ defined in (6.21) with $\theta_1(T)$ which corresponds to the $\Gamma = \{e\}$ case. That is,

$$\theta_1(T) = 2\int_0^1 \left(\mathrm{Tr}_s[N\exp(-t^2 D'^2_T|_{L^2_{[1,+\infty),T}(\Omega^*(M,F))})] - \frac{a_{-1}}{t} - a_0 + \widetilde{\chi}'(F)\right)\frac{\mathrm{d}t}{t}. \tag{6.43}$$

From (6.21), (6.26), (6.27) and (6.43), one finds that when $T > 0$ is large enough,

$$\begin{aligned}\widetilde{\theta}_1(T) - \theta_1(T) &= 2\int_0^1 (\mathrm{Tr}_{\mathcal{N},s}[N\exp(-t^2\widetilde{D}''^2_T)] - \mathrm{Tr}_s[N\exp(-t^2 D''^2_T)])\frac{\mathrm{d}t}{t}\\ &= 2\int_0^1 (\mathrm{Tr}_{\mathcal{N},s}[NG_t(t\widetilde{D}''_T)] - \mathrm{Tr}_s[NG_t(tD''_T)])\frac{\mathrm{d}t}{t}\\ &\quad + 2\int_0^1 (\mathrm{Tr}_{\mathcal{N},s}[NH_t(t\widetilde{D}''_T)] - \mathrm{Tr}_s[NH_t(tD''_T)])\frac{\mathrm{d}t}{t}.\end{aligned} \tag{6.44}$$

Now we write

$$\begin{aligned}\mathrm{Tr}_{\mathcal{N},s}[NH_t(t\widetilde{D}''_T)] - \mathrm{Tr}_s[NH_t(tD''_T)] &= \mathrm{Tr}_{\mathcal{N},s}[NH_t(t\widetilde{D}'_T)] - \mathrm{Tr}_s[NH_t(tD'_T)]\\ &\quad - (\mathrm{Tr}_{\mathcal{N},s}[NH_t(t\widetilde{D}'_T|_{L^2_{[0,1),T}(\Omega^*(\widetilde{M},\widetilde{F}))})]\\ &\quad - \mathrm{Tr}_s[NH_t(tD'_T|_{L^2_{[0,1),T}(\Omega^*(M,F))})]).\end{aligned} \tag{6.45}$$

Let $H_t(t\widetilde{D}'_T)(\widetilde{x},\widetilde{x}')$ (resp. $H_t(tD'_T)(x,x')$) denote the smooth kernel associated to $H_t(t\widetilde{D}'_T)$ (resp. $H_t(tD'_T)$) on $\widetilde{M}\times\widetilde{M}$ (resp. $M\times M$) with respect to $\mathrm{dvol}_{g^{T\widetilde{M}}}$ (resp. $\mathrm{dvol}_{g^{TM}}$).

Let W be a fundamental domain of the covering space $\Gamma \to \widetilde{M} \xrightarrow{\pi} M$. Then one has (cf. [1])

$$\mathrm{Tr}_{\mathcal{N},s}[NH_t(t\widetilde{D}'_T)] = \int_W \mathrm{Tr}_s[NH_t(t\widetilde{D}'_T)(\widetilde{x},\widetilde{x})]\,\mathrm{dvol}(\widetilde{x}). \tag{6.46}$$

By proceeding in exactly the same way as in [3, Remark 13.5], which uses in an essential way the finite propagation speed property of hyperbolic equations, one sees that one can choose $\alpha > 0$ in Section 6.3 small enough (but still fixed) so that for any $\widetilde{x} \in \widetilde{M}$, $H_t(t\widetilde{D}'_T)(\widetilde{x},\widetilde{x})$ depends only on the behavior of $\widetilde{D}'_T$ in a sufficiently small neighborhood of $\widetilde{x}$ in $\widetilde{M}$.

In particular, one gets

$$\mathrm{Tr}_s[NH_t(t\widetilde{D}'_T)(\widetilde{x},\widetilde{x})] = \mathrm{Tr}_s[NH_t(tD'_T)(\pi(\widetilde{x}),\pi(\widetilde{x}))]. \tag{6.47}$$

From (6.46), (6.47), one gets

$$\mathrm{Tr}_{\mathcal{N},s}[NH_t(t\widetilde{D}'_T)] = \int_M \mathrm{Tr}_s[NH_t(tD'_T)(x,x)]\,\mathrm{dvol}(x) = \mathrm{Tr}_s[NH_t(tD'_T)]. \tag{6.48}$$

On the other hand, by (4.16) and (6.4), one deduces easily that when $T > 0$ is large enough,

$$|\mathrm{Tr}_{\mathcal{N},s}[NH_t(t\widetilde{D}'_T|_{L^2_{[0,1),T}(\Omega^*(\widetilde{M},\widetilde{F}))})] - \mathrm{Tr}_s[NH_t(tD'_T|_{L^2_{[0,1),T}(\Omega^*(M,F))})]| \leqslant c\sqrt{t}\,\mathrm{e}^{-CT} \tag{6.49}$$

for some constants $c, C > 0$.

From (6.28), (6.44), (6.45), (6.48), (6.49) and the dominant convergence theorem, one gets

$$\lim_{T\to+\infty} (\widetilde{\theta}_1(T) - \theta_1(T)) = 0. \tag{6.50}$$

From (6.9) and (6.50), one gets

$$\lim_{T\to+\infty} (\widetilde{\theta}_1(T) + \widetilde{\theta}_2(T) - (\theta_1(T) + \theta_2(T))) = 0. \tag{6.51}$$

From (6.2), (6.6), (6.7) and (6.51), one gets (4.23).

The proof of Theorem 4.6 is completed. □

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