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QUANTIZATION FORMULA FOR SYMPLECTIC MANIFOLDS WITH BOUNDARY

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Abstract

We extend our earlier work in [TiZ1], where an analytic approach to the Guillemin-Sternberg geometric quantization conjecture [GuSt] was developed, to the case of manifolds with boundary. We also give a general quantization formula that works for both regular and singular reductions. As simple applications, we prove an analytic analogue of the relative residue formula of Guillemin-Kalkman [GuK] and Martin [M], as well as a Guillemin-Sternberg type formula for singular reductions under circle actions.

0 Introduction

In a previous paper [TiZ1], we have developed a direct analytic approach to the Guillemin-Sternberg geometric quantization conjecture [GuSt], which has been proved in various generalities in [DuGMW], [Gu], [GuSt], [JKi], [Me1,2], [V]. The purpose of this paper is to extend this quantization formula to manifolds with boundary by using the methods in [TiZ1].

Let (M,ω) be a compact symplectic manifold with boundary ∂M such that there is a Hermitian line bundle L over M admitting a Hermitian connection ∇^L with the property that $\frac{\sqrt{-1}}{2\pi}(\nabla^L)^2 = \omega$. Let J be an almost complex structure on TM such that $g^{TM}(u,v) = \omega(u, Jv)$ defines a Riemannian metric on TM. With these data in hand, one can construct canonically a Spin^c-Dirac operator

$$D_+^L : \Omega^{0,\mathrm{even}}(M,L) \to \Omega^{0,\mathrm{odd}}(M,L)\,, \tag{0.1}$$

and thus also a canonically induced formally self-adjoint Spin^c-Dirac operator

$$D_{\partial M,+}^L : \Omega^{0,\mathrm{even}}(M,L)|_{\partial M} \to \Omega^{0,\mathrm{even}}(M,L)|_{\partial M} \tag{0.2}$$

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on ∂M (cf. [APS1], [G]). One can use $D^L_{\partial M,+}$ to impose the Atiyah-Patodi-Singer boundary condition [APS1] to get an elliptic operator $D^L_{+,APS}$, which gives rise to the finite dimensional virtual vector space

$$Q_{APS}(M,L) = \ker D^L_{+,APS} - \operatorname{coker} D^L_{+,APS}\,. \tag{0.3}$$

Suppose (M,ω) admits a Hamiltonian action of a compact connected Lie group G with Lie algebra $\mathbf{g}$. Clearly, this action preserves the boundary ∂M. Let $\mu : M \to \mathbf{g}^*$ be the corresponding moment map. A formula due to Kostant [Ko] (cf. [TiZ1, (1.13)]) induces a natural $\mathbf{g}$ action on L. One makes the basic assumption that this $\mathbf{g}$ action can be lifted to a G action on L. Then this G action preserves ∇^L. One can also assume, after an integration over G if necessary, that G preserves the Hermitian metric on L, the almost complex structure J and thus also the Riemannian metric g^{TM}.

Now we make the assumption that $0 \in \mathbf{g}^*$ is a regular value of μ, and, for simplicity, that G acts freely on $\mu^{-1}(0)$. Furthermore, we assume in this paper that $\mu^{-1}(0) \cap \partial M = \emptyset$. Then $M_G = \mu^{-1}(0)/G$ is a smooth closed symplectic manifold on which one can construct the virtual vector space $Q(M_G, L_G)$ (cf. e.g. [TiZ1]).

Since G preserves everything, it commutes with D^L_+ and $D^L_{\partial M,+}$. Thus $Q_{APS}(M,L)$ is a virtual G representation. Denote by $Q_{APS}(M,L)^G$ its G-trivial component.

Let $\mathbf{g}$ (and thus $\mathbf{g}^*$ also) be equipped with an $\operatorname{Ad} G$ invariant metric. Let $\mathcal{H} = |\mu|^2$ be the norm square of the moment map μ. We denote by $X^{\mathcal{H}}$ the Hamiltonian vector field associated to $\mathcal{H}$. Then one verifies easily that $X^{\mathcal{H}}|_{\partial M} \in \Gamma(T\partial M)$.

If $X \in \Gamma(T\partial M)$, we use the notation $\widetilde{c}(X)$ to denote the canonical Clifford action of X on $\Omega^{0,\mathrm{even}}(M,L)|_{\partial M}$. Also we identify $T^*(\partial M)$ with $T\partial M$ by using the metric $g^{T\partial M} = g^{TM}|_{\partial M}$.

For any $T \in \mathbf{R}$, we define an odd dimensional analogue of the deformation introduced in [TiZ1, Definition 1.2] as

$$D^L_{\partial M,+,T} = D^L_{\partial M,+} + \frac{\sqrt{-1}T}{2}\widetilde{c}(X^{\mathcal{H}})\,. \tag{0.4}$$

It is again a formally self-adjoint elliptic operator.

Let e_m, $m = \dim M$, be the inward unit normal vector field perpendicular to ∂M. Let y_m be the associated geodesic distance coordinate to ∂M. Let $d^{\partial M}$ be the exterior derivation on ∂M.

Denote by $\mathcal{B} \subset M$ the set of critical points of $\mathcal{H}$. Set $B = \mathcal{B} \cap \partial M$.

We can now state the main result of this paper as follows.

Theorem 0.1. *If the following inequality holds on B,*

$$4\pi\mathcal{H} - \frac{\sqrt{-1}}{2}\widetilde{c}\left(d^{\partial M}\frac{\partial\mathcal{H}}{\partial y_m}\right)\widetilde{c}(Je_m) > 0\,, \tag{0.5}$$

then there exists $T_0 > 0$ such that for any $T \geq T_0$,

(i) *when restricted to the G-invariant subspace of $\Omega^{0,\mathrm{even}}(M,L)|_{\partial M}$, $D^L_{\partial M,+,T}$ is invertible;*

(ii) *the following identity holds,*

$$\dim Q(M_G, L_G) = \dim Q_{APS}(M,L)^G - \mathrm{sf}^G\{D^L_{\partial M,+}, D^L_{\partial M,+,T}\}\,, \tag{0.6}$$

where $\mathrm{sf}^G\{D^L_{\partial M,+}, D^L_{\partial M,+,T}\}$ is a G-invariant version of the spectral flow associated to the linear path $\{(1-u)D^L_{\partial M,+} + uD^L_{\partial M,+,T}\}_{0\leq u\leq 1}$.

The condition (0.5) holds when $G = S^1$ (see section 3 for more details). Also, it is clear that when $\partial M = \emptyset$, (0.6) reduces to the Guillemin-Sternberg conjecture [GuSt].

The basic strategy of the proof of Theorem 0.1 is the same as that in [TiZ1]. That is, we deform the Spin^c-Dirac operator D^L by using the Clifford action of $X^{\mathcal{H}}$, and then apply the methods and techniques of Bismut-Lebeau [BLe] to complete the proof. The analysis outside of ∂M is the same as that in [TiZ1]. While near ∂M, we encounter extra difficulties arising from the non-product nature of the metrics on TM, L, etc. near the boundary. Here we rely on the refined analysis contained in [G]. On the other hand, a G-invariant version of a result of Dai and Zhang [DZ, Theorem 1.1] also plays an important role in obtaining (0.6).

We want to emphasize that in obtaining the localization estimates the situation is quite different depending on whether $B = \emptyset$ or not. If $B = \emptyset$, the localization estimates can be proved quite easily. Otherwise, the analysis near ∂M is by no means trivial. As a manifestation, the condition (0.5) emerges naturally through the process. Since in this paper we only present applications in situations where $B = \emptyset$, we give separate proofs of these two cases in sections 2 and 3 respectively.

The ideas and methods in the proof of Theorem 0.1 have further implications. In one of the two applications given in this paper, we establish a general quantization formula that works for both regular and singular reductions. To be more specific, consider the situation that M is *closed* and that $0 \in \mathbf{g}^*$ may not be a regular value of μ. Let $c > 0$ be a regular value of $\mathcal{H}$ so that the preimage M^c_+ under $\mathcal{H}$ of $[0,c]$ contains no critical points of $\mathcal{H}$ other than $\mu^{-1}(0)$. Then our second main result is the following universal asymptotic quantization formula.

Theorem 0.2. *There exists $T_0 > 0$ such that for any $T \geq T_0$,*

1°. *when restricted to the G-invariant subspace of $\Omega^{0,\mathrm{even}}(M_+^c, L)|_{\partial M_+^c}$, $D^L_{\partial M_+^c, T}$ is invertible;*

2°. *the following identity holds,*

$$\dim Q(M,L)^G = \dim Q_{APS,T}(M_+^c, L|_{M_+^c})^G \,. \tag{0.7}$$

This theorem allows us to prove a Guillemin-Sternberg type formula for singular reductions under circle actions. When L is twisted by a S^1-equivariant Hermitian vector bundle, we get a similar asymptotic quantization formula. As an interesting phenomenon, a nontrivial correction term naturally arises in our formula, see section 6c) for more details. In the other application, we prove an analytic analogue of the relative residue formula of Guillemin-Kalkman [GuK] and Martin [M].

This paper is organized as follows. In section 1, we recall the construction of Spin^c-Dirac operators on symplectic manifolds and the key deformation under the appearance of Hamiltonian group actions. We also introduce the corresponding Atiyah-Patodi-Singer type boundary value problems. In section 2, we prove part (i) of Theorem 0.1, as well as an estimate which enables us to localize the problem of proving (0.6) to sufficiently small neighborhoods of $\mu^{-1}(0)$, under an extra assumption that $B = \emptyset$. This is sufficient for our applications in sections 5 and 6. In section 3, we analyze carefully the general case where B is not empty. We prove the first part of Theorem 0.1 and establish the crucial estimates needed for the localization procedure. The arguments involved here are more delicate than those in section 2. In section 4, we introduce the concept of invariant spectral flow and complete the proof of (0.6). Section 5 contains the applications leading to an analytic analogue of the relative residue formula of Guillemin-Kalkman [GuK] and Martin [M]. In section 6, we present the proof of Theorem 0.2 as well as a Guillemin-Sternberg type formula for singular reductions under circle actions. There is also an Appendix in which we give precise forms of the fixed point set contributions appearing in sections 5 and 6.

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1 Spin^c-Dirac Operators and Their Deformations on Symplectic Manifolds with Boundary

In this section, we recall the basic properties of Spin^c-Dirac operators on symplectic manifolds with boundary and the associated generalized Atiyah-Patodi-Singer boundary conditions [APS1]. In particular, in dealing with the case of non-product metric near boundary, we will follow the analysis in [G]. We also recall a deformation introduced in [TiZ1] and consider the corresponding generalized Atiyah-Patodi-Singer boundary condition.

This section is organized as follows. In a), we review the construction of Spin^c-Dirac operators on symplectic manifolds. In b), following [APS1] and [G], we examine the generalized Atiyah-Patodi-Singer type boundary conditions for the Spin^c-Dirac operators constructed in a). In c), we examine the above mentioned deformation and the associated generalized Atiyah-Patodi-Singer type boundary problem.

a) Spin^c-Dirac operators on symplectic manifolds. Let (M,ω) be a compact symplectic manifold with boundary ∂M. Let J be an almost complex structure on TM such that

$$g^{TM}(v,w)=\omega(v,Jw) \tag{1.1}$$

defines a Riemannian metric on TM. Let $TM_{\mathbf{C}}=TM\otimes\mathbf{C}$ denote the complexification of the tangent bundle TM. Then one has the canonical splittings

$$\begin{aligned} TM_{\mathbf{C}}&=T^{(1,0)}M\oplus T^{(0,1)}M\,,\\ \wedge^{*,*}(T^*M)&=\bigoplus_{i,j=0}^{\dim_{\mathbf{C}}M}\wedge^{i,j}(T^*M)\,, \end{aligned}\tag{1.2}$$

where

$$\begin{aligned} T^{(1,0)}M&=\left\{z\in TM_{\mathbf{C}};Jz=\sqrt{-1}z\right\},\\ T^{(0,1)}M&=\left\{z\in TM_{\mathbf{C}};Jz=-\sqrt{-1}z\right\},\\ \wedge^{i,j}\,(T^*M)&=\wedge^i(T^{(1,0)*}M)\otimes\wedge^j(T^{(0,1)*}M)\,, \end{aligned}\tag{1.3}$$

and $\dim_{\mathbf{C}}M=\frac12\dim M$ is the complex dimension of M.

The almost complex structure J determines a canonical Spin^c-structure on TM (cf. [LMi, Appendix D]). Furthermore, with g^{TM}, the fundamental $\mathbf{Z}_2$-graded Spin^c-bundle is given by

$$\wedge^{0,*}(T^*M)=\wedge^{0,\mathrm{even}}(T^*M)\oplus\wedge^{0,\mathrm{odd}}(T^*M)\,. \tag{1.4}$$

For any $X \in TM$ whose complexification has the decomposition $X = X_1 + X_2 \in T^{(1,0)}M \oplus T^{(0,1)}M$, let ${\overline{X}_1}^* \in T^{(0,1)*}M$ be the metric dual of X_1 (cf. [BLe, Sect. 5]). Then $c(X) = \sqrt{2}{\overline{X}_1}^* \wedge -\sqrt{2}i_{X_2}$ defines the canonical Clifford action of X on $\wedge^{0,*}(T^*M)$ (cf. [LMi, Appendix D]). It interchanges $\wedge^{0,\mathrm{even}}(T^*M)$ and $\wedge^{0,\mathrm{odd}}(T^*M)$.

Let λ be the complex line bundle

$$\lambda = \det(T^{(1,0)}M)\,. \tag{1.5}$$

We now temporarily assume that M is spin. In this case one can construct a square root $\lambda^{1/2}$ of λ, which together with the canonically induced Spin^c-structure on TM determine a Spin structure on TM. Let $S(TM) = S_+(TM) \oplus S_-(TM)$ be the corresponding $\mathbf{Z}_2$-graded bundle of spinors associated to (M, g^{TM}). Then, one has the following canonical identifications of Clifford modules (cf. [LMi, Appendix D]),

$$\begin{aligned} S_+(TM) \otimes \lambda^{1/2} &= \wedge^{0,\mathrm{even}}(T^*M)\,, \\ S_-(TM) \otimes \lambda^{1/2} &= \wedge^{0,\mathrm{odd}}(T^*M)\,, \\ S(TM) \otimes \lambda^{1/2} &= \wedge^{0,*}(T^*M)\,. \end{aligned} \tag{1.6}$$

Let ∇^{TM} be the Levi-Civita connection of g^{TM}. Then ∇^{TM} together with the almost complex structure J induce via projection a canonical Hermitian connection $\nabla^{T^{(1,0)}M}$ on $T^{(1,0)}M$. This, in turn, induces a Hermitian connection ∇^{λ} on λ and thus a Hermitian connection $\nabla^{\lambda^{1/2}}$ on $\lambda^{1/2}$.

Also, ∇^{TM} lifts to a Hermitian connection $\nabla^{S(TM)}$ on $S(TM)$ preserving $S_\pm(TM)$. Let $\nabla^{S(TM)\otimes\lambda^{1/2}}$ be the tensor product connection on $S(TM) \otimes \lambda^{1/2}$ defined by

$$\nabla^{S(TM)\otimes\lambda^{1/2}} = \nabla^{S(TM)} \otimes \mathrm{Id}_{\lambda^{1/2}} + \mathrm{Id}_{S(TM)} \otimes \nabla^{\lambda^{1/2}}\,. \tag{1.7}$$

Then $\nabla^{S(TM)\otimes\lambda^{1/2}}$ is a well-defined Hermitian connection on $\wedge^{0,*}(T^*M) = S(TM) \otimes \lambda^{1/2}$ and preserves the $\mathbf{Z}_2$-grading. We will also denote this connection by $\nabla^{\wedge^{0,*}(T^*M)}$.

For the general case without the assumption that M is spin, it is well-known that although $\lambda^{1/2}$ and $S(TM)$ might not exist, one can still construct their product which does exist (cf. [LMi, Appendix D]). Furthermore, one can still construct the tensor product connection as above locally and get in fact a globally well-defined connection $\nabla^{\wedge^{0,*}(T^*M)}$ on $\wedge^{0,*}(T^*M)$. In particular, when doing local computations, one can use the above identifications just as in the spin case. From now on, we will drop the spin condition on M and adopt the above convention.

Now assume that there is a Hermitian line bundle L over M with a Hermitian connection ∇^L such that

$$\frac{\sqrt{-1}}{2\pi}(\nabla^L)^2 = \omega\,. \tag{1.8}$$

The tensor product connection

$$\nabla^{\wedge^{0,*}(T^*M)\otimes L} = \nabla^{\wedge^{0,*}(T^*M)} \otimes \mathrm{Id}_L + \mathrm{Id}_{\wedge^{0,*}(T^*M)} \otimes \nabla^L \tag{1.9}$$

defines a Hermitian connection on $\wedge^{0,*}(T^*M) \otimes L$. Denote by $\Omega^{0,*}(M,L)$ the set of smooth sections of $\wedge^{0,*}(T^*M) \otimes L$.

Let $e_1, \dots, e_{\dim M}$ be an oriented orthonormal base of TM.

Definition 1.1. *The Spin^c-Dirac operator D^L is defined by*

$$D^L = \sum_{i=1}^{\dim M} c(e_i)\nabla_{e_i}^{\wedge^{0,*}(T^*M)\otimes L} : \Omega^{0,*}(M,L) \to \Omega^{0,*}(M,L)\,. \tag{1.10}$$

Also denote by D_+^L (resp. D_-^L) the restriction of D^L on $\Omega^{0,\mathrm{even}}(M,L)$ (resp. $\Omega^{0,\mathrm{odd}}(M,L)$).

Clearly, D^L is a formally self-adjoint operator. However, when $\partial M \neq \emptyset$, it is not elliptic. To get an elliptic operator, one should impose the boundary conditions of Atiyah-Patodi-Singer [APS1] type. This will be examined in the next subsection.

b) Geometry near the boundary. We follow the convention as in the paper of Gilkey [G].

Let $\varepsilon_0 > 0$ be less than the injectivity radius of g^{TM}. We use the inward geodesic flow to identify a neighborhood of the boundary with the collar $\partial M \times [0, \varepsilon_0)$. Let $e_{\dim M}$ be the inward unit normal vector field perpendicular to ∂M. Let $e_1, \dots, e_{\dim M-1}$ be an oriented orthonormal base of $T\partial M$ so that $e_1, \dots, e_{\dim M-1}, e_{\dim M}$ is an oriented orthonormal base of $TM|_{\partial M}$. Then using parallel transport with respect to ∇^{TM} along the unit speed geodesics perpendicular to ∂M, $e_1, \dots, e_{\dim M}$ forms an oriented orthonormal base of TM over $\partial M \times [0, \varepsilon_0)$.

Let

$$\pi_{ij} = \langle \nabla_{e_i}^{TM} e_j, e_{\dim M}\rangle\big|_{\partial M} \tag{1.11}$$

be the second fundamental form of the isometric embedding $i_{\partial M} : \partial M \hookrightarrow M$.

Definition 1.2. *Let $D_{\partial M}^L : \Omega^{0,*}(M,L)|_{\partial M} \to \Omega^{0,*}(M,L)|_{\partial M}$ be the differential operator on ∂M defined by*

$$D_{\partial M}^L = -\sum_{i=1}^{\dim M-1} c(e_{\dim M})c(e_i)\nabla_{e_i}^{\wedge^{0,*}(T^*M)\otimes L} + \frac{1}{2}\sum_{i=1}^{\dim M-1} \pi_{ii}\,. \tag{1.12}$$

Let $D^L_{\partial M,+}$ *(resp.* $D^L_{\partial M,-}$*) be the restriction of* $D^L_{\partial M}$ *to* $\Omega^{0,\mathrm{even}}(M,L)|_{\partial M}$ *(resp.* $\Omega^{0,\mathrm{odd}}(M,L)|_{\partial M}$*).*

By [G, Lemmas 2.1 and 2.2], $D^L_{\partial M}$ is a formally self-adjoint first order elliptic differential operator intrinsically defined on ∂M. Also, it preserves the natural $\mathbf{Z}_2$-grading of $\Omega^{0,*}(M,L)|_{\partial M}$.

For any $\lambda \in \mathrm{Sp}\{D^L_{\partial M}\}$, the spectrum of $D^L_{\partial M}$, denote by E_λ the eigenspace corresponding to λ. For any $a \in \mathbf{R}$, let $P_{\geq a}$ (resp. $P_{>a}$) be the orthogonal projection from the L^2-completion of $\Omega^{0,*}(M,L)|_{\partial M}$ onto $\oplus_{\lambda\geq a}E_\lambda$ (resp. $\oplus_{\lambda>a}E_\lambda$). Let $P_{\geq a,\pm}$ (resp. $P_{>a,\pm}$) be the restrictions of $P_{\geq a}$ (resp. $P_{>a}$) on the L^2-completions of $\Omega^{0,\mathrm{even}}(M,L)|_{\partial M}$ and $\Omega^{0,\mathrm{odd}}(M,L)|_{\partial M}$ respectively.

From [APS1], [G], one knows that $(D^L_+, P_{\geq 0,+})$ defines an elliptic boundary value problem whose adjoint is $(D^L_-, P_{>0,-})$.

Definition 1.3. *Let* $Q_{APS}(M,L)$ *be the finite dimensional virtual vector space*

$$Q_{APS}(M,L) = \ker(D^L_+, P_{\geq 0,+}) - \ker(D^L_-, P_{>0,-})\,. \tag{1.13}$$

c) Hamiltonian group action and the deformation of the generalized Atiyah-Patodi-Singer problem. From now on, we suppose that (M,ω) admits a Hamiltonian action of a compact connected Lie group G with Lie algebra $\mathbf{g}$. Clearly, this G action preserves ∂M.

Denote by

$$\mu : M \to \mathbf{g}^* \tag{1.14}$$

the associated moment map. Let $\mathbf{g}$ (and thus $\mathbf{g}^*$ also) be equipped with an $\mathrm{Ad}\,G$ invariant metric. A formula due to Kostant [Ko] (cf. [TiZ1, (1.13)]) induces a natural $\mathbf{g}$ action on L. We make the assumption that this $\mathbf{g}$ action can be lifted to a G action on L. Then one verifies easily that this G action preserves ∇^L. We also assume, after an integration over G if necessary, that G preserves the Hermitian metric on L, the almost complex structure J and thus also the Riemannian metric g^{TM}.

Let $h_1,\ldots,h_{\dim G}$ be an orthonormal base of $\mathbf{g}^*$. Then μ has the expression

$$\mu = \sum_{i=1}^{\dim G} \mu_i h_i\,, \tag{1.15}$$

where each μ_i is a real function on M. Denote by V_i the Killing vector field on M induced by the dual of h_i. Then one verifies easily that (cf. [TiZ1, (1.18)])

$$J(d\mu_i)^* = -V_i\,. \tag{1.16}$$

Thus, denoting by $\mathcal{H} = |\mu|^2$ the norm square of μ and $X^{\mathcal{H}}$ its associated Hamiltonian vector field, one has (cf. [TiZ1, (1.19)])

$$X^{\mathcal{H}} = -2J \sum_{i=1}^{\dim G} \mu_i (d\mu_i)^* = 2 \sum_{i=1}^{\dim G} \mu_i V_i \,. \tag{1.17}$$

As G preserves ∂M, one gets the important fact that

$$X^{\mathcal{H}}\big|_{\partial M} \in T\partial M \,. \tag{1.18}$$

By (1.18) and following [TiZ1, Definition 1.2] and [G, Lemma 2.2], we set, for any $T \in \mathbf{R}$,

$$\begin{aligned} D^L_{+,T} &= D^L_+ + \frac{\sqrt{-1}T}{2} c(X^{\mathcal{H}}) : \Omega^{0,\mathrm{even}}(M,L) \to \Omega^{0,\mathrm{odd}}(M,L) \,, \\ D^L_{-,T} &= D^L_- + \frac{\sqrt{-1}T}{2} c(X^{\mathcal{H}}) : \Omega^{0,\mathrm{odd}}(M,L) \to \Omega^{0,\mathrm{even}}(M,L) \,, \\ D^L_T &= D^L + \frac{\sqrt{-1}T}{2} c(X^{\mathcal{H}}) : \Omega^{0,*}(M,L) \to \Omega^{0,*}(M,L) \end{aligned} \tag{1.19}$$

and

$$\begin{gathered} D^L_{\partial M,+,T} = D^L_{\partial M,+} - \frac{\sqrt{-1}T}{2} c(e_{\dim M}) c(X^{\mathcal{H}}) : \\ \Omega^{0,\mathrm{even}}(M,L)|_{\partial M} \to \Omega^{0,\mathrm{even}}(M,L)|_{\partial M} \,; \\ D^L_{\partial M,-,T} = D^L_{\partial M,-} - \frac{\sqrt{-1}T}{2} c(e_{\dim M}) c(X^{\mathcal{H}}) : \\ \Omega^{0,\mathrm{odd}}(M,L)|_{\partial M} \to \Omega^{0,\mathrm{odd}}(M,L)|_{\partial M} \,; \\ D^L_{\partial M,T} = D^L_{\partial M} - \frac{\sqrt{-1}T}{2} c(e_{\dim M}) c(X^{\mathcal{H}}) : \\ \Omega^{0,*}(M,L)|_{\partial M} \to \Omega^{0,*}(M,L)|_{\partial M} \,. \end{gathered} \tag{1.20}$$

One verifies easily that $D^L_{\partial M,T}$ is also formally self-adjoint on ∂M. For any $\lambda \in \mathrm{Sp}\{D^L_{\partial M,T}\}$, let $E_{\lambda,T}$ be the corresponding eigenspace. For any $a \in \mathbf{R}$, denote by $P_{\geq a,T}$ (resp. $P_{>a,T}$) the orthogonal projection from the L^2-completion of $\Omega^{0,*}(M,L)|_{\partial M}$ onto $\oplus_{\lambda \geq a} E_{\lambda,T}$ (resp. $\oplus_{\lambda > a} E_{\lambda,T}$). Let $P_{\geq a,\pm,T}$ (resp. $P_{>a,\pm,T}$) be the restrictions of $P_{\geq a,T}$ (resp. $P_{>a,T}$) on the L^2-completions of $\Omega^{0,\mathrm{even}}(M,L)|_{\partial M}$ and $\Omega^{0,\mathrm{odd}}(M,L)|_{\partial M}$ respectively.

Definition 1.4. *For any $T \in \mathbf{R}$, let $(D^L_{+,T}, P_{\geq 0,+,T})$ (resp. $(D^L_{-,T}, P_{>0,-,T})$) be the natural deformation of the boundary value problem $(D^L_+, P_{\geq 0,+})$ (resp. $(D^L_-, P_{>0,-})$).*

One verifies easily again that both $(D^L_{+,T}, P_{\geq 0,+,T})$ and $(D^L_{-,T}, P_{>0,-,T})$ are elliptic, and that $(D^L_{-,T}, P_{>0,-,T})$ is the adjoint of $(D^L_{+,T}, P_{\geq 0,+,T})$ (cf. [G]).

Definition 1.5. *For any $T \in \mathbf{R}$, set*

$$Q_{APS,T}(M,L) = \ker(D^L_{+,T}, P_{\geq 0,+,T}) - \ker(D^L_{-,T}, P_{>0,-,T})\,. \tag{1.21}$$

Since G preserves everything, $Q_{APS,T}(M,L)$ is a finite dimensional virtual G representation. We denote by $Q_{APS,T}(M,L)^G$ its G-trivial component.

2 Localization to Neighborhoods of $\mu^{-1}(0)$: the Easy Case

In what follows, we assume that $\mu^{-1}(0) \cap \partial M = \emptyset$. We further assume in this section that ∂M contains no critical points of $\mathcal{H} = |\mu|^2$, that is, the subset $B \subset \partial M$ defined in the Introduction is empty.

The purpose of this section is first to prove the part (i) of Theorem 0.1, and then to show that the proof of (0.6) can be localized to sufficiently small neighborhoods of $\mu^{-1}(0)$. The basic idea in this section is the same as in [TiZ1, Sect. 2] where we treated the $\partial M = \emptyset$ case. What we still need to do is to carry out the analysis near the boundary.

This section is organized as follows. In a), we prove the first part of Theorem 0.1, that is, the invertibility of the restriction of $D^L_{\partial M,T}$ to the G-invariant subspace, when $T > 0$ is sufficiently large. In b), we carry out the localization principle near the boundary. In c), we glue the obtained results to get the localization principle outside of $\mu^{-1}(0)$. We will use the same assumptions and notation as in section 1.

a) The invertibility of $D^L_{\partial M,T}$ as $T \to +\infty$. In this subsection, we prove the following result which implies the invertibility of $D^L_{\partial M,T}$ for sufficiently large T.

Theorem 2.1. *There exist $C > 0$, $b > 0$ such that for any $T \geq 1$ and any G-invariant element s of $\Omega^{0,*}(M,L)|_{\partial M}$, the following estimate holds,*

$$\|D^L_{\partial M,T} s\|^2_{\partial M,0} \geq C\big(\|s\|^2_{\partial M,1} + (T-b)\|s\|^2_{\partial M,0}\big)\,. \tag{2.1}$$

Proof. By (1.17), one knows that $X^{\mathcal{H}}$ is nowhere zero on ∂M. So (2.1) follows easily from an odd dimensional analogue of [TiZ1, Sect. 2]. The main observation here is that, when regarded as a Spin^c-Dirac operator, $D^L_{\partial M}$ is exactly the intrinsic Spin^c-Dirac operator on ∂M associated to $(g^{TM}|_{\partial M}, \nabla^L|_{\partial M}, \nabla^\lambda|_{\partial M}, J)$ (cf. [G]). While the operator $D^L_{\partial M,T}$ is exactly the odd dimensional analogue of the deformation used in [TiZ1]. A simple computation of $D^{L,2}_{\partial M,T}$, along with the fact that $X^{\mathcal{H}}$ is nowhere zero on ∂M, then gives (2.1). □

REMARK 2.2. In section 3, we will deal with the more delicate case that ∂M may contain critical points of $\mathcal{H}$ and prove the corresponding extension of Theorem 2.1. In particular, an explicit Bochner type formula for $D^{L,2}_{\partial M,T}$ will be given.

COROLLARY 2.3. *There exists $T_0>0$ such that for any $T\geq T_0$, the restriction of $D^L_{\partial M,T}$ to the G-invariant subspace of $\Omega^{0,*}(M,L)|_{\partial M}$ is invertible.*

b) An estimate near ∂M. The purpose of this subsection is to prove the following estimate near ∂M.

PROPOSITION 2.4. *There exists an open neighborhood U of ∂M and positive constants T_0, C and b such that for any $T\geq T_0$ and G-invariant element s of $\Omega^{0,*}(M,L)$ with* Supp $s\subset U$ *and $P_{\geq 0,T}s|_{\partial M}=0$, the following inequality holds,*

$$\|D^L_Ts\|^2_{M,0}\geq C\big(\|s\|^2_{M,1}+(T-b)\|s\|^2_{M,0}\big)\,. \tag{2.2}$$

Proof. Let dv_M (resp. $dv_{\partial M}$) be the volume element on M (resp. ∂M) of g^{TM} (resp. $g^{TM}|_{\partial M}$). From Green's formula (cf. [G, (2.28)]), one has

$$\begin{aligned}\|D^L_Ts\|^2_{M,0}&=\int_M\langle D^L_Ts,D^L_Ts\rangle dv_M\\&=\int_M\langle s,D^{L,2}_Ts\rangle dv_M+\int_{\partial M}\big\langle s,c(e_{\dim M})D^L_Ts\big\rangle dv_{\partial M}\,.\end{aligned}\tag{2.3}$$

From (1.18), one knows that $c(X^{\mathcal{H}})$ anti-commutes with $c(e_{\dim M})$. Thus following [G, (2.26), (2.27)], one verifies that on the boundary ∂M,

$$c(e_{\dim M})D^L_T=-\nabla^{\wedge^{0,*}(T^*M)\otimes L}_{e_{\dim M}}-D^L_{\partial M,T}+\frac{1}{2}\sum_{i=1}^{\dim M-1}\pi_{ii}\,. \tag{2.4}$$

From (2.3) and (2.4), one gets

$$\begin{aligned}\|D^L_Ts\|^2_{M,0}=&-\int_{\partial M}\langle s,D^L_{\partial M,T}s\rangle dv_{\partial M}+\frac{1}{2}\int_{\partial M}\left\langle s,\sum_{i=1}^{\dim M-1}\pi_{ii}s\right\rangle dv_{\partial M}\\&-\int_{\partial M}\big\langle s,\nabla^{\wedge^{0,*}(T^*M)\otimes L}_{e_{\dim M}}s\big\rangle dv_{\partial M}+\int_M\big\langle s,D^{L,2}_Ts\big\rangle dv_M\,.\end{aligned}\tag{2.5}$$

We now recall from [TiZ1, Theorem 1.6] the following Bochner type formula for $D^{L,2}_T$,

$$\begin{aligned}D^{L,2}_T=&\,D^{L,2}+\frac{\sqrt{-1}T}{4}\sum_{j=1}^{\dim M}c(e_j)c(\nabla^{TM}_{e_j}X^{\mathcal{H}})-\frac{\sqrt{-1}T}{2}\mathrm{Tr}[\nabla^{T^{(1,0)}M}_{\cdot}X^{\mathcal{H}}]\\&+\frac{T}{2}\sum_{i=1}^{\dim G}\big(\sqrt{-1}c(JV_i)c(V_i)+|V_i|^2\big)+4\pi T\mathcal{H}-2\sqrt{-1}T\sum_{i=1}^{\dim G}\mu_iL_{V_i}+\frac{T^2}{4}|X^{\mathcal{H}}|^2\,,\end{aligned}\tag{2.6}$$

where L_{V_i} is the infinitesimal action of the Killing vector field V_i on $\Omega^{0,*}(M,L)$. As in [TiZ1, Definition 1.8], set

$$F_T^L = D_T^{L,2} + 2T\sqrt{-1}\sum_{i=1}^{\dim G}\mu_i L_{V_i}\,. \tag{2.7}$$

We now prove the following simple estimate.

LEMMA 2.5. *There exists an open neighborhood U_1 of ∂M and $T_0>0$, $C_1>0$, $b_1>0$ such that for any $s\in\Omega^{0,*}(M,L)$ with* $\operatorname{Supp} s\subset U_1$ *and $T\geq T_0$, one has*

$$\int_M \langle s, F_T^L s\rangle dv_M - \int_{\partial M}\big\langle s, \nabla_{e_{\dim M}}^{\wedge^{0,*}(T^*M)\otimes L}s\big\rangle dv_{\partial M} \geq C_1\big(\|s\|_{M,1}^2 + (T-b_1)\|s\|_{M,0}^2\big)\,. \tag{2.8}$$

Proof. Let

$$\Delta = \sum_{i=1}^{\dim M}\Big((\nabla_{e_i}^{\wedge^{0,*}(T^*M)\otimes L})^2 - \nabla_{\nabla_{e_i}^{TM}e_i}^{\wedge^{0,*}(T^*M)\otimes L}\Big) \tag{2.9}$$

be the Bochner Laplacian. Then for any $s\in\Omega^{0,*}(M,L)$ which is supported near ∂M, one has by Green's formula that

$$\int_M\langle s,-\Delta s\rangle dv_M - \int_{\partial M}\big\langle s,\nabla_{e_{\dim M}}^{\wedge^{0,*}(T^*M)\otimes L}s\big\rangle dv_{\partial M} = \sum_{i=1}^{\dim M}\int_M\big\langle\nabla_{e_i}^{\wedge^{0,*}(T^*M)\otimes L}s,\nabla_{e_i}^{\wedge^{0,*}(T^*M)\otimes L}s\big\rangle dv_M\,. \tag{2.10}$$

From (2.6), (2.7), (2.10) and the Lichnerowicz formula (cf. [LMi, Appendix D], [TiZ1, (2.5)])

$$D^{L,2} = -\Delta + O(1)\,, \tag{2.11}$$

one deduces that

$$\int_M\langle s,F_T^L s\rangle dv_M - \int_{\partial M}\big\langle s,\nabla_{e_{\dim M}}^{\wedge^{0,*}(T^*M)\otimes L}s\big\rangle dv_{\partial M} = \int_M\sum_{i=1}^{\dim M}\big\langle\nabla_{e_i}^{\wedge^{0,*}(T^*M)\otimes L}s,\nabla_{e_i}^{\wedge^{0,*}(T^*M)\otimes L}s\big\rangle dv_M + \int_M\langle A_T s,s\rangle dv_M\,, \tag{2.12}$$

where

$$A_T = \frac{\sqrt{-1}T}{4}\sum_{j=1}^{\dim M}c(e_j)c(\nabla_{e_j}^{TM}X^{\mathcal H}) - \frac{\sqrt{-1}T}{2}\operatorname{Tr}[\nabla_{\cdot}^{T^{(1,0)}M}X^{\mathcal H}] + \frac{T}{2}\sum_{i=1}^{\dim G}\big(\sqrt{-1}c(JV_i)c(V_i)+|V_i|^2\big) + 4\pi T\mathcal H + \frac{T^2}{4}|X^{\mathcal H}|^2 + O(1)\,. \tag{2.13}$$

Since ∂M contains no critical point of $\mathcal{H}$, by (1.17), one knows that there exists a constant $A_1 > 0$ such that on ∂M,

$$|X^{\mathcal{H}}|^2 \geq 8A_1 . \tag{2.14}$$

From (2.13) and (2.14), one deduces that there exists an open neighborhood U_1 of ∂M and constants $A_2 > 0$, $A_3 > 0$ such that on U_1,

$$A_T \geq A_1 T^2 - A_2 T - A_3 . \tag{2.15}$$

From (2.12) and (2.15), one gets

$$\begin{aligned} \int_M \langle s, F_T^L s\rangle dv_M &- \int_{\partial M} \left\langle s, \nabla_{e_{\dim M}}^{\wedge^{0,*}(T^*M)\otimes L} s\right\rangle dv_{\partial M} \\ &\geq \int_M \sum_{i=0}^{\dim M} \left\langle \nabla_{e_i}^{\wedge^{0,*}(T^*M)\otimes L} s, \nabla_{e_i}^{\wedge^{0,*}(T^*M)\otimes L} s\right\rangle dv_M \\ &\qquad + (A_1T^2 - A_2 T - A_3) \int_M \langle s, s\rangle dv_M , \end{aligned} \tag{2.16}$$

from which (2.8) follows. □

We now complete the proof of Proposition 2.4 by establishing an estimate for the remaining boundary contribution on the right hand side of (2.5).

Since s satisfies

$$P_{\geq 0,T} s|_{\partial M} = 0 , \tag{2.17}$$

one sees from Corollary 2.3 that for T large enough,

$$\int_{\partial M} \langle s, -D_{\partial M,T}^L s\rangle dv_{\partial M} \geq -\lambda_{1,T} \int_{\partial M} \langle s, s\rangle dv_{\partial M} , \tag{2.18}$$

where $\lambda_{1,T}$ is the first negative eigenvalue of $D_{\partial M,T}^L$. Now by Theorem 2.1, one finds easily that there exist constants $T_1 > 0$, $A_4 > 0$ such that for any $T \geq T_1$,

$$\lambda_{1,T} \leq -A_4 \sqrt{T} . \tag{2.19}$$

From (2.18) and (2.19), one gets for $T \geq T_1$ that, Ml

$$\int_{\partial M} \langle s, -D_{\partial M,T}^L s\rangle dv_{\partial M} \geq A_4\sqrt{T} \int_{\partial M} \langle s, s\rangle dv_{\partial M} . \tag{2.20}$$

On the other hand, $\sum_{i=1}^{\dim M-1} \pi_{ii}$ is clearly a bounded function on ∂M. Thus by (2.20), one sees that there exists $T_2 > 0$ such that for any $T \geq T_2$,

$$\int_{\partial M} \langle s, -D_{\partial M,T}^L s\rangle dv_{\partial M} + \frac{1}{2} \int_{\partial M} \left\langle s, \sum_{i=1}^{\dim M-1} \pi_{ii} s\right\rangle dv_{\partial M} \geq 0 . \tag{2.21}$$

Formula (2.2) follows from (2.5), (2.7), (2.8) and (2.21). □

c) An estimate outside of $\mu^{-1}(0)$. In this subsection we prove the following result which will play a role analogous to what [TiZ1, Theorem 2.1] plays in [TiZ1].

Theorem 2.6. *For any open neighborhood U of $\mu^{-1}(0)$ with $\overline{U}\cap\partial M=\emptyset$, there exist $T_0>0$, $C>0$, $b>0$ such that for any $T\geq T_0$ and any G-invariant element s of $\Omega^{0,*}(M,L)$ with* $\operatorname{Supp} s\subset M\setminus U$ *and $P_{\geq 0,T}s|_{\partial M}=0$, one has*

$$\|D_T^L s\|_{M,0}^2\geq C\big(\|s\|_{M,1}^2+(T-b)\|s\|_{M,0}^2\big)\,. \tag{2.22}$$

Proof. The critical observation is, as in [TiZ1], that on the G-invariant subspace $\Omega^{0,*}(M,L)^G$ of $\Omega^{0,*}(M,L)$, one has by (2.7) that

$$D_T^{L,2}=F_T^L\,. \tag{2.23}$$

Now since the estimate in Proposition 2.4 holds on a (fixed) open neighborhood U_1 of ∂M, by a simple gluing procedure as in [TiZ1, Sect. 2c)] and [BLe, pp. 115–117], which can be adapted here easily, one only needs to prove Theorem 2.6 on $M\setminus(U\cup U_1)$.

Clearly, the local estimate established in [TiZ1, Prop. 2.2] still holds at every point $x\in M\setminus(U\cup U_1)$. The proof of Theorem 2.6 then follows from the compactness of $M\setminus(U\cup U_1)$ and the gluing procedure. □

3 Localization to Neighborhoods of $\mu^{-1}(0)$: The General Case

In this section, we extend the result in section 2 to the case where ∂M might contain critical points of $\mathcal{H}$. The results of this section are not used in the applications given in sections 5 and 6.

Clearly, the main difficulty in this general situation appears near the boundary. As before, let y_m, $m=\dim M$, be the geodesic distance coordinate to the boundary. Recall that the subset $B\subset\partial M$ of the critical points of $\mathcal{H}$ has been defined in the Introduction.

We first state a natural condition under which the principal estimates in this section will hold.

CONDITION 3.0. *We assume that the following inequality holds on B,*

$$4\pi\mathcal{H}-\frac{\sqrt{-1}}{2}c\left(d^{\partial M}\frac{\partial\mathcal{H}}{\partial y_m}\right)c(Je_m)>0\,, \tag{3.0}$$

where $d^{\partial M}$ is the exterior derivation on ∂M, and $T^*(\partial M)$ is identified with $T\partial M$ via the metric $g^{T\partial M}=g^{TM}|_{\partial M}$.

The geometric meaning of this condition will be discussed briefly at the end of this section.

Recall that $\Omega^{0,*}(M,L)^G$ is the notation for the G-invariant subspace of $\Omega^{0,*}(M,L)$.

We state the main estimate near ∂M as follows.

Theorem 3.1. *If Condition 3.0 holds, then there exists an open neighborhood U of ∂M and positive constants T_0, C and b such that for any $T \geq T_0$ and $s \in \Omega^{0,*}(M,L)^G$ with $\operatorname{Supp} s \subset U$ and $P_{\geq 0,T} s|_{\partial M} = 0$, one has*

$$\|D_T^L s\|_{M,0}^2 \geq C\big(\|s\|_{M,1}^2 + (T-b)\|s\|_{M,0}^2\big)\,. \tag{3.1}$$

We have seen in section 2 that Theorem 3.1 holds when ∂M contains no critical points of $\mathcal{H}$. In order to deal with the general situation, we shall take a closer look at the behavior of $\mathcal{H}$ near ∂M.

First of all, near ∂M, $X^{\mathcal{H}}$ has the following unique (local) decomposition:

$$X^{\mathcal{H}} = -2\sum_{j=1}^m f_j J e_j\,. \tag{3.2}$$

In particular,

$$f_m|_{\partial M} = -\tfrac{1}{2}\langle X^{\mathcal{H}}, Je_m\rangle\big|_{\partial M} \tag{3.3}$$

is a globally defined function on ∂M.

Clearly, $x \in B$ if and only if $f_j(x) = 0$ for all $1 \leq j \leq m$.

Let $\alpha > 0$ be defined by

$$\alpha = \max\left\{\rho : 2\pi\mathcal{H} - \frac{\sqrt{-1}}{4}c\left(d^{\partial M}\frac{\partial\mathcal{H}}{\partial y_m}\right)c(Je_m) \geq \rho\,\mathrm{Id}_{\Omega^{0,*}(M,L)|_{\partial M}} \text{ on } B\right\},$$

under Condition 3.0.

In order to prove Theorem 3.1 in general, we need the following refinement of Theorem 2.1. It implies part (i) of Theorem 0.1 as a consequence.

Theorem 3.2. *If Condition 3.0 holds, then there exists a G-invariant open neighborhood $U(B)$ of B in ∂M and positive constants T_0, C such that for any $T \geq T_0$ and $s \in \Omega^{0,*}(M,L)^G|_{\partial M}$,*

$$\|D_{\partial M,T}^L s\|_{\partial M,0}^2 \geq C\|s\|_{\partial M,1}^2 + \tfrac{1}{2}\alpha T\|s\|_{\partial M,0}^2 + T^2\int_{U(B)}\langle s, |f_m|^2 s\rangle dv_{\partial M}\,. \tag{3.4}$$

Proof. The difference of (3.4) in comparing with (2.1) is the extra term $T^2\int_{U(B)}\langle s,|f_m|^2 s\rangle dv_{\partial M}$ which requires a detailed treatment.

Following [G], for any $X \in \Gamma(T\partial M)$, set

$$\widetilde{c}(X) = -c(e_m)c(X)\,. \tag{3.5}$$

Then by (1.20), one can write $D^L_{\partial M,T}$ as

$$D^L_{\partial M,T} = D^L_{\partial M} + \frac{\sqrt{-1}T}{2}\widetilde{c}(X^{\mathcal{H}})\,, \tag{3.6}$$

From (1.12), (3.5), (3.6) and by proceeding similarly as in the proof of [TiZ1, Theorem 1.6], one gets the following Bochner type formula for $D^{L,2}_{\partial M,T}$.

Theorem 3.3. *The following identity holds on ∂M,*

$$\begin{aligned} D^{L,2}_{\partial M,T} = D^{L,2}_{\partial M} &+ \frac{\sqrt{-1}T}{2}\sum_{j=1}^{m-1}\sum_{i=1}^{\dim G}\widetilde{c}(e_j)\widetilde{c}\big(\nabla^{TM}_{e_j}(\widetilde{\mu}_iV_i)\big) \\ &- \frac{\sqrt{-1}T}{2}\sum_{i=1}^{\dim G}\widetilde{c}(e_m)\widetilde{c}(\mu_i\nabla^{TM}_{e_m}V_i) \\ -\sqrt{-1}T\sum_{i=1}^{\dim G}\mathrm{Tr}\big[\nabla^{T^{(1,0)}M}_{\cdot}(\widetilde{\mu}_iV_i)\big] &+ \frac{T}{2}\sum_{i=1}^{\dim G}\big(\sqrt{-1}\widetilde{c}(d^{\partial M}\mu_i)\widetilde{c}(V_i)+|d^{\partial M}\mu_i|^2\big) \\ + 4\pi T\mathcal{H} + \frac{\sqrt{-1}T}{2}\Big(\sum_{j=1}^{m-1}\pi_{jj}\Big)\widetilde{c}(X^{\mathcal{H}}) &- 2\sqrt{-1}T\sum_{i=1}^{\dim G}\mu_iL_{V_i} + \frac{T^2}{4}|X^{\mathcal{H}}|^2\,, \end{aligned} \tag{3.7}$$

where by $\widetilde{\mu}_i$ we mean that it is the restriction of μ_i on ∂M with the obvious convention that the directional derivative $e_m(\widetilde{\mu}_i) = 0$.

Set

$$\widetilde{F}^L_T = D^{L,2}_{\partial M,T} + 2T\sqrt{-1}\sum_{i=1}^{\dim G}\mu_iL_{V_i} \tag{3.8}$$

on ∂M. The following pointwise estimate follows immediately from (3.7), (3.8) and the argument in section 2.

Lemma 3.4. *If $x\in\partial M\setminus B$, then there exists an open neighborhood U_x of x in ∂M and $T_x>0$, $C_x>0$ such that for any $T\geq T_x$ and $s\in\Omega^{0,*}(M,L)|_{\partial M}$ with* Supp $s\subset U_x$, *one has*

$$\int_{\partial M}\langle s,\widetilde{F}^L_Ts\rangle dv_{\partial M} \geq C_x\|s\|^2_{\partial M,1} + \alpha T\|s\|^2_{\partial M,0}\,. \tag{3.9}$$

Now we establish the pointwise estimates around each critical point of $\mathcal{H}$ contained in ∂M.

Proposition 3.5. *If* (3.0) *holds at $x\in B$, then there exists an open neighborhood U_x of x in ∂M and $T_x>0$, $C_x>0$ such that for any $T\geq T_x$ and $s\in\Omega^{0,*}(M,L)|_{\partial M}$ with* Supp $s\subset U_x$, *one has*

$$\int_{\partial M}\langle s,\widetilde{F}^L_Ts\rangle dv_{\partial M} \geq C_x\|s\|^2_{\partial M,1} + \alpha T\|s\|^2_{\partial M,0} + T^2\int_{U_x}\big\langle s,|f_m|^2s\big\rangle dv_{\partial M}\,. \tag{3.10}$$

Proof. Since x is a critical point of $\mathcal{H}$, one can choose an orthonormal base $e_1,\dots,e_{m-1}$ of $T\partial M$ near x so that if $(y_1,\dots,y_{m-1})$ is the normal coordinate system associated to $e_1,\dots,e_{m-1}$, then in a sufficiently small open neighborhood of x in M one has that

$$\mathcal{H}(y)=\mathcal{H}(x)+\sum_{j=1}^{m}a_jy_j^2+\sum_{j=1}^{m-1}b_jy_jy_m+O\big(|y|^3\big)\,, \tag{3.11}$$

where the a_j's and b_j's may possibly be zero. From (1.17) and (3.11), one verifies easily that

$$X^{\mathcal{H}}(y)=-2\sum_{j=1}^{m}f_j(y)Je_j \tag{3.12}$$

with

$$f_j(y)=a_jy_j+\tfrac{1}{2}b_jy_m+O\big(|y|^2\big)\,,\qquad 1\le j\le m-1 \tag{3.13}$$

and

$$f_m(y)=a_my_m+\frac{1}{2}\sum_{j=1}^{m-1}b_jy_j+O\big(|y|^2\big)\,. \tag{3.14}$$

For clarity, we shall sometimes write

$$\widetilde{f}_j(y)=f_j(y)\big|_{\partial M} \tag{3.15}$$

near $x\in\partial M$ for $1\le j\le m$, $y\in\partial M$. Thus, near x,

$$\widetilde{f}_j(y)=a_jy_j+O\big(|y|^2\big)\,,\qquad 1\le j\le m-1 \tag{3.16}$$

and

$$\widetilde{f}_m(y)=\frac{1}{2}\sum_{j=1}^{m-1}b_jy_j+O\big(|y|^2\big)\,. \tag{3.17}$$

We now state an odd dimensional analogue of [TiZ1, Lemma 2.3].

Lemma 3.6. *The following inequality holds at $x\in B$,*

$$\begin{aligned}&\frac{\sqrt{-1}}{2}\sum_{j=1}^{m-1}\sum_{i=1}^{\dim G}\widetilde{c}(e_j)\widetilde{c}\big(\nabla_{e_j}^{TM}(\widetilde{\mu}_iV_i)\big)-\frac{\sqrt{-1}}{2}\sum_{i=1}^{\dim G}\widetilde{c}(e_m)\widetilde{c}(\mu_i\nabla_{e_m}^{TM}V_i)\\&-\sqrt{-1}\sum_{i=1}^{\dim G}\mathrm{Tr}\big[\nabla_{\cdot}^{T^{(1,0)}M}(\widetilde{\mu}_iV_i)\big]+\frac{1}{2}\sum_{i=1}^{\dim G}\big(\sqrt{-1}\widetilde{c}(d^{\partial M}\mu_i)\widetilde{c}(V_i)+|d^{\partial M}\mu_i|^2\big)\\&\qquad\ge-\sum_{j=1}^{m-1}|a_j|-\frac{\sqrt{-1}}{2}\widetilde{c}\left(d^{\partial M}\frac{\partial\mathcal{H}}{\partial y_m}\right)\widetilde{c}(Je_m)\,,\end{aligned} \tag{3.18}$$

where the inequality is strict if and only if at least one of the a_j, $1\le j\le m-1$, is negative.

Proof. From (3.12)–(3.17), one deduces directly that

$$\frac{\sqrt{-1}}{2}\sum_{j=1}^{m-1}\sum_{i=1}^{\dim G}\widetilde{c}(e_j)\widetilde{c}\big(\nabla^{TM}_{e_j}(\widetilde{\mu}_i V_i)\big) = -\frac{\sqrt{-1}}{2}\sum_{j=1}^{m-1}a_j\widetilde{c}(e_j)\widetilde{c}(Je_j) - \frac{\sqrt{-1}}{4}\sum_{j=1}^{m-1}\widetilde{c}(b_je_j)\widetilde{c}(Je_m) \quad (3.19)$$

and that

$$-\sqrt{-1}\sum_{i=1}^{\dim G}\mathrm{Tr}\big[\nabla^{T^{(1,0)}M}_{\cdot}(\widetilde{\mu}_iV_i)\big] = -\frac{1}{2}\sum_{j=1}^{m-1}a_j + \frac{\sqrt{-1}}{4}\left\langle\sum_{j=1}^{m-1}b_je_j, Je_m\right\rangle - \sqrt{-1}\sum_{i=1}^{\dim G}\left\langle\mu_i\nabla^{TM}_{e_m}V_i, \frac{1}{2}\left(1+\frac{J}{\sqrt{-1}}\right)e_m\right\rangle . \quad (3.20)$$

One also verifies that

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{\dim G}&\big(\sqrt{-1}\widetilde{c}(d^{\partial M}\mu_i)\widetilde{c}(V_i) + |d^{\partial M}\mu_i|^2\big)\\ &= \frac{1}{2}\sum_{i=1}^{\dim G}\big(-\sqrt{-1}\widetilde{c}(d^{\partial M}\mu_i)\widetilde{c}(Jd^{\partial M}\mu_i) + |d^{\partial M}\mu_i|^2\big)\\ &\quad - \frac{\sqrt{-1}}{2}\sum_{i=1}^{\dim G}\widetilde{c}\left(\frac{\partial\mu_i}{\partial y_m}d^{\partial M}\mu_i\right)\widetilde{c}(Je_m) . \end{aligned}\quad (3.21)$$

Now a direct calculation shows that for each $1\le j\le m-1$,

$$-\tfrac{\sqrt{-1}}{2}a_j\widetilde{c}(e_j)\widetilde{c}(Je_j) - \tfrac{1}{2}a_j \ge -|a_j| , \quad (3.22)$$

where the inequality is strict if and only if a_j is negative, and that

$$-\sqrt{-1}\widetilde{c}(d^{\partial M}\mu_i)\widetilde{c}(Jd^{\partial M}\mu_i) + |d^{\partial M}\mu_i|^2 \ge 0 . \quad (3.23)$$

On the other hand, by (3.11) one deduces that, at x,

$$\begin{aligned}\sum_{i=1}^{\dim G}\frac{\partial\mu_i}{\partial y_m}d^{\partial M}\mu_i &= \frac{1}{2}d^{\partial M}\frac{\partial\mathcal{H}}{\partial y_m} - \sum_{i=1}^{\dim G}\mu_i d^{\partial M}\frac{\partial\mu_i}{\partial y_m}\\ &= \frac{1}{2}\sum_{j=1}^{m-1}b_jdy_j - \sum_{i=1}^{\dim G}\mu_i d^{\partial M}\frac{\partial\mu_i}{\partial y_m} . \end{aligned}\quad (3.24)$$

Formula (3.18) will follow from (3.19)–(3.24) and the following lemma.

LEMMA 3.7. *The following identities hold at $x\in B$,*

$$\sum_{i=1}^{\dim G}\mu_i d^{\partial M}\frac{\partial\mu_i}{\partial y_m} = 0 , \quad (3.25)$$

$$\sum_{i=1}^{\dim G} \mu_i \nabla_{e_m}^{TM} V_i = 0 \tag{2.25$'$}$$

and

$$\left\langle d^{\partial M} \tfrac{\partial \mathcal{H}}{\partial y_m}, Je_m \right\rangle = 0\,. \tag{3.26}$$

Proof. Clearly, the left-hand side of (3.25) does not depend on the choice of the base h_i, $1 \leq i \leq \dim G$, of $\mathbf{g}^*$. Since $\mu(x) \neq 0$, one can choose

$$h_1 = \tfrac{\mu(x)}{|\mu(x)|}\,. \tag{3.27}$$

Thus, evaluated at x,

$$\sum_{i=1}^{\dim G} \mu_i d^{\partial M} \frac{\partial \mu_i}{\partial y_m} = \mu_1 d^{\partial M} \frac{\partial \mu_1}{\partial y_m}\,. \tag{3.28}$$

By a result of Kirwan [Ki, 3.7, 3.10], one knows that x is a fixed point of the action of the torus generated by h_1. In other words, x is a zero point of the Killing vector field $V_1 = -J(d\mu_1)^*$.

Since $V_1|_{\partial M} \in \Gamma(T\partial M)$, one sees easily that $e_m(x)$ lies in the tangent space at x of the zero set of V_1, and so does $Je_m(x)$. Thus, one gets (3.25)$'$ and that, near x,

$$\frac{\partial \mu_1(y)}{\partial y_m} = -\big\langle V_1(y), Je_m \big\rangle = O(|y|^2)\,. \tag{3.29}$$

Formula (3.25) follows from (3.28) and (3.29).

Now since $\langle V_i, e_m \rangle = 0$ on ∂M, by (1.16) one verifies that

$$\langle d^{\partial M} \mu_i, Je_m \rangle = 0\,. \tag{3.30}$$

Formula (3.26) follows from (3.24), (3.25) and (3.30). □

The proof of Lemma 3.6 is completed. □

Now from (1.12) one deduces easily that, near x,

$$D_{\partial M}^{L,2} = \sum_{j=1}^{m-1} (\nabla_{e_j}^{\wedge^{0,*}(T^*M)\otimes L})^* \nabla_{e_j}^{\wedge^{0,*}(T^*M)\otimes L} + O(\partial^{\partial M} + 1)\,, \tag{3.31}$$

where $(\nabla^{\wedge^{0,*}(T^*M)\otimes L})^*$ is the formal adjoint of $\nabla^{\wedge^{0,*}(T^*M)\otimes L}$, while

$$O(\partial^{\partial M}) \tag{3.32}$$

is the notation for the first order differential operators on ∂M with coefficients of type $O(1)$. For any $1 \leq j \leq m-1$, set $\varepsilon_j = \operatorname{sgn}(a_j)$. Then from (3.0), (3.7), (3.8), (3.12), (3.31) and Lemma 3.6, one deduces that, near x,

$$\widetilde{F}_T^L = \sum_{j=1}^{m-1} \big((\nabla_{e_j}^{\wedge^{0,*}(T^*M)\otimes L})^* + T\varepsilon_j \widetilde{f}_j\big)(\nabla_{e_j}^{\wedge^{0,*}(T^*M)\otimes L} + T\varepsilon_j \widetilde{f}_j) + 4\pi T\mathcal{H}$$

$$
\begin{aligned}
&- \frac{\sqrt{-1}T}{2}\widetilde{c}\left(d^{\partial M}\frac{\partial \mathcal{H}}{\partial y_m}\right)\widetilde{c}(Je_m) + T^2\widetilde{f}_m^2 + O\big(\partial^{\partial M} + 1 + T|y|\big) \\
&\geq \frac{1}{k}\sum_{j=1}^{m-1}\big((\nabla_{e_j}^{\wedge^{0,*}(T^*M)\otimes L})^* + T\varepsilon_j\widetilde{f}_j\big)(\nabla_{e_j}^{\wedge^{0,*}(T^*M)\otimes L} + T\varepsilon_j\widetilde{f}_j) \\
&\qquad\qquad + 2\alpha T + T^2\widetilde{f}_m^2 + O\big(\partial^{\partial M} + 1 + T|y|\big) \qquad (3.33)
\end{aligned}
$$

for $k \geq 1$.

Taking k large enough, one sees easily that Proposition 3.5 holds (Compare with [TiZ1, Sect. 2b)]). □

From Proposition 3.5, the fact that $\widetilde{f}_m$ is globally defined on ∂M, and from the gluing arguments similar to those in [BLe, pp. 115–117], one deduces the following estimate which holds globally around $B \subset \partial M$.

PROPOSITION 3.8. *If Condition 3.0 holds, then there exists a G-invariant open neighborhood $U(B)$ of B in ∂M and $T_B > 0$, $C_B > 0$ such that for any $T \geq T_B$ and $s \in \Omega^{0,*}(M,L)|_{\partial M}$ with* $\operatorname{Supp} s \subset U(B)$, *one has*

$$\int_{\partial M}\langle s, \widetilde{F}_T^L s\rangle dv_{\partial M} \geq C_B\|s\|_{\partial M,1}^2 + \tfrac{2}{3}\alpha T\|s\|_{\partial M,0}^2 + T^2\int_{U(B)}\big\langle s, |f_m|^2 s\big\rangle dv_{\partial M}\,. \tag{3.34}$$

Since ∂M is compact, by Lemma 3.4 and Proposition 3.8 one can proceed as in [BLe, pp. 115–117] to show, after shrinking $U(B)$ a little bit if necessary, that there exist $T_0 > 0$, $C > 0$ such that for any $T \geq T_0$, $s \in \Omega^{0,*}(M,L)|_{\partial M}$, one has

$$\int_{\partial M}\langle s, \widetilde{F}_T^L s\rangle dv_{\partial M} \geq C\|s\|_{\partial M,1}^2 + \frac{1}{2}\alpha T\|s\|_{\partial M,0}^2 + T^2\int_{U(B)}\big\langle s, |f_m|^2 s\big\rangle dv_{\partial M}\,. \tag{3.35}$$

By (3.8), one has the following identity on $\Omega^{0,*}(M,L)^G|_{\partial M}$,

$$\widetilde{F}_T^L = D_{\partial M,T}^{L,2}\,. \tag{3.36}$$

Formula (3.4) follows from (3.35) and (3.36). This completes the proof of Theorem 3.2. □

Part (i) of Theorem 0.1 follows.

In view of (2.5) and (3.4), to prove Theorem 3.1 one also needs the following estimate.

PROPOSITION 3.9. *If Condition 3.0 holds, then there exists a G-invariant open neighborhood $U(B)$ of B in ∂M, which can be made arbitrarily small, an open neighborhood U of ∂M in M, and $T_0 > 0$, $C > 0$, $b > 0$ such that* *for any* $T \geq T_0$, $s \in \Omega^{0,*}(M,L)$ *with* $\operatorname{Supp} s \subset U$, *one has*

$$\int_M \langle s, F_T^L s\rangle dv_M - \int_{\partial M} \langle s, \nabla_{e_m}^{\wedge^{0,*}(T^*M)\otimes L} s\rangle dv_{\partial M}$$
$$\geq C\big(\|s\|_{M,1}^2 + (T-b)\|s\|_{M,0}^2\big) - T\int_{U(B)} \big\langle s, |f_m|s\big\rangle dv_{\partial M}\,. \quad (3.37)$$

Proof. As it is clear by now, one should first establish the following estimates, the first of which follows directly from the arguments in the proof of Lemma 2.5.

LEMMA 3.10. *For any open neighborhood* $U \subset M$ *of* B, *there exists an open neighborhood* U' *of* ∂M *and* $T_0 > 0$, $C > 0$, $b > 0$ *such that for any* $T \geq T_0$ *and* $s \in \Omega^{0,*}(M,L)$ *with* $\operatorname{Supp} s \subset U' \setminus U$, *one has*

$$\int_M \langle s, F_T^L s\rangle dv_M - \int_{\partial M} \langle s, \nabla_{e_m}^{\wedge^{0,*}(T^*M)\otimes L} s\rangle dv_{\partial M}$$
$$\geq C\big(\|s\|_{M,1}^2 + (T-b)\|s\|_{M,0}^2\big)\,. \quad (3.38)$$

LEMMA 3.11. *If* $x \in B$ *verifies Condition 3.0, then there exists an open neighborhood* U_x *of* x *in* M, *which can be made arbitrarily small, and* $T_0 > 0$, $C > 0$, $b > 0$ *such that for any* $T \geq T_0$ *and* $s \in \Omega^{0,*}(M,L)$ *with* $\operatorname{Supp} s \subset U_x$, *one has*

$$\int_M \langle s, F_T^L s\rangle dv_M - \int_{\partial M} \langle s, \nabla_{e_m}^{\wedge^{0,*}(T^*M)\otimes L} s\rangle dv_{\partial M}$$
$$\geq C\big(\|s\|_{M,1}^2 + (T-b)\|s\|_{M,0}^2\big) - T\int_{\partial M} \big\langle s, |f_m|s\big\rangle dv_{\partial M}\,. \quad (3.39)$$

Proof. Without loss of generality we assume that $\mathcal{H}$ is of the form (3.11) near $x \in M$. Then by (3.12)–(3.14) and (3.26), one verifies easily the following formula at x for the corresponding terms appearing on the right-hand side of (2.13),

$$\frac{\sqrt{-1}}{4}\sum_{j=1}^m c(e_j)c(\nabla_{e_j}^{TM}X^{\mathcal{H}}) - \frac{\sqrt{-1}}{2}\operatorname{Tr}[\nabla_{\cdot}^{T^{(1,0)}M}X^{\mathcal{H}}]$$
$$= -\frac{\sqrt{-1}}{2}\sum_{j=1}^m a_j c(e_j)c(Je_j) - \frac{1}{2}\sum_{j=1}^m a_j - \frac{\sqrt{-1}}{4}\sum_{j=1}^{m-1} c(b_je_j)c(Je_m)$$
$$- \frac{\sqrt{-1}}{4}\sum_{j=1}^{m-1} c(e_m)c(b_jJe_j)$$
$$\geq -\sum_{j=1}^m |a_j| - \frac{\sqrt{-1}}{4}\sum_{j=1}^{m-1} c(b_je_j)c(Je_m) - \frac{\sqrt{-1}}{4}\sum_{j=1}^{m-1} c(e_m)c(b_jJe_j)\,. \quad (3.40)$$

For ease of notation, we shall also use J to denote the natural extension of the almost complex structure J to a unitary automorphism on $\Omega^{0,*}(M,L)$. Then by (3.26) again, one verifies that, at x,

$$\sum_{j=1}^{m-1} c(e_m)c(b_jJe_j) = J^{-1}\sum_{j=1}^{m-1} c(b_je_j)c(Je_m)J\,. \tag{3.41}$$

From (3.41) and the definition of $\alpha > 0$, one deduces that, at x,

$$\begin{aligned} 2\pi\mathcal{H} - \frac{\sqrt{-1}}{4}\sum_{j=1}^{m-1} c(e_m)c(b_jJe_j) &= J^{-1}\bigg(2\pi\mathcal{H} - \frac{\sqrt{-1}}{4}\sum_{j=1}^{m-1} c(b_je_j)c(Je_m)\bigg)J \\ &\geq \alpha\,. \end{aligned} \tag{3.42}$$

From (2.12), (2.13), (3.11), (3.13), (3.14), (3.40), (3.42) and the calculations in [TiZ1, Sect. 2b)], one deduces easily that there is an open neighborhood U_x of x in M such that for any $s \in \Omega^{0,*}(M,L)$ with $\operatorname{Supp} s \subset U_x$, one has

$$\begin{aligned} \int_M \langle s, F_T^L s\rangle dv_M - \int_{\partial M} \langle s, \nabla_{e_m}^{\wedge^{0,*}(T^*M)\otimes L} s\rangle dv_{\partial M} &\geq \sum_{j=1}^{m} \|\nabla_{e_j}^{\wedge^{0,*}(T^*M)\otimes L} s\|_{M,0}^2 \\ &\quad - T\bigg(\sum_{j=1}^{m}|a_j|\bigg)\|s\|_{M,0}^2 + T^2\sum_{j=1}^{m}\|f_js\|_{M,0}^2 + 4\pi T\mathcal{H}(x)\|s\|_{M,0}^2 \\ - \frac{\sqrt{-1}}{4}\sum_{j=1}^{m-1}\big\langle s, (c(b_je_j)c(Je_m) + c(e_m)c(b_jJe_j))s\big\rangle_M &+ \big\langle s, O(1+T|y|)s\big\rangle_M \\ &\geq \sum_{j=1}^{m}\big\|\nabla_{e_j}^{\wedge^{0,*}(T^*M)\otimes L} s + T(\operatorname{sgn} a_j)f_js\big\|_{M,0}^2 + 2\alpha T\|s\|_{M,0}^2 \\ &\quad + T\int_{\partial M}\big\langle s, (\operatorname{sgn} a_m)f_ms\big\rangle dv_{\partial M} + \big\langle s, O(1+T|y|)s\big\rangle_M \\ &\geq \frac{1}{k}\sum_{j=1}^{m}\big\|\nabla_{e_j}^{\wedge^{0,*}(T^*M)\otimes L} s + T(\operatorname{sgn} a_j)f_js\big\|_{M,0}^2 + 2\alpha T\|s\|_{M,0}^2 \\ &\quad + T\int_{\partial M}\big\langle s, (\operatorname{sgn} a_m)f_ms\big\rangle dv_{\partial M} + \big\langle s, O(1+T|y|)s\big\rangle_M \end{aligned} \tag{3.43}$$

for $k \geq 1$.

By making k sufficiently large, one obtains Lemma 3.11 from (3.43) easily (compare with [TiZ1, Sect. 2b)]). □

Following the arguments in [BLe, pp. 115–116], we now glue together the pointwise estimates in Lemma 3.11 to an estimate valid on sufficiently small open neighborhoods of B.

Lemma 3.12. *Assume that Condition 3.0 holds, then there exists an open neighborhood $U \subset M$ of B, which can be made arbitrarily small, and positive constants T_0, C, b such that for any $T \geq T_0$ and $s \in \Omega^{0,*}(M,L)$ with* $\operatorname{Supp} s \subset U$, *the estimate* (3.39) *holds.*

Proof. Let U_1, U_2 be two open subsets of M, on which the estimate (3.39) holds. Following [BLe, pp. 115], one can construct two nonnegative smooth functions τ_1, τ_2 on M such that $\operatorname{Supp} \tau_j \subset U_j$, $j = 1, 2$, and that $\tau_1^2 + \tau_2^2 = 1$ on $U_1 \cup U_2$.

Now let $s \in \Omega^{0,*}(M,L)$ be such that $\operatorname{Supp} s \subset U_1 \cup U_2$, then $\operatorname{Supp}(\tau_j s) \subset U_j$, $j = 1, 2$. Thus, $s_j = \tau_j s$, $j = 1, 2$, verify the estimate (3.39).

Using (2.12), (2.13) and adopting the notation there, one deduces that

$$\begin{aligned}
&\int_M \langle s, F_T^L s\rangle dv_M - \int_{\partial M} \langle s, \nabla_{e_{\dim M}}^{\wedge^{0,*}(T^*M)\otimes L} s\rangle dv_{\partial M} \\
&\qquad = \sum_{j=1}^{2} \bigg(\int_M \sum_{i=1}^{\dim M} \langle \tau_j \nabla_{e_i}^{\wedge^{0,*}(T^*M)\otimes L} s, \tau_j \nabla_{e_i}^{\wedge^{0,*}(T^*M)\otimes L} s\rangle dv_M \\
&\qquad\qquad + \int_M \langle A_T \tau_j s, \tau_j s\rangle dv_M \bigg) \\
&\qquad = \sum_{j=1}^{2} \bigg(\int_M \langle s_j, F_T^L s_j\rangle dv_M - \int_{\partial M} \langle s_j, \nabla_{e_{\dim M}}^{\wedge^{0,*}(T^*M)\otimes L} s_j\rangle dv_{\partial M} \bigg) \\
&\qquad - \sum_{j=1}^{2} \sum_{i=1}^{\dim M} \int_M \big(\langle e_i(\tau_j) s, e_i(\tau_j) s\rangle + 2\langle \tau_j e_i(\tau_j) s, \nabla_{e_i}^{\wedge^{0,*}(T^*M)\otimes L} s\rangle \big) dv_M\,.
\end{aligned} \tag{3.44}$$

It is clear that for any $\eta > 0$, there exists $C_\eta > 0$ such that

$$\begin{aligned}
\sum_{j=1}^{2} \sum_{i=1}^{\dim M} \bigg| \int_M \big(\langle e_i(\tau_j) s, e_i(\tau_j) s\rangle + 2\langle \tau_j e_i(\tau_j) s, \nabla_{e_i}^{\wedge^{0,*}(T^*M)\otimes L} s\rangle \big) dv_M \bigg| \\
\leq \eta \|s\|_{M,1}^2 + C_\eta \|s\|_{M,0}^2\,.
\end{aligned} \tag{3.45}$$

Since s_j, $j = 1, 2$, verify the estimate (3.39), by (3.44), (3.45) and taking η small enough, one deduces easily that s also verifies the estimate (3.39).

Now as B is compact, by using Lemma 3.11 and the above arguments, one can perform a finite number of gluings to get Lemma 3.12. □

By proceeding as in the proof of Lemma 3.12, one can glue the estimates in Lemmas 3.10 and 3.12 together to get Proposition 3.9. □

Proof of Theorem 3.1. Clearly, we can choose $U(B)$ and U to satisfy both Propositions 3.8 and 3.9. From (2.5), (2.7) and Proposition 3.9, one verifies that for any G-invariant element $s \in \Omega^{0,*}(M,L)$ with $\operatorname{Supp} s \subset U$,

$$\|D_T^L s\|_{M,0}^2 \geq -\int_{\partial M} \langle s, D_{\partial M,T}^L s\rangle dv_{\partial M} + \frac{1}{2}\int_{\partial M} \left\langle s, \left(\sum_{j=1}^{m-1} \pi_{jj}\right) s\right\rangle dv_{\partial M}$$
$$+ C\big(\|s\|_{M,1}^2 + (T-b)\|s\|_{M,0}^2\big) - T\int_{U(B)} \big\langle s, |f_m| s\big\rangle dv_{\partial M} . \quad (3.46)$$

We now estimate the term

$$-\int_{\partial M} \langle s, D_{\partial M,T}^L s\rangle dv_{\partial M} + \frac{1}{2}\int_{\partial M} \left\langle s, \left(\sum_{j=1}^{m-1} \pi_{jj}\right) s\right\rangle dv_{\partial M}$$
$$- T\int_{U(B)} \big\langle s, |f_m| s\big\rangle dv_{\partial M} \quad (3.47)$$

under the condition

$$P_{\geq 0,T}\, s|_{\partial M} = 0 . \quad (3.48)$$

For any G-invariant element $s \in \Omega^{0,*}(M,L)$ verifying condition (3.48), one has

$$-\int_{\partial M} \langle s, D_{\partial M,T}^L s\rangle dv_{\partial M} = \int_{\partial M} \big\langle s, |D_{\partial M,T}^L| s\big\rangle dv_{\partial M} . \quad (3.49)$$

Set

$$\beta = \sup_{x\in\partial M} \left\{\frac{1}{2}\left|\sum_{j=1}^{m-1} \pi_{jj}(x)\right|\right\},$$
$$\gamma = \sup_{x\in U(B)} \big\{|f_m(x)|\big\} . \quad (3.50)$$

Let $G_{U(B)}$ be the bounded operator acting on $L^2(\Omega^{0,*}(M,L)|_{\partial M})$, the L^2-completion of $\Omega^{0,*}(M,L)|_{\partial M}$, defined by

$$G_{U(B)} s = \psi_{U(B)} |f_m| s,\ \ s \in L^2\big(\Omega^{0,*}(M,L)|_{\partial M}\big) , \quad (3.51)$$

where

$$\psi_{U(B)} = \begin{cases} 1 & \text{on } U(B) \\ 0 & \text{on } \partial M \setminus U(B) \end{cases} \quad (3.52)$$

is the characteristic function of the subset $U(B) \subset \partial M$. Clearly, $G_{U(B)}$ preserves the G invariant subspace of $L^2(\Omega^{0,*}(M,L)|_{\partial M})$.

From Theorem 3.2, one knows that for T_0 large enough and $T \geq T_0$,

$$\|D_{\partial M,T}^L s\|_{\partial M,0}^2 \geq \tfrac{\alpha T}{2}\|s\|_{\partial M,0}^2 + T^2\|G_{U(B)} s\|_{\partial M,0}^2 . \quad (3.53)$$

That is,

$$\int_{\partial M} \langle s, D_{\partial M,T}^{L,2} s\rangle dv_{\partial M} \geq \int_{\partial M} \left\langle s, \left(\tfrac{\alpha T}{2} + T^2 G_{U(B)}^2\right) s\right\rangle dv_{\partial M} . \quad (3.54)$$

Thus we have the following inequality between positive operators on the G-invariant subspace of $L^2(\Omega^{0,*}(M,L)|_{\partial M})$,

$$D^{L,2}_{\partial M,T} \geq \tfrac{\alpha T}{2} + T^2 G^2_{U(B)}\,, \tag{3.55}$$

which implies that

$$|D^{L}_{\partial M,T}| \geq \sqrt{\tfrac{\alpha T}{2} + T^2 G^2_{U(B)}}\,. \tag{3.56}$$

Now observe that since $f_m|_B = 0$, we can take $U(B)$ in Proposition 3.8 small enough so that γ verifies that

$$\beta\gamma \leq \tfrac{\alpha}{6}\,. \tag{3.57}$$

From (3.57), one deduces easily that for $T_0 > 0$ large enough, one has for any $T \geq T_0$ the following inequality

$$\sqrt{\tfrac{\alpha T}{2} + T^2 G^2_{U(B)}} \geq \beta + T G_{U(B)}\,. \tag{3.58}$$

By (3.46), (3.49) through (3.52), (3.56) and (3.58), one can complete the proof of Theorem 3.1 easily. □

Theorem 3.1 gives the crucial estimate for the localization procedure near the boundary. Combining this with the pointwise estimates on $M \setminus (\partial M \cup \mu^{-1}(0))$, which hold by [TiZ1, Prop. 2.2], a simple gluing argument gives the following main result of this section, which extends Theorem 2.6 to the general case.

Theorem 3.13. *If Condition 3.0 holds, then for any open neighborhood U of $\mu^{-1}(0)$ such that $\overline{U} \cap \partial M = \emptyset$, there exist $T_0 > 0$, $C > 0$, $b > 0$ such that for any $T \geq T_0$ and $s \in \Omega^{0,*}(M,L)^G$ with* $\operatorname{Supp} s \subset M \setminus U$ *and $P_{\geq 0,T}\, s|_{\partial M} = 0$, one has*

$$\|D^L_T s\|^2_{M,0} \geq C\big(\|s\|^2_{M,1} + (T-b)\|s\|^2_{M,0}\big)\,. \tag{3.59}$$

We conclude this section with the following result, which shows that Condition 3.0 holds when G is the circle.

PROPOSITION 3.14. *If $G = S^1$, then the following identity holds on B,*

$$d^{\partial M} \tfrac{\partial \mathcal{H}}{\partial y_m} = 0\,. \tag{3.60}$$

Proof. Take $x \in B$. Since $\mu(x) \neq 0$, by (1.16), (1.17) one deduces that

$$d^{\partial M} \mu = 0 \tag{3.61}$$

at x.

Formula (3.60) follows from (3.24), (3.25) and (3.61). □

REMARK 3.15. Formula (3.60) is equivalent to the condition that e_m is an eigenvector of the Hessian of $\mathcal{H}$ at $x \in B$.

4 Invariant Spectral Flow and the Quantization Formula

In this section, we introduce an invariant version of the Atiyah-Patodi-Singer concept of spectral flow [APS2], which appears in our quantization formula, and examine its basic properties. These properties are invariant versions of the corresponding properties of the usual spectral flow and can be proved by entirely the same methods. We also prove the quantization formulae for symplectic manifolds with boundary, which are the main results of this paper, including both an asymptotic quantization formula and the formula stated in Theorem 0.1.

As in [TiZ1], the methods and techniques of Bismut and Lebeau [BLe, Sects. 8, 9] are essential to this section.

This section is organized as follows. In a), we introduce the invariant version of the spectral flow for our specific symplectic situation and consider its basic properties. In b), we state and prove an asymptotic quantization formula. In c), we give a proof of the second part of Theorem 0.1.

a) Invariant spectral flow. For simplicity and especially for the exclusive use in this paper, we will only consider operators acting on $\Omega^{0,\mathrm{even}}(M,L)|_{\partial M}$.

By a Dirac type operator acting on $\Omega^{0,\mathrm{even}}(M,L)|_{\partial M}$ we will mean a first order elliptic differential operator which has the same symbol as that of $D^L_{\partial M,+}$. For any G-equivariant Dirac type operator D, we denote by D^G its restriction to $\Omega^{0,\mathrm{even}}(M,L)^G|_{\partial M}$, the G-invariant subspace of $\Omega^{0,\mathrm{even}}(M,L)|_{\partial M}$. Clearly, if D is (formally) self-adjoint, so is D^G.

Definition 4.1. *Let $\{D_t, 0 \le t \le 1\}$ be a one parameter family of self-adjoint G-equivariant Dirac type operators acting on $\Omega^{0,\mathrm{even}}(M,L)|_{\partial M}$. The G-invariant spectral flow of $\{D_t, 0 \le t \le 1\}$, denoted by $\mathrm{sf}^G\{D_t, 0 \le t \le 1\}$, is defined to be the spectral flow of the family of self-adjoint Fredholm operators $\{D^G_t, 0 \le t \le 1\}$ in the sense of Atiyah-Patodi-Singer [APS2].*

As it should be clear from the above definition, many properties satisfied by the usual spectral flow still hold for the invariant spectral flow. For the use of this paper, in what follows we shall state an invariant version of a result of Dai and Zhang [DZ, Theorem 1.1].

Let $\{\widetilde{D}_t : \Omega^{0,\mathrm{even}}(M,L) \to \Omega^{0,\mathrm{odd}}(M,L), 0 \le t \le 1\}$ be a one parameter family of G-equivariant Dirac type operators (that is, the symbol of each $\widetilde{D}_t$ is the same as that of D^L_+), with the canonically associated boundary operators $\{D_t, 0 \le t \le 1\}$ acting on $\Omega^{0,\mathrm{even}}(M,L)|_{\partial M}$ (cf. [APS1], [G]). Let

Q_t be the Atiyah-Patodi-Singer projection produced in [APS1] associated to D_t. That is, Q_t is the orthogonal projection from the L^2-completion of $\Omega^{0,\mathrm{even}}(M,L)|_{\partial M}$ to the space of direct sum of nonnegative eigenspaces of D_t.

Let $\widetilde{D}_t^G$ and Q_t^G be the restrictions of $\widetilde{D}_t$ and Q_t to the corresponding G-invariant subspaces respectively. Then since each $(\widetilde{D}_t, Q_t)$ is an elliptic boundary value problem ([APS1], [G]), $(\widetilde{D}_t^G, Q_t^G)$ satisfies all the properties an elliptic boundary value problem verifies. In particular, it has a well-defined index, denoted by $\mathrm{ind}(\widetilde{D}_t^G, Q_t^G)$.

We can now state the following G-invariant version of [DZ, Theorem 1.1].

Theorem 4.2. *The following identity holds,*

$$\mathrm{ind}(\widetilde{D}_1^G, Q_1^G) - \mathrm{ind}(\widetilde{D}_0^G, Q_0^G) = -\mathrm{sf}^G\{D_t, 0 \le t \le 1\}\,. \tag{4.1}$$

Proof. Theorem 4.2 can be proved in the same way as in [DZ, Theorem 1.1]. The only thing to be noted is that in [DZ, Theorem 1.1], Dai and Zhang only stated their result for the situation that the metric near the boundary is a product metric. However, as already explained in [APS1], the non-product metric nature near the boundary has no effect on the Fredholm nature of the problem. Therefore, the proof of (4.1) type formulas can be easily reduced to the product metric situation near the boundary. We leave the details to the interested reader. □

b) An asymptotic quantization formula. Recall from the Introduction the assumption that $0 \in \mathbf{g}^*$ is a regular value of μ and that $\mu^{-1}(0) \cap \partial M = \emptyset$. For simplicity, we have also assumed that G acts on $\mu^{-1}(0)$ freely. Thus one can construct the Marsden-Weinstein symplectic reduction (M_G, ω_G), where $M_G = \mu^{-1}(0)/G$ is smooth and ω_G is the symplectic form on M_G induced naturally from ω. One also has the corresponding virtual vector space $Q(M_G, L_G)$ as is explained in the Introduction.

The main result of this subsection can be stated as follows.

Theorem 4.3. *If Condition 3.0 holds, then there exists $T_0 > 0$ such that*

(i) $\dim Q_{APS,T}(M,L)^G$ *does not depend on* $T \ge T_0$;

(ii) *the following identity holds for* $T \ge T_0$,

$$\dim Q_{APS,T}(M,L)^G = \dim Q(M_G, L_G)\,. \tag{4.2}$$

Proof. (i) Recall that $D^L_{\partial M,+,T}$ is the canonical boundary operator associated to $D^L_{+,T}$, and that $P_{\ge 0,+,T}$ is the Atiyah-Patodi-Singer projection associated to $D^L_{\partial M,+,T}$. In particular, one has obviously that

$$\dim Q_{APS,T}(M,L)^G = \mathrm{ind}\big((D^L_{+,T})^G, P^G_{\ge 0,+,T}\big)\,. \tag{4.3}$$

On the other hand, from Theorem 3.2[1] one knows that there exists $T_0 > 0$ such that for any $T \geq T_0$, the restriction of $D^L_{\partial M,+,T}$ to the G-invariant subspace of $\Omega^{0,\mathrm{even}}(M,L)|_{\partial M}$ is invertible. Thus for any $T \geq T_0$, the G-invariant spectral flow associated to the natural path $\{D^L_{\partial M,+,t}, T_0 \leq t \leq T\}$ vanishes,

$$\mathrm{sf}^G\{D^L_{\partial M,+,t}, T_0 \leq t \leq T\} = 0\,. \tag{4.4}$$

From Theorem 4.2, (4.3) and (4.4), one sees that $\dim Q_{APS,T}(M,L)^G$ is constant for $T \geq T_0$.

(ii) We proceed as in [TiZ1, Sect. 3], which in turn relies heavily on [BLe, Sects. 8, 9]. Since [BLe] and [TiZ1] only deal with the boundaryless case, we shall now discuss the modifications needed to fit to our situation.

Let

$$D^L_{T,APS} = (D^L_{+,T}, P_{\geq 0,+,T}) + (D^L_{-,T}, P_{>0,-,T}) \tag{4.5}$$

be the Spin^c-Dirac operator with the specified boundary condition. Then $D^L_{T,APS}$ is elliptic and self-adjoint (cf. [G]).

We now apply the analysis and methods in [TiZ1], [BLe] to $D^{L,G}_{T,APS}$, the restriction of $D^L_{T,APS}$ to the G-invariant part of its domain.

Since $\mu^{-1}(0) \cap \partial M = \emptyset$, one sees that the analysis near $\mu^{-1}(0)$ is the same as those in [TiZ1], [BLe]. In particular, one constructs the Dirac type operator $D^{L_G}_Q$ on M_G as in [TiZ1, Definition 3.12].

On the other hand, the analysis outside of $\mu^{-1}(0)$ is certainly different from that in [TiZ1] and [BLe] due to the appearance of the boundary ∂M. Thus, to ensure that everything still works, one must examine carefully each of the steps involved. In doing so, we find that all those to be checked beyond [BLe] and [TiZ1] are concerned with the following two ingredients.

The first one is exactly Theorem 3.13[2], which can be viewed as the analogue of [BLe, Prop. 9.13] and [TiZ1, Theorem 2.1]. It enables us to localize the problem to arbitrary small neighborhoods of $\mu^{-1}(0)$. Furthermore, after proving Theorem 3.13, we find that all the elliptic estimates needed in order to apply the techniques of [BLe] can either be covered by Theorem 3.13, or be reduced to the elliptic estimates, which hold clearly, on sections with compact support lying outside of an open neighborhood (fixed by Theorem 3.13) of ∂M. In particular, from Theorem 3.13 and the analysis near $\mu^{-1}(0)$, one can apply the gluing arguments in [BLe, pp. 115–116] easily to obtain an analogue of [BLe, Theorem 9.14].

[1] Or Corollary 2.3, if $B = \emptyset$.
[2] Or Theorem 2.6, if $B = \emptyset$.

The second one comes up when we need to modify the formal arguments in [BLe, Sect. 9c)–f)]. Here, as one of the referees pointed out, one does not obtain the analogue of the second inequality in [BLe, (9.110)] as easily as there. However, since we are only interested in the small eigenvalues involved, we actually need not pursue the full analogue of [BLe, Sect. 9c)–f)]. In what follows we will present an analogue of a simplified version of [BLe, Sect. 9c)–f)] in some detail. The main purpose is to prove the following analogue of [TiZ1, Theorem 3.13] and/or [BLe, (9.156)].

PROPOSITION 4.4. *If Condition 3.0 holds, then there exist $c > 0$, $T_0 > 0$ such that there are no nonzero eigenvalues of $D_Q^{L_G,2}$ in $[0, c]$, and that for any $T \geq T_0$, the number of eigenvalues in $[0, c]$ of $(D_{T,APS}^{L,G})^2$ is equal to* $\dim(\ker D_Q^{L_G})$.

Proof. As in [BLe], we first introduce some notation of Sobolev spaces. For $q \geq 0$, let E^q (resp. F^q) be the set of sections of $\wedge^{0,*}(T^*M) \otimes L$ over M (resp. $\wedge^{0,*}(T^*M_G) \otimes L_G$ over M_G) which lie in the q-th Sobolev space. As before, we always use $\|\cdot\|_0$ as the notation for the standard L^2-norms. We will denote the G-invariant part of E^q by $E^{q,G}$. For any $T \geq 0$, set

$$E^{1,G}(T) = \{s \in E^{1,G} : P_{\geq 0,T}\, s|_{\partial M} = 0\} . \tag{4.6}$$

Then $E^{1,G}(T)$ is a Hilbert space with respect to $\|\cdot\|_1$. Also, since when T is large enough the G-invariant restriction of $D_{\partial M,T}^{L}$ is invertible, one sees that $E^{1,G}(T)$ is the exactly the domain of $D_{T,APS}^{L,G}$ for sufficiently large T.

Following the discussions above, we now combine the results in sections 2, 3 as well as the analysis near $\mu^{-1}(0)$, which is the same as in [TiZ1, Sect. 3] and [BLe, Sect. 9], to obtain the following assertions.

There exists a sufficiently small G-invariant open neighborhood $U \subset M$ of $\mu^{-1}(0)$ with $\overline{U} \cap \partial M = \emptyset$, and a linear map $J_T : F^q \to E^{q,G}$ for $T > 0$, which is the analogue of the map defined in [BLe, Definition 9.4][3], such that for any $u \in \Omega^{0,*}(M_G, L_G)$, one has $J_T u \in \Omega^{0,*}(M, L)^G$ with Supp $J_T u \subset U$. Let $E_T^{q,G}$ be the image of F^q in $E^{q,G}$ by J_T. Then $J_T : F^0 \to E_T^{0,G}$ is an isometry. It is clear that

$$E_T^{1,G} \subset E^{1,G}(T) .$$

Let $E_{T,\perp}^{0,G}$ be the orthogonal space to $E_T^{0,G}$ in $E^{0,G}$. Set

$$E_\perp^{1,G}(T) = E^{1,G}(T) \cap E_{T,\perp}^{0,G} .$$

[3] Clearly, one should use [TiZ1, Theorem 3.10(i)] to replace [BLe, Theorem 7.4] when constructing J_T in our context.

Let p_T, $p_{T,\perp}$ be the orthogonal projection operators from $E^{0,G}$ to $E^{0,G}_T$, $E^{0,G}_{T,\perp}$ respectively. Then we have the following decomposition of $D^{L,G}_{T,APS}$:

$$D^{L,G}_{T,APS} = \sum_{i=1}^{4} D_{T,i} \tag{4.7}$$

with

$$\begin{aligned} D_{T,1} = p_T D^{L,G}_{T,APS} p_T\,, \quad D_{T,2} = p_T D^{L,G}_{T,APS} p_{T,\perp}\,, \\ D_{T,3} = p_{T,\perp} D^{L,G}_{T,APS} p_T\,, \quad D_{T,4} = p_{T,\perp} D^{L,G}_{T,APS} p_{T,\perp}\,. \end{aligned} \tag{4.8}$$

By summarizing the preceding arguments, we get the following proposition, which consists of the analogues of [BLe, Theorems 9.8, 9.10 and 9.14] in our situation.

PROPOSITION 4.5. a) *As* $T \to +\infty$,

$$J_T^{-1} D_{T,1} J_T = D_Q^{L_G} + O\big(\tfrac{1}{\sqrt{T}}\big)\,; \tag{4.9}$$

b) *there exist* $C_1 > 0$, $C_2 > 0$ *and* $T_0 > 0$ *such that for any* $T \geq T_0$, *any* $s \in E^{1,G}_{\perp}(T)$, $s' \in E^{1,G}_T$, *then*

$$\begin{aligned} \|D_{T,2}s\|_0 \leq C_1 \left(\tfrac{\|s\|_1}{\sqrt{T}} + \|s\|_0\right)\,, \\ \|D_{T,3}s'\|_0 \leq C_1 \left(\tfrac{\|s'\|_1}{\sqrt{T}} + \|s'\|_0\right) \end{aligned} \tag{4.10}$$

and

$$\|D_{T,4}s\|_0 \geq C_2\big(\|s\|_1 + \sqrt{T}\|s\|_0\big)\,. \tag{4.11}$$

We will now proceed to prove Proposition 4.4 by using Proposition 4.5.

Let $c > 0$ be such that

$$\mathrm{Sp}(D_Q^{L_G}) \cap [-2c, 2c] = \{0\}\,. \tag{4.12}$$

Let $\delta = \{\lambda \in \mathbf{C} : |\lambda| = c\}$ be counter-clockwise oriented.

By using Proposition 4.5 and proceeding similarly as in [BLe, Sect. 9c)–e)], one proves easily that when T is large enough, both $\lambda - D^{L,G}_{T,APS}$ and $\lambda - D_{T,4}$ are invertible for $\lambda \in \delta$.

Let $E^G_c(T)$ denote the direct sum of the eigenspaces of $D^{L,G}_{T,APS}$ associated with the eigenvalues lying in $[-c, c]$. Then $E^G_c(T)$ is a finite dimensional subspace of $E^{0,G}$. Let $P_{T,c}$ denote the orthogonal projection operator from $E^{0,G}$ to $E^G_c(T)$.

LEMMA 4.6. *There exist* $C_3 > 0$, $T_1 > 0$ *such that for any* $T \geq T_1$ *and* $\sigma \in \ker(D_Q^{L_G})$,

$$\|P_{T,c} J_T \sigma - J_T \sigma\|_0 \leq \tfrac{C_3}{\sqrt{T}} \|\sigma\|_0\,. \tag{4.13}$$

Proof. Clearly, one has for T large enough that

$$P_{T,c}J_T\sigma - J_T\sigma = \frac{1}{2\pi\sqrt{-1}}\int_\delta \left((\lambda - D^{L,G}_{T,APS})^{-1} - \lambda^{-1}\right) J_T\sigma d\lambda \,. \tag{4.14}$$

As in [BLe, Definition 9.20], let $M_T(\lambda)$, $\lambda \in \delta$, be the map from $E^{1,G}_T$ to $E^{0,G}_T$ defined by

$$M_T(\lambda) = \lambda - D_{T,1} - D_{T,2}(\lambda - D_{T,4})^{-1} D_{T,3} \,. \tag{4.15}$$

By proceeding as in [BLe, Theorem 9.21], one proves easily that when T is large enough, $M_T(\lambda)$, $\lambda \in \delta$, is invertible. Furthermore, the norm of its inverse, $M^{-1}_T(\lambda)$, is uniformly[4] bounded from above (compare with the first formula in [BLe, (9.124)]).

Now one verifies directly that (cf. [BLe, (12.3)])

$$\begin{aligned}\left((\lambda - D^{L,G}_{T,APS})^{-1} - \lambda^{-1}\right) J_T\sigma &= \left(M^{-1}_T(\lambda) - \lambda^{-1}\right) J_T\sigma \\ &\quad + (\lambda - D_{T,4})^{-1} D_{T,3} M^{-1}_T(\lambda) J_T\sigma \,. \end{aligned}\tag{4.16}$$

As an analogue of the second inequality with $j = 3$ in [BLe, (9.140)], one proves easily that

$$\left\|(\lambda - D_{T,4})^{-1} D_{T,3} M^{-1}_T(\lambda) J_T\sigma\right\|_0 \le \tfrac{C_4}{\sqrt{T}} \|J_T\sigma\|_0 \tag{4.17}$$

for $\lambda \in \delta$, T large enough and some constant $C_4 > 0$.

On the other hand, since

$$M^{-1}_T(\lambda) - \lambda^{-1} = \frac{M^{-1}_T(\lambda)}{\lambda}\left(D_{T,1} + D_{T,2}(\lambda - D_{T,4})^{-1} D_{T,3}\right), \tag{4.18}$$

by using the easy analogues of [BLe, (9.125), (9.128) and (9.129)], as well as the uniformly boundedness of $M^{-1}_T(\lambda)$, one deduces from the assumption $D^{L_G}_Q \sigma = 0$ that

$$\left\|(M^{-1}_T(\lambda) - \lambda^{-1}) J_T\sigma\right\|_0 \le \tfrac{C_5}{\sqrt{T}} \|J_T\sigma\|_0 \tag{4.19}$$

for $\lambda \in \delta$, T large enough and some constant $C_5 > 0$.

Formula (4.13) follows from (4.14), (4.16), (4.17) and (4.19). □

By taking T_1 large enough in Lemma 4.6, one deduces easily that for any $T \ge T_1$,

$$\dim E^G_c(T) \ge \dim(\ker D^{L_G}_Q) \,. \tag{4.20}$$

Now let $E^{1,G}_{0,\perp}(T)$ denote the orthogonal space to $J_T(\ker D^{L_G}_Q)$ in $E^{1,G}(T)$ with respect to $\|\cdot\|_0$. Clearly, $E^{1,G}_{0,\perp}(T)$ is closed with respect to $\|\cdot\|_1$ in $E^{1,G}(T)$.

[4] That is, not depending on T.

LEMMA 4.7. *There exists $T_2 > 0$ such that for any $T \geq T_2$, any $s \in E_{0,\perp}^{1,G}(T)$, then*

$$\left\| D_{T,APS}^{L,G} s \right\|_0 \geq \tfrac{3c}{2} \|s\|_0 \,. \tag{4.21}$$

Proof. Write s as $s = s' + s''$ with $s' \in E_T^{1,G}$ and $s'' \in E_\perp^{1,G}(T)$. Then one has

$$\left\| D_{T,APS}^{L,G} s \right\|_0^2 = \|D_{T,1} s' + D_{T,2} s''\|_0^2 + \|D_{T,3} s' + D_{T,4} s''\|_0^2 \,, \tag{4.22}$$

from which it follows that for any sufficiently small $\eta > 0$, one has

$$\begin{aligned} \left\| D_{T,APS}^{L,G} s \right\|_0 &\geq \tfrac{7}{8} \|D_{T,1} s' + D_{T,2} s''\|_0 + \eta \|D_{T,3} s' + D_{T,4} s''\|_0 \\ &\geq \tfrac{7}{8} \|D_{T,1} s'\|_0 - \tfrac{7}{8} \|D_{T,2} s''\|_0 + \eta \|D_{T,4} s''\|_0 - \eta \|D_{T,3} s'\|_0 \,. \end{aligned} \tag{4.23}$$

In view of (4.12), one sees easily that

$$\|J_T D_Q^{L_G} J_T^{-1} s'\|_0 \geq 2c \|s'\|_0 \,. \tag{4.24}$$

From (4.24) and Proposition 4.5a), one deduces that there exists $C_6 > 0$ such that when T is sufficiently large, one has

$$\tfrac{7}{8} \|D_{T,1} s'\|_0 \geq \tfrac{3c}{2} \|s'\|_0 + \tfrac{1}{8} \|J_T D_Q^{L_G} J_T^{-1} s'\|_0 - \tfrac{C_6}{\sqrt{T}} \big(\|J_T D_Q^{L_G} J_T^{-1} s'\|_0 + \|s'\|_0 \big) \,. \tag{4.25}$$

From (4.24), (4.25) one finds that when T is sufficiently large,

$$\tfrac{7}{8} \|D_{T,1} s'\|_0 \geq \tfrac{3c}{2} \|s'\|_0 + \tfrac{1}{16} \|J_T D_Q^{L_G} J_T^{-1} s'\|_0 \,. \tag{4.26}$$

On the other hand, by standard elliptic estimates as well as an obvious analogue of [BLe, (9.7)], there exists constant $C_7 > 0$ such that

$$\|s'\|_1 \leq C_7 \big(\|J_T D_Q^{L_G} J_T^{-1} s'\|_0 + \sqrt{T} \|s'\|_0 \big) \,. \tag{4.27}$$

By (4.23)–(4.27) and Proposition 4.5b), one deduces that when T is sufficiently large,

$$\begin{aligned} \|D_{T,APS}^{L,G} s\|_0 &\geq \frac{3c}{2} \|s'\|_0 + \frac{1}{32} \|J_T D_Q^{L_G} J_T^{-1} s'\|_0 + \frac{c}{16} \|s'\|_0 - \frac{7C_1}{8} \left(\frac{\|s''\|_1}{\sqrt{T}} + \|s''\|_0 \right) \\ &\quad + \eta C_2 \big(\|s''\|_1 + \sqrt{T} \|s''\|_0 \big) - \eta C_1 \left(\frac{C_7 \|J_T D_Q^{L_G} J_T^{-1} s'\|_0}{\sqrt{T}} + (C_7 + 1) \|s'\|_0 \right) \\ &\geq \frac{3c}{2} \|s' + s''\|_0 + \left(\frac{1}{32} - \frac{\eta C_1 C_7}{\sqrt{T}} \right) \|J_T D_Q^{L_G} J_T^{-1} s'\|_0 + \Big(\frac{c}{16} - \eta C_1 (C_7 + 1) \Big) \|s'\|_0 \\ &\quad + \left(\eta C_2 - \frac{7C_1}{8\sqrt{T}} \right) \|s''\|_1 + \left(\eta C_2 \sqrt{T} - \frac{7C_1}{8} - \frac{3c}{2} \right) \|s''\|_0 \,. \end{aligned} \tag{4.28}$$

Now if we choose $\eta > 0$ so that one also has

$$\tfrac{c}{16} - \eta C_1 (C_7 + 1) \geq 0 \,, \tag{4.29}$$

then from (4.28) one deduces easily that when T is sufficiently large, (4.21) holds.

The proof of Lemma 4.7 is completed. $\square$

From Lemma 4.7, one deduces easily that for T large enough,

$$\dim E_c^G(T) \leq \dim(\ker D_Q^{L_G}) \,. \tag{4.30}$$

By (4.20) and (4.30), the proof of Proposition 4.4 is completed. $\square$

Formula (4.2) follows from Proposition 4.4 through an easy parity consideration. $\square$

c) Proof of (0.6) We can now complete the proof of the second part of Theorem 0.1. In fact, by Theorem 4.2 and (4.3), one has

$$\dim Q_{APS,T}(M,L)^G - \dim Q_{APS}(M,L)^G = -\mathrm{sf}^G\{D^L_{\partial M,+,t}, 0 \leq t \leq T\} \,. \tag{4.31}$$

Formula (0.6) then follows from (4.2) and (4.31) when T is large enough. $\square$

5 Applications to Circle Actions

In this section we specify to the case where $G = S^1$. Our main concern is to apply the basic ideas and methods in the previous sections to prove an analytic analogue of a relative residue formula of Guillemin-Kalkman [GuK] and Martin [M]. As we will see, to obtain such a formula we need only to consider a very special case of quantization formulas. In particular, one no longer needs the existence of the prequantized line bundle L and the final formula in fact holds for any auxiliary bundle. This also fits with the argument in [GuK] where the symplectic form does not appear in the forms to be evaluated over the symplectic quotient.

This section is organized as follows. In a), we prove a very special quantization formula which holds for any auxiliary bundle. In b), we introduce a trick which reduces the calculation of the analytic index on the symplectic quotient to a certain kind of invariant spectral flow which no longer involves the symplectic conditions. In c), we identify the above invariant spectral flow in a specific situation which implies the analytic analogue of the Guillemin-Kalkman-Martin ([GuK], [M]) relative residue formula. There is also an appendix to this section after section 6 in which we explain certain operators on the fixed point set of the S^1 action appearing in the relative index formula proved in c).

a) A simple quantization formula for manifolds with boundary. Results in this subsection are not restricted to the case that $G = S^1$.

We assume that (M,ω) is a compact symplectic manifold with boundary ∂M which admits a Hamiltonian G action such that $0 \in \mathbf{g}^*$ is a regular value of the moment map μ with $\mu^{-1}(0) \cap \partial M = \emptyset$ and, for simplicity, that G acts on $\mu^{-1}(0)$ freely as before.

Let E be a G-equivariant Hermitian vector bundle over M, admitting a G-equivariant Hermitian connection ∇^E. Then (E, ∇^E) induces a corresponding Hermitian vector bundle with a Hermitian connection (E_G, ∇^{E_G}) over M_G. Replacing L by E in section 1, we can define the virtual vector spaces $Q_{APS,T}(M,E)^G$, $Q(M_G, E_G)$, etc.

The main result of this subsection is as follows.

Theorem 5.1. *If no critical point of $\mathcal{H} = |\mu|^2$ is contained in $M \setminus \mu^{-1}(0)$, then there exists $T_0 \geq 0$ such that for any $T \geq T_0$,*

$$\dim Q_{APS,T}(M,E)^G = \dim Q(M_G, E_G)\,. \tag{5.1}$$

Proof. Since now $X^{\mathcal{H}}$ is never zero on $M \setminus \mu^{-1}(0)$, the same arguments as in section 2 show that the the problem can be localized directly to arbitrary small neighborhoods of $\mu^{-1}(0)$, while the analysis near $\mu^{-1}(0)$ is the same as that in [TiZ1, Sects. 3, 4a)]. One can then proceed as in section 4b) to complete the proof of (5.1). □

b) Specialization to $G = S^1$ case. We now consider the special case where $G = S^1$. Note that μ is now a real function on M. We have clearly

$$\begin{aligned} d|\mu|^2 &= 2\mu d\mu\,, \\ X^{\mathcal{H}} &= -2J(\mu d\mu)^* = 2\mu V\,, \end{aligned} \tag{5.2}$$

where we use V to denote the Killing vector field generated by the unit base of the Lie algebra of S^1.

For any $T \in \mathbf{R}$, set

$$\begin{aligned} \widetilde{D}_T^E &= D^E + \sqrt{-1}Tc(V) : \Omega^{0,*}(M,E) \to \Omega^{0,*}(M,E)\,, \\ \widetilde{D}_{\partial M,T}^E &= D_{\partial M}^E + \sqrt{-1}T\widetilde{c}(V) : \Omega^{0,*}(M,E)|_{\partial M} \to \Omega^{0,*}(M,E)|_{\partial M}\,. \end{aligned} \tag{5.3}$$

It is straightforward to define the generalized Atiyah-Patodi-Singer boundary value problem for $\widetilde{D}_T^E$ and the associated virtual vector space which we shall denote by $Q^{\sim}_{APS,T}(M,E)$. Let $Q^{\sim}_{APS,T}(M,E)^G$ be its G-invariant part. We then have the following easy result.

Proposition 5.2. *Under the same condition as in Theorem 5.1 for the $G = S^1$ case, there exists $T_0 > 0$ such that for any $T \in \mathbf{R}$ with $|T| \geq T_0$,*

$$\dim Q^{\sim}_{APS,T}(M,E)^G = 0\,. \tag{5.4}$$

Proof. Since $0 \in \mathbf{R} = \mathbf{g}^*$ is a regular value of μ, one sees that V has no zero point on $\mu^{-1}(0)$. Thus from (5.2) and the condition of the proposition, one sees that V has no zero point on the whole M. For any $T \in \mathbf{R}$, one verifies that

$$\widetilde{D}_T^{E,2} = D^{E,2} + 2\sqrt{-1}Tr_V^E + T^2|V|^2$$
$$+ \sqrt{-1}T\left(\frac{1}{2}\sum_{j=1}^{\dim M} c(e_j)c(\nabla_{e_j}V) - \mathrm{Tr}\big[\nabla_{\cdot}^{T^{(1,0)}M}V|_{T^{(1,0)}M}\big] - 2L_V\right), \quad (5.5)$$

where L_V is the infinitesimal action of V on $\Omega^{0,*}(M,E)$ and $r_V^E = L_V^E - \nabla_V^E$ with L_V^E the infinitesimal action of V on E.

Now using the obvious fact that L_V vanishes when acting on the G-invariant subspace of $\Omega^{0,*}(M,E)$, that V is never zero on M, and following the arguments in section 2, one sees easily that there exists $T_0 > 0$ such that for any $|T| \geq T_0$,

$$Q_{\widetilde{APS},T}(M,E)^G = \{0\}\,. \quad (5.6)$$

The proof of Proposition 5.2 is completed. □

Theorems 4.2, 5.1 and Proposition 5.2 have the following important consequence.

Corollary 5.3. *Under the assumption of Theorem 5.1 and that $G = S^1$, there exists $T_0 > 0$ such that for any $T \geq T_0$, one has*

$$\dim Q(M_G,E_G) = -\mathrm{sf}^G\{\check{D}_{\partial M,u}^E, 0 \leq u \leq 1\}$$
$$= -\mathrm{sf}^G\{\widehat{D}_{\partial M,u}^E, 0 \leq u \leq 1\}\,, \quad (5.7)$$

with

$$\widehat{D}_{\partial M,u}^E = (1-u)\widetilde{D}_{\partial M,+,T}^E + uD_{\partial M,+,T}^E\,, \quad (5.8)$$

and

$$\check{D}_{\partial M,u}^E = (1-u)\widetilde{D}_{\partial M,+,-T}^E + uD_{\partial M,+,T}^E\,, \quad (5.9)$$

where $\widetilde{D}_{\partial M,+,T}^E$ and $D_{\partial M,+,T}^E$ are the restrictions of $\widetilde{D}_{\partial M,T}^E$ and $D_{\partial M,T}^E$ to $\Omega^{0,\mathrm{even}}(M,L)|_{\partial M}$ respectively.

Rewriting (5.8) and (5.9) more explicitly, one has

$$\widehat{D}_{\partial M,u}^E = D_{\partial M,+}^E + \sqrt{-1}T(1-u+u\mu)\widetilde{c}(V)\,, \quad (5.10)$$

and

$$\check{D}_{\partial M,u}^E = D_{\partial M,+}^E + \sqrt{-1}T(u-1+u\mu)\widetilde{c}(V)\,. \quad (5.11)$$

We now decompose ∂M into two disjoint parts:

$$\partial M = (\partial M)_+ \cup (\partial M)_- \quad (5.12)$$

such that

$$\mu|_{(\partial M)_+} > 0\,, \quad \mu|_{(\partial M)_-} < 0\,. \tag{5.13}$$

Obviously, one has that

$$\begin{aligned} &1 - u + u\mu > 0 \text{ on } (\partial M)_+\,, \\ &u - 1 + u\mu < 0 \text{ on } (\partial M)_- \end{aligned} \tag{5.14}$$

for any $u \in [0,1]$.

From (5.14) and from the fact that $L_V|_{\partial M}$ vanishes on the G-invariant subspace of $\Omega^{0,*}(M,E)|_{\partial M}$, one deduces easily the following result just by a simple calculation of $\widehat{D}^{E,2}_{\partial M,u}$ and $\check{D}^{E,2}_{\partial M,u}$.

LEMMA 5.4. *There exists $T_0 > 0$ such that for any $T \geq T_0$ and $u \in [0,1]$, when restricted to the G-invariant subspace, $\widehat{D}^E_{\partial M,u}$ (resp. $\check{D}^E_{\partial M,u}$) is invertible on $(\partial M)_+$ (resp. $(\partial M)_-$).*

From Corollary 5.3 and Lemma 5.4, we get the main result of this subsection as follows.

Theorem 5.5. *Under the same condition as in Theorem 5.1 and that $G = S^1$, there exists $T_0 > 0$ such that for any $T \geq T_0$,*

$$\begin{aligned} \dim Q(M_G, E_G) &= -\mathrm{sf}^G_{(\partial M)_+}\{\widetilde{D}^E_{\partial M,+,t}, -T \leq t \leq T\} \\ &= -\mathrm{sf}^G_{(\partial M)_-}\{\widetilde{D}^E_{\partial M,+,t}, T \geq t \geq -T\}\,. \end{aligned} \tag{5.15}$$

REMARK 5.6. A remarkable feature of Theorem 5.5 is that it expresses the quantities on the symplectic quotients through quantities on the boundary involving no symplectic nature at all. Another feature of this formula is that it works for any auxiliary bundle E.

c) A relative index theorem for symplectic quotients of circle actions. In this subsection, we assume again that $G = S^1$ but we no longer assume that M is compact. However, we will assume that the moment map $\mu : M \to \mathbf{R}$ is proper.

From the Hamiltonian action condition

$$i_V\omega = d\mu\,, \tag{5.16}$$

one sees that for any number $c \in \mathbf{R}$, $\mu - c$ is also a moment map for the S^1 action. The point here is that we do not assume the existence of the prequantized line bundle L so that we need not make prequantized restriction on the moment map. So any $c \in \mathbf{R}$ can be chosen. Thus, for any regular value $c \in \mathbf{R}$ of μ one can construct the symplectic quotient $(M_c = \mu^{-1}(c)/S^1, \omega_c)$. Here for simplicity we shall still make the assumption that S^1 acts on $\mu^{-1}(c)$ freely.

Let $a<b$ be two regular values of μ and assume that S^1 acts on both $\mu^{-1}(a)$ and $\mu^{-1}(b)$ freely. Let E_a and E_b be the induced Hermitian vector bundles over M_a and M_b respectively. In what follows, we want to express the difference $\dim Q(M_b,E_b)-\dim Q(M_a,E_a)$ through quantities on the S^1-fixed point set in $\mu^{-1}([a,b])$.

Since μ is proper, one knows that $\mu^{-1}([a,b])$ is compact. One also knows that $\mu^{-1}(a),\mu^{-1}(b)$ are connected (cf. [A]). Let $F_1(a,b),\cdots,F_q(a,b)$ be the connected components of the fixed point set $F(a,b)$ of the S^1-action in $\mu^{-1}([a,b])$. In other words, $F_i(a,b)$, $1\le i\le q$, are the connected components of the zero set $F(a,b)$ of the Killing vector field V. According to Appendix, to any $F_i(a,b)$ one can associate two naturally constructed elliptic differential operators $D^E_{F_i(a,b),+}(V)$ and $D^E_{F_i(a,b),+}(-V)$ which give the local contributions of $F_i(a,b)$ to the S^1-invariant index. Here $D^E_{F_i(a,b),+}(V)$ (resp. $D^E_{F_i(a,b),+}(-V)$) is obtained from the deformation $D^E_+ + \sqrt{-1}Tc(V)$ by letting $T\to+\infty$ (resp. $T\to-\infty$) (see the Appendix for more details).

We can now state the main result of this subsection.

Theorem 5.7. *The following identity holds,*

$$\begin{aligned}&\dim Q(M_b,E_b)-\dim Q(M_a,E_a)\\&\qquad=\sum_{i=1}^{q}\operatorname{ind}\big(D^E_{F_i(a,b),+}(V)\big)-\sum_{i=1}^{q}\operatorname{ind}\big(D^E_{F_i(a,b),+}(-V)\big)\,. \qquad (5.17)\end{aligned}$$

Proof. Without loss of generality we assume that a (resp. b) is not the minimal (resp. the maximal) value of μ on M. Thus there exists a sufficiently small $\varepsilon>0$ such that no critical value of μ lies in $[a-\varepsilon,a+\varepsilon]$ (resp. $[b-\varepsilon,b+\varepsilon]$), and that $\mu^{-1}(a-\varepsilon)$ (resp. $\mu^{-1}(b+\varepsilon)$) is nonempty. By applying Theorem 5.5 to $\mu^{-1}([a-\varepsilon,a+\varepsilon])$ and $\mu^{-1}([b-\varepsilon,b+\varepsilon])$, and by viewing $\mu-a$ and $\mu-b$ as the moment maps respectively, one derives easily the following equalities for any $T\ge T_0$ with $T_0>0$ large enough,

$$\begin{aligned}\dim Q(M_a,E_a)&=-\mathrm{sf}^G_{\mu^{-1}(a-\varepsilon)}\{\widetilde{D}^E_{\mu^{-1}(a-\varepsilon),+,t},T\ge t\ge -T\}\,,\\ \dim Q(M_b,E_b)&=-\mathrm{sf}^G_{\mu^{-1}(b+\varepsilon)}\{\widetilde{D}^E_{\mu^{-1}(b+\varepsilon),+,t},-T\le t\le T\}\,. \qquad (5.18)\end{aligned}$$

Thus one gets

$$\begin{aligned}&\dim Q(M_b,E_b)-\dim Q(M_a,E_a)\\&\qquad=-\mathrm{sf}^G_{\mu^{-1}(b+\varepsilon)}\{\widetilde{D}^E_{\mu^{-1}(b+\varepsilon),+,t},-T\le t\le T\}\\&\qquad\quad+\mathrm{sf}^G_{\mu^{-1}(a-\varepsilon)}\{\widetilde{D}^E_{\mu^{-1}(a-\varepsilon),+,t},T\ge t\ge -T\}\end{aligned}$$

$$= -\mathrm{sf}^G_{\mu^{-1}(b+\varepsilon)}\{\widetilde{D}^E_{\mu^{-1}(b+\varepsilon),+,t}, -T \le t \le T\} - \mathrm{sf}^G_{\mu^{-1}(a-\varepsilon)}\{\widetilde{D}^E_{\mu^{-1}(a-\varepsilon),+,t}, -T \le t \le T\}. \quad (5.19)$$

Now consider the deformation operator

$$\widetilde{D}^E_T = D^E + \sqrt{-1}Tc(V) : \Omega^{0,*}\big(\mu^{-1}(a-\varepsilon, b+\varepsilon), E\big) \to \Omega^{0,*}\big(\mu^{-1}(a-\varepsilon, b+\varepsilon), E\big) \quad (5.20)$$

and the associated generalized Atiyah-Patodi-Singer boundary value problem. One deduces easily, by following the localization principle in the Appendix, that there exists $T_0 > 0$ such that for any $T \ge T_0$,

$$\dim Q^{\sim}_{APS,T}\big(\mu^{-1}(a-\varepsilon, b+\varepsilon), E\big)^G = \sum_{i=1}^q \mathrm{ind}\big(D^E_{F_i(a,b),+}(V)\big) \quad (5.21)$$

and

$$\dim Q^{\sim}_{APS,-T}\big(\mu^{-1}(a-\varepsilon, b+\varepsilon), E\big)^G = \sum_{i=1}^q \mathrm{ind}\big(D^E_{F_i(a,b),+}(-V)\big). \quad (5.22)$$

From (5.21), (5.22) and Theorem 4.2, one deduces that

$$\sum_{i=1}^q \mathrm{ind}\big(D^E_{F_i(a,b),+}(V)\big) - \sum_{i=1}^q \mathrm{ind}\big(D^E_{F_i(a,b),+}(-V)\big) = -\mathrm{sf}^G\{\widetilde{D}^E_{\mu^{-1}(a-\varepsilon),+,t}, -T \le t \le T\} - \mathrm{sf}^G\{\widetilde{D}^E_{\mu^{-1}(b+\varepsilon),+,t}, -T \le t \le T\}. \quad (5.23)$$

From (5.19) and (5.23), one gets

$$\dim Q(M_b, E_b) - \dim Q(M_a, E_a) = \sum_{i=1}^q \mathrm{ind}\big(D^E_{F_i(a,b),+}(V)\big) - \sum_{i=1}^q \mathrm{ind}\big(D^E_{F_i(a,b),+}(-V)\big), \quad (5.24)$$

which is exactly (5.17). □

Remark 5.8. It is interesting to observe that since a and b are regular values of μ, by taking ε in (5.18) small enough one obtains actually that,

$$\dim Q(M_a, E_a) = \mathrm{sf}^G_{\mu^{-1}(a)}\{\widetilde{D}^E_{\mu^{-1}(a),+,t}, -T \le t \le T\},$$
$$\dim Q(M_b, E_b) = -\mathrm{sf}^G_{\mu^{-1}(b)}\{\widetilde{D}^E_{\mu^{-1}(b),+,t}, -T \le t \le T\}, \quad (5.25)$$

which hold for $T > 0$ large enough. (Be aware of the convention for orientations of $\mu^{-1}(a)$ and $\mu^{-1}(b)$ used here.) (5.25) is in fact a universal formula for circle actions. It will play a role in the next section when we study quantization formula for singular reductions under circle actions.

6 A General Asymptotic Quantization Formula on Closed Manifolds

In this section, we apply our quantization formula for manifolds with boundary to give a general quantization formula which works also for singular reductions. As an application, combining with the ideas in section 5 we prove a Guillemin-Sternberg type formula for singular reductions under circle actions.

This section is organized as follows. In a), we prove a general asymptotic quantization formula. In b), we apply the results of a) and section 5 to study quantization formula for singular reductions under circle actions. In c), we generalize the results in b) to cover an arbitrary auxiliary bundle.

a) A universal asymptotic quantization formula. In this subsection, we assume that M is closed. The other assumptions and notation are as in sections 1 and 2, except that we no longer assume that $0 \in \mathbf{g}^*$ is a regular value of the moment map μ.

Let $\mathcal{B} \subset M$ be the set of critical points of $\mathcal{H} = |\mu|^2$. Then there exists $\delta > 0$ such that

$$\mathcal{H}^{-1}((0,\delta]) \cap \mathcal{B} = \emptyset \,. \tag{6.1}$$

Thus for any $c \in (0,\delta)$, a regular value of $\mathcal{H}$, $\mathcal{H}^{-1}(c)$ is a G-invariant hypersurface of M, cutting M into two parts $M = M_+^c \cup M_-^c$ with common boundary $M_+^c \cap M_-^c = \mathcal{H}^{-1}(c)$. We assume that M_+^c contains $\mu^{-1}(0)$ and $\mathcal{B} \setminus \mu^{-1}(0) \subset M_-^c$. We can construct the deformations of the canonical Spin^c-Dirac operators as in section 1 for M, M_+^c and M_-^c respectively, as well as the corresponding virtual G-invariant vector spaces.

The main result of this subsection is as follows.

Theorem 6.1. *There exists $T_0 > 0$ such that for any $T \geq T_0$,*

1° *when restricted to the G-invariant subspace of its domain, $D^L_{\partial M_+^c,T}$ is invertible;*

2° *the following identity holds,*

$$\dim Q(M,L)^G = \dim Q_{APS,T}(M_+^c, L|_{M_+^c})^G \,. \tag{6.2}$$

Proof. 1° is a corollary of Theorem 2.1. From 1° and Theorem 4.2, one knows that for $T \geq T_0$, $\dim Q_{APS,T}(M_\pm^c, L_{M_\pm^c})^G$ does not depend on T. Let $Q_T(M,L)^G$ be the G-invariant virtual vector space associated to D_T^L. Then by 1°, one deduces easily the following splitting formula of G-invariant indices for $T \geq T_0$,

$$\dim Q_T(M,L)^G = \dim Q_{APS,T}(M_+^c, L|_{M_+^c})^G + \dim Q_{APS,T}(M_-^c, L|_{M_-^c})^G \,. \tag{6.3}$$

Since $\mu^{-1}(0)\cap M_-^c=\emptyset$, Theorem 0.1 implies that there exists $T_1>0$ such that for $T\geq T_1$,

$$\dim Q_{APS,T}(M_-^c, L|_{M_-^c})^G=0\,. \tag{6.4}$$

Now (6.2) follows from (6.3), (6.4) and the following obvious identity,

$$\dim Q_T(M,L)^G=\dim Q(M,L)^G\,. \tag{6.5}$$

□

REMARK 6.2. Formula (6.2) shows that $\dim Q(M,L)^G$ depends only on the behavior of μ near $\mu^{-1}(0)$. If $0\in\mathbf{g}^*$ is a regular value of μ, then one can apply Theorems 0.1 and 6.1 to obtain the Guillemin-Sternberg conjecture. In general, if $0\in\mathbf{g}^*$ is a critical value of μ, then (6.2) provides a universal formula which reduces the computation of $\dim Q(M,L)^G$ to sufficiently small neighborhoods of $\mu^{-1}(0)$.

REMARK 6.3. Guillemin-Sternberg type formula for singular reductions has been studied by Meinrenken and Sjamaar in [MeSj]. In the next subsection we will give a treatment for circle actions from our point of view. (Compare also with [TiZ2] where analytic treatments of some of the results of Meinrenken-Sjamaar [MeSj] for (possibly) non-abelian group actions were given.)

b) Singular reductions and quantization for circle actions. We now specialize to the case $G=S^1$, and we shall also assume that $0\in\mathbf{R}$ is a singular value of $\mu: M\to\mathbf{R}$. In this particular situation, one can use μ instead of $\mathcal{H}=|\mu|^2$ to obtain the same result as in a) with obvious modifications.

Thus, let $\delta>0$ be such that $\mu^{-1}([-\delta,\delta])\cap(\mathcal{B}\setminus\mu^{-1}(0))=\emptyset$ and take $0<c<\delta$. By Theorem 6.1, there exists $T_0>0$ such that for any $T\geq T_0$,

$$\dim Q(M,L)^G=\dim Q_{APS,T}(M_+^c, L|_{M_+^c})^G\,, \tag{6.6}$$

where now $M_+^c=\mu^{-1}([-c,c])$. Since $0\in\mathbf{R}$ is a critical value of μ, by (5.2), $\mu^{-1}(0)$ contains fixed points of the S^1 action which we denote by

$$F_0=\mu^{-1}(0)\cap\big\{x\in M; V(x)=0\big\}\,. \tag{6.7}$$

Using the deformation by V as in (5.3),

$$\widetilde{D}_T^L=D^L+\sqrt{-1}Tc(V):\Omega^{0,*}(M,L)\to\Omega^{0,*}(M,L)\,, \tag{6.8}$$

one deduces, by the results described in the Appendix (and using the similar notation there), that for any $T\geq T_0$ with some $T_0>0$,

$$\dim Q^{\sim}_{APS,T}(M_+^c, L|_{M_+^c})^G=\operatorname{ind}\big(D_{F_0,+}(V)\big) \tag{6.9}$$

and

$$\dim Q^{\sim}_{APS,-T}(M_+^c, L|_{M_+^c})^G=\operatorname{ind}\big(D_{F_0,+}(-V)\big)\,. \tag{6.10}$$

Clearly, the boundary of M_+^c decomposes into two components:

$$\partial M_+^c = \mu^{-1}(c) \cup \mu^{-1}(-c)\,. \tag{6.11}$$

We give $\mu^{-1}(\pm c)$ their induced orientations from M_+^c respectively.

From (6.6), (6.9) and the trick used to prove Theorem 5.5, one gets

$$\dim Q(M,L)^G = \mathrm{ind}\big(D_{F_0,+}(V)\big) + \mathrm{sf}^G\{\widetilde{D}^L_{\mu^{-1}(-c),+,t}, -T \le t \le T\}\,, \tag{6.12}$$

for $T > 0$ sufficiently large. Similarly, one gets from (6.6) and (6.10) that

$$\dim Q(M,L)^G = \mathrm{ind}\big(D_{F_0,+}(-V)\big) - \mathrm{sf}^G\{\widetilde{D}^L_{\mu^{-1}(c),+,t}, -T \le t \le T\}\,. \tag{6.13}$$

Now by (5.25), one has

$$\dim Q(M_c, L_c) = -\mathrm{sf}^G\{\widetilde{D}^L_{\mu^{-1}(c),+,t}, -T \le t \le T\}$$

and

$$\dim Q(M_{-c}, L_{-c}) = \mathrm{sf}^G\{\widetilde{D}^L_{\mu^{-1}(-c),+,t}, -T \le t \le T\} \tag{6.14}$$

respectively with respect to the corresponding induced orientations.

On the other hand, it is clear that $\dim Q(M_{\pm c}, L_{\pm c})$ do not depend on $c > 0$ as long as c is sufficiently small. Thus it is natural to denote them by $\dim Q(M_{G,0^\pm}, L_{G,0^\pm})$ respectively.

To summarize (6.12) through (6.14), we obtain the following main result of this subsection.

Theorem 6.4. *The following identities hold for circle actions,*

$$\begin{aligned}\dim Q(M,L)^G &= \dim Q(M_{G,0^-}, L_{G,0^-}) + \mathrm{ind}\big(D_{F_0,+}(V)\big)\\ &= \dim Q(M_{G,0^+}, L_{G,0^+}) + \mathrm{ind}\big(D_{F_0,+}(-V)\big)\,.\end{aligned} \tag{6.15}$$

REMARK 6.5. If $0 \in \mathbf{R}$ is a regular value of μ, then $F_0 = \emptyset$. Thus (6.15) reduces to the Guillemin-Sternberg conjecture [GuSt] in this case.

For $0 \in \mathbf{R}$ being a critical value of μ, there are three possibilities: (i) 0 is the minimum of μ; (ii) 0 is the maximum of μ; and (iii) 0 is neither the minimum nor the maximum of μ. Let us discuss them separately.

Case (i) In this case, one clearly has that $\dim Q(M_{G,0^-}, L_{G,0^-}) = 0$ as $M_{G,0^-} = \emptyset$. Thus one needs to compute $\mathrm{ind}(D^L_{F_0,+}(V))$.

Let N be the normal bundle to F_0 in M. Then since $V = -J(d\mu)^*$ and 0 is the minimum of μ, one verifies that $N_+ = 0$, using the notation in the Appendix. On the other hand, by using the Kostant formula ([Ko], cf. [TiZ1, (1.13)]) one verifies that

$$\sqrt{-1}L_V|_{L|_{F_0}} = 0\,. \tag{6.16}$$

From (6.16) and the observation in Remark A.3, one deduces that

$$D^L_{F_0,+}(V) = D^{L|_{F_0}}_{F_0,+} : \Omega^{0,\mathrm{even}}(F_0, L|_{F_0}) \to \Omega^{0,\mathrm{odd}}(F_0, L|_{F_0})\,, \tag{6.17}$$

from which we have, via the Atiyah-Singer index theorem [AS] and (6.15),

$$\dim Q(M,L)^G = \big\langle \mathrm{Td}(TF_0)\exp(c_1(L|_{F_0})), [F_0]\big\rangle\,. \tag{6.18}$$

Case (ii) Similarly, if 0 is the maximum of μ, one verifies that

$$\dim Q(M,L)^G = \left\langle \mathrm{Td}(TF_0)\exp(c_1(L|_{F_0})), [F_0] \right\rangle . \tag{6.19}$$

Case (iii) In this case, both $N_\pm \neq 0$. Thus by the observation in Remark A.3, one sees that $\mathrm{ind}(D^L_{F_0,+}(\pm V)) = 0$, and (6.15) becomes

$$\dim Q(M,L)^G = \dim Q(M_{G,0^+}, L_{G,0^+}) = \dim Q(M_{G,0^-}, L_{G,0^-}) . \tag{6.20}$$

REMARK 6.6. Formulas (6.18) and (6.19) have been obtained in [DuGMW] as applications of the equivariant index formula [AS], and (6.20) fits with the discussion in [MeSj]. However, as we will see next, new phenomenon occurs when one allows general auxiliary vector bundles.

c) Singular reductions and asymptotic quantization formula for general coefficients. Let E be a G-equivariant Hermitian vector bundle over M, carrying a G-equivariant Hermitian connection. Let m be a positive integer. In this subsection we consider the quantization formula for S^1 singular reductions associated to the twisted Spin^c-Dirac operator

$$D_+^{L^m\otimes E} : \Omega^{0,\mathrm{even}}(M, L^m\otimes E) \to \Omega^{0,\mathrm{odd}}(M, L^m\otimes E) . \tag{6.21}$$

Recall that we have shown in [TiZ1, Sect. 4] that there exists $m_0 > 0$ such that for each positive integer $m \geq m_0$, all the pointwise estimates needed to localize the quantization problem still hold for $D^{L^m\otimes E}$. Furthermore, one sees easily that the arguments in this paper all work for $D^{L^m\otimes E}$ with $m \geq m_0$.

Still consider the circle action case, we get the following result which extends Theorem 6.4 to an asymptotic formula valid for general twisted coefficients.

Theorem 6.7. *If 0 is a critical value of $\mu : M \to \mathbf{R}$, then the following identities hold for $m \geq m_0$,*

$$\begin{aligned}\dim Q(M, L^m\otimes E)^G &= \dim Q\big(M_{G,0^-}, (L^m\otimes E)_{G,0^-}\big) + \mathrm{ind}\big(D^{L^m\otimes E}_{F_0,+}(V)\big)\\ &= \dim Q\big(M_{G,0^+}, (L^m\otimes E)_{G,0^+}\big) + \mathrm{ind}\big(D^{L^m\otimes E}_{F_0,+}(-V)\big) .\end{aligned} \tag{6.22}$$

Now the eigenvalues of $\sqrt{-1}L_V|_{E|_{F_0}}$ might contain both positive and negative ones. Therefore, even if 0 is neither the maximum nor the minimum, the indices $\mathrm{ind}(D^{L^m\otimes E}_{F_0,+}(\pm V))$ might well be nonzero. Thus there are nontrivial correction terms to $\dim Q(M_{G,0^\pm}, (L^m\otimes E)_{G,0^\pm})$ in order to make up $\dim Q(M, L^m\otimes E)^G$. See [TiZ3] for more details.

Appendix. The Contributions of the Fixed Point Components

The purpose of this appendix is to make explicit constructions of the operators $D^E_{F_i(a,b),+}(\pm V)$'s appeared in section 5c).

Without loss of generality, we can and we will fix one component of the fixed point set and denote it by F. Let N be the normal bundle to F, then N inherits naturally an almost complex structure J_N, a Hermitian metric g^N as well as a Hermitian connection ∇^N.

Since V is a generator of the S^1-action, $\sqrt{-1}L_V$ acts on N as a covariantly constant invertible self-adjoint operator commuting with J_N. Let N_+, N_- be the positive and negative eigenbundles of $\sqrt{-1}L_V|_N$ respectively. Then J_N preserves $N_\pm$, and one has the canonical splittings

$$N_\pm \otimes \mathbf{C} = N_\pm^{(1,0)} \oplus N_\pm^{(0,1)} . \tag{A.1}$$

Let $\mathrm{Sym}(N_+^{(1,0)})$ (resp. $\mathrm{Sym}(N_-^{(0,1)})$) be the total symmetric power of $N_+^{(1,0)}$ (resp. $N_-^{(0,1)}$). Then $\mathrm{Sym}(N_-^{(0,1)}) \otimes \mathrm{Sym}(N_+^{(1,0)}) \otimes \det(N_+^{(1,0)}) \otimes E|_F$ is an infinite dimensional vector bundle over F, on which $\sqrt{-1}L_V$ acts as a covariantly constant self-adjoint operator. Furthermore, its zero eigenbundle, denoted by $(\mathrm{Sym}(N_-^{(0,1)}) \otimes \mathrm{Sym}(N_+^{(1,0)}) \otimes \det(N_+^{(1,0)}) \otimes E|_F)^G$, is finite dimensional.

Definition A.1. *The operator $D^E_{F,+}(V)$ is defined as the (twisted) Spin^c-Dirac operator on F,*

$$\begin{aligned} D^E_{F,+}(V) : \Omega^{0,\mathrm{even}}\big(F, (\mathrm{Sym}(N_-^{(0,1)}) \otimes \mathrm{Sym}(N_+^{(1,0)}) \otimes \det(N_+^{(1,0)}) \otimes E|_F)^G\big) \\ \to \Omega^{0,\mathrm{odd}}\big(F, (\mathrm{Sym}(N_-^{(0,1)}) \otimes \mathrm{Sym}(N_+^{(1,0)}) \otimes \det(N_+^{(1,0)}) \otimes E|_F)^G\big) . \end{aligned} \tag{A.2}$$

If we change V to $-V$, we get the similar definition of $D^E_{F,+}(-V)$.

Formulas (5.21), (5.22) can be proved by proceeding as in sections 2 and 4. The first step is again to show that the problem can be localized to sufficiently small neighborhoods of the fixed point set. This causes no difficulty as V is nowhere zero on the boundary. While near the fixed point set, one can proceed as in section 4 and [BLe] to complete the proof. In particular, one can identify the operators $D^E_{F,+}(\pm V)$. We leave the details to the interested reader.

Remark A.2. The paper [WuZ] contains the needed analysis near F in a holomorphic context. However, one sees easily that the analysis in [WuZ] applies here directly. Compare also with Taubes [T, Sect. 2] which goes back to Witten [W].

REMARK A.3. One verifies easily that the restriction of $\sqrt{-1}L_V$ to $\mathrm{Sym}(N_+^{(1,0)}) \otimes \det(N_+^{(1,0)})$ is positive, while its restriction to $\mathrm{Sym}(N_-^{(0,1)})$ is nonnegative with $\ker(\sqrt{-1}L_V|_{\mathrm{Sym}(N_-^{(0,1)})})$ being the trivial line bundle. This observation plays important roles in concrete applications.

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