C. R. Acad. Sci. Paris, t. 324, Série I, p. 433-438, 1997
Géométrie différentielle/*Differential Geometry*

Symplectic reduction and quantization

Youliang TIAN and Weiping ZHANG

Y. T. : Courant Institute of Mathematical Sciences,
New York, NY 10012, USA.
E-mail: ytian@cims.nyu.edu
W. Z. : Nankai Institute of Mathematics,
Tianjin, 300071, P. R. China.

Abstract. We present a direct analytic proof of the Guillemin-Sternberg geometric quantization conjecture [2]. Further extensions are also obtained.

Réduction symplectique et quantification

Résumé. *Nous présentons une preuve analytique d'une conjecture de Guillemin-Sternberg* [2], *ainsi que des extensions de ce résultat.*

Version française abrégée

Soit G un groupe de Lie compact connexe agissant sur une variété symplectique compacte (M, ω) par une action hamiltonienne. Soit (L, ∇^L) un fibré en droites hermitien muni d'une connexion hermitienne, supposé G-équivariant, et tel que $\nabla^{L,2} = \frac{2\pi}{\sqrt{-1}}\omega$. Soit $\mu : M \to g^*$ l'application moment associée. Soit J une structure presque complexe G-invariante sur TM, telle que $g^{TM}(u,v) = \omega(u, Jv)$ est une métrique riemannienne sur TM.

Soit $D^L : \Omega^{0,*}(M, L) \to \Omega^{0,*}(M, L)$ l'opérateur Spin^c de Dirac associé (*voir* [4]). Alors, on a une représentation virtuelle $RR(M, L)$ de G donnée par

$$RR(M, L) = \Omega^{0,\,\mathrm{pair}}(M, L) \cap \ker D^L - \Omega^{0,\,\mathrm{impair}}(M, L) \cap \ker D^L.$$

Supposons que $0 \in g^*$ soit une valeur régulière de μ, et que G agisse librement sur $\mu^{-1}(0)$. On note $M_G = \mu^{-1}(0)/G$ la réduction symplectique de Marsden-Weinstein. Le fibré $L_G = (L|_{\mu^{-1}(0)}/G)$ est un fibré hermitien en droites sur M_G. On obtient ainsi un espace virtuel $RR(M_G, L_G)$.

Conjecture (Guillemin-Sternberg, [2]) *On a*

$$\dim RR(M, L)^G = \dim RR(M_G, L_G).$$

Note présentée par Jean-Michel Bismut.

Dans cette Note, nous présentons une preuve analytique de cette conjecture et on obtient également des extensions de ce résultat. Ainsi, si (M, ω) est kählérienne, on montre des inégalités de type Morse relative à la partie invariante de la cohomologie de L

In this Note, we present a direct analytic proof of the Guillemin-Sternberg geometric quantization conjecture [2]. Besides deriving an alternative proof of this conjecture in the full nonabelian group action case, our methods also lead to immediate generalizations in various contexts. Details and further applications will appear in [7].

1. The Guillemin-Sternberg conjecture

Let (M, ω) be a closed symplectic manifold such that there is a Hermitian line bundle L over M admitting a Hermitian connection ∇^L with the property that $\nabla^{L,2} = \frac{2\pi}{\sqrt{-1}}\omega$. Let J be an almost complex structure on TM so that $g^{TM}(u,v) = \omega(u, Jv)$ defines a riemannian metric on TM.

With these data, one can construct canonically a Spin^c-Dirac operator (*see* [4, Appendix D])

$$D^L : \Omega^{0,*}(M, L) \to \Omega^{0,*}(M, L), \tag{1.1}$$

which gives rise to the finite dimensional virtual vector space

$$RR(M, L) = \Omega^{0,\mathrm{even}}(M, L) \cap \ker D^L - \Omega^{0,\mathrm{odd}}(M, L) \cap \ker D^L. \tag{1.2}$$

Now suppose that a compact connected Lie group G acts on (M, ω) in a Hamiltonian way, which lifts to L naturally and preserves J, ∇^L, etc. Let $\mu : M \to \mathrm{g}^*$ be the corresponding moment map. We assume that $0 \in \mathrm{g}^*$ is a regular value of μ, and for simplicity, that G acts on $\mu^{-1}(0)$ freely. Then $M_G = \mu^{-1}(0)/G$ is a smooth manifold. On the other hand, ω descends to a symplectic form ω_G on M_G. Thus we get the Marsden-Weinstein symplectic reduction space (M_G, ω_G). The pair (L, ∇^L) also descends to a pair (L_G, ∇^{L_G}) over M_G. Then one defines the corresponding Spin^c-Dirac operator and in particular the virtual vector space $RR(M_G, L_G)$.

Since G preserves everything, it commutes with D^L. Thus $RR(M, L)$ is a virtual representation space of G. Denote by $RR(M, L)^G$ the G-trivial representation component of $RR(M, L)$.

THEOREM 1.1. – $\dim RR(M, L)^G = \dim RR(M_G, L_G)$.

Theorem 1.1 was first proved by Guillemin-Sternberg [2] in the holomorphic category when (M, g^{TM}) is Kähler. They raised it as a conjecture for general symplectic manifolds. When G is abelian, this conjecture was proved by Meinrenken [5] and Vergne ([8], [9]). A proof for the full nonabelian case was given by Meinrenken [6].

2. Quantized Witten deformation and its Laplacian

For any $X \in \Gamma(TM)$ with complexification $X = X_1 + X_2 \in \Gamma(T^{(1,0)}M \oplus T^{(0,1)}M)$, set $c(X) = \sqrt{2}\bar{X}_1^* \wedge -\sqrt{2}i_{X_2}$, where $\bar{X}_1^* \in \Gamma(T^{*(0,1)}M)$ is the metric dual of X_1 (*see* [1], Section 5). Then $c(X)$ extends to an action on $\Omega^{0,*}(M, L)$.

Let g and thus g^* be equipped with an Ad G-invariant metric. Let $|\mu|^2$ be the norm square of the moment map. Let $Jd|\mu|^2 \in \Gamma(T^*M) \simeq \Gamma(TM)$ be the 1-form introduced by Witten [10].

DEFINITION 2.1. – For any $T \in \mathbf{R}$, the quantized Witten symplectic deformation operator D_T is the formally self-adjoint first order elliptic differential operator given by

$$D_T = D^L - \frac{\sqrt{-1}T}{2} c(Jd|\mu|^2) : \Omega^{0,*}(M, L) \to \Omega^{0,*}(M, L). \tag{2.1}$$

Remark 2.2. – If J is integrable, so that (M, g^{TM}) is Kähler, one has

$$D_T = \sqrt{2}\,(e^{-T|\mu|^2/2}\,\bar\partial^L\, e^{T|\mu|^2/2} + e^{T|\mu|^2/2}\,(\bar\partial^L)^*\, e^{-T|\mu|^2/2}). \tag{2.2}$$

Also, a similar deformation has been used by Vergne [9] on the symbol level.

Let $h_1, \ldots, h_{\dim G}$ be an orthonormal base of $\mathfrak{g}^*$. Then μ has the expression $\mu = \sum_{i=1}^{\dim G} \mu_i h_i$, where each μ_i is a real function on M. Let V_i be the killing vector field on M induced by the dual of h_i. Using (2.1) and the Kostant formula [3] for the infinitesimal action of G on L, one obtains the following Bochner type formula.

THEOREM 2.3. – *The following identity holds,*

$$\begin{aligned} D_T^2 = D^{L,2} &+ \sqrt{-1}T \sum_{i=1}^{\dim G} c(d\mu_i)\, c(V_i) + 4\pi T|\mu|^2 + T^2 \left| \sum_{i=1}^{\dim G} \mu_i V_i \right|^2 \\ &+ \sqrt{-1}T \sum_{i=1}^{\dim G} \mu_i \left(\frac{1}{2} \sum_{j=1}^{\dim M} c(e_j)\, c(\nabla_{e_j} V_i) - \operatorname{Tr}[\nabla^{(1,0)} V_i|_{T^{(1,0)}M}] - 2\,\hat{L}_{V_i} \right), \end{aligned} \tag{2.3}$$

where $\hat{L}_{V_i}$ denotes the infinitesimal action of V_i on $\Omega^{0,*}(M, L)$, and $\nabla^{(1,0)}$ denotes the connection on $T^{(1,0)}M$ induced from the Levi-Civita connection ∇ of g^{TM}.

3. Localization to neighbourhoods of $\mu^{-1}(0)$

In this section, we show that the proof of Theorem 1.1 can be localized to arbitrary small neighbourhoods of $\mu^{-1}(0)$. The main difficulty arises from the fact that the nonzero critical point set of $|\mu|^2$ may not be nondegenerate in the sense of Bott. We overcome this difficulty by doing pointwise estimates instead of global estimates used in the standard analytic Morse theory.

Let $\Omega_G^{0,*}(M, L)$ denote the G-invariant part of $\Omega^{0,*}(M, L)$.

THEOREM 3.1. – *For any open neighbourhood U of $\mu^{-1}(0)$, there exist constants $C > 0$, $b > 0$ such that for any $T \geq 1$ and any $s \in \Omega_G^{0,*}(M, L)$ with* $\operatorname{Supp} s \subset M \backslash U$,

$$\|D_T s\|_0^2 \geq C(\|s\|_1^2 + (T - b)\|s\|_0^2). \tag{3.1}$$

We prove Theorem 3.1 in two steps. The first step is to prove the following key pointwise estimate.

PROPOSITION 3.2. – *Let*

$$Q_T = D_T^2 + 2\sqrt{-1}T \sum_{i=1}^{\dim G} \mu_i\, \hat{L}_{V_i} \tag{3.2}$$

act on $\Omega^{0,*}(M, L)$. For any $x \in M \backslash U$, there exist an open neighborhood W of x and constants $C_x > 0$, $b_x > 0$ such that if $s \in \Omega^{0,*}(M, L)$ with $\operatorname{Supp} s \subset W$, *then for any $T \geq 1$,*

$$\langle Q_T s, s\rangle \geq C_x(\|s\|_1^2 + (T - b_x)\|s\|_0^2). \tag{3.3}$$

If x is not a critical point of $|\mu|^2$, the proof of (3.3) is trivial. We now assume that x is a *nonzero critical* point of $|\mu|^2$. Then one can find an orthonormal basis $f_1, \ldots, f_{\dim M}$ of $T_x M$ with the corresponding normal coordinates $y_1, \ldots, y_{\dim M}$ such that near x, $|\mu|^2$ can be written as

$$|\mu(y)|^2 = |\mu(x)|^2 + \sum_{j=1}^{\dim M} a_j y_j^2 + O(|y|^3), \tag{3.4}$$

where the constants a_j's may possibly be zero.

From (3.4), one can see directly that at x,

$$\begin{aligned}\sqrt{-1}\sum_{i=1}^{\dim G} c(d\mu_i)c(V_i) + \sqrt{-1}T\sum_{i=1}^{\dim G}\mu_i\left(\frac{1}{2}\sum_{j=1}^{\dim M} c(f_j)c(\nabla_{f_j}V_i) - \operatorname{Tr}[\nabla^{(1,0)}V_i]\right)\\ \geq -\sum_{j=1}^{\dim M}|a_j|.\end{aligned} \tag{3.5}$$

From (3.5), (3.4), (3.2) and (2.3), one gets (3.3).

The second step of the proof of Theorem 3.1 is to glue together the pointwise estimates in Proposition 3.2. The key point is that when restricted to $\Omega_G^{0,*}(M, L)$, one has $\hat{L}_{V_i} = 0$. Thus $D_T^2 = Q_T$ on $\Omega_G^{0,*}(M, L)$. On the other hand, since $M\backslash U$ is compact, finitely many glueing suffice.

4. The analysis near $\mu^{-1}(0)$ and a proof of Theorem 1.1

Theorem 3.1 allows us to reduce the proof of Theorem 1.1 to a sufficiently small open neighbourhood U of $\mu^{-1}(0)$. We take U to be equivariant.

Since $0 \in \mathbf{g}^*$ is a regular value of μ, $\mu^{-1}(0)$ is a nondegenerate critical submanifold of $|\mu|^2$ in the sense of Bott. One can then apply directly here methods and techniques of the paper of Bismut-Lebeau [1, Sections 8, 9] and localize everything at $\mu^{-1}(0)$. As G acts on $\mu^{-1}(0)$ freely, $G \to \mu^{-1}(0) \xrightarrow{\pi} M_G = M/G$ is a principal fibration. Furthermore, the vertical G-direction covariant derivatives are bounded operators when restricted to G-invariant subspaces. This eventually pushes everything down to M_G.

In summary, we get a self-adjoint Spin^c-Dirac type operator D_Q on M_G acting on $\Omega^{0,*}(M_G, L_G)$, having the properties given in Theorem 4.1. To our surprise, it turns out to be non identical to the Spin^c-Dirac operator D^{L_G}.

THEOREM 4.1. – *There exist $c > 0$, $T_0 > 0$ such that there are no nonzero eigenvalues of D_Q^2 in $[0, c]$, and such that for any $T \geq T_0$, the number of eigenvalues of $D_T^2|_{\Omega_G^{0,*}(M, L)}$ in $[0, c]$ is equal to* $\dim(\ker D_Q)$.

Now, all arguments used to prove Theorem 4.1 preserve the $\mathbf{Z}_2$-grading of the Spin^c-bundles. Theorem 1.1 then follows from Theorem 4.1 easily.

Remark 4.2. – For a precise form of D_Q in the holomorphic category, *see* (6.1).

Remark 4.3. – If G does not act freely on $\mu^{-1}(0)$, then M_G is an orbifold. In this case, the above arguments can be modified easily to prove the orbifold version of Theorem 1.1.

Remark 4.4. – Alternatively, one can first take the principal fibration $G \to U \to U/G$ and then apply [1] to U/G to prove Theorem 4.1.

5. Two immediate extensions

Arguments in Sections 2 to 4 also lead immediately to further extensions of Theorem 1.1. Here we only state two of them. The first is a dual version of Theorem 1.1.

THEOREM 5.1. – *The following identity holds:*

$$\dim RR(M,\, L^{-1}\otimes \det(T^{(0,1)}M))^G = (-1)^{\dim G} \dim RR(M_G,\, L_G^{-1}\otimes \det(T^{(0,1)}M_G)).$$

The second result can be viewed as an invariance property of symplectic quotients. It has also been obtained independently by Meinrenken and Sjamaar.

THEOREM 5.2. – *If* $\mu^{-1}(0)$ *is not empty, then we have the equality of Todd genus,* $\langle \mathrm{Td}(TM),\, [M]\rangle = \langle \mathrm{Td}(TM_G),\, [M_G]\rangle$.

6. Holomorphic Morse inequalities

We now assume that $(M,\, \omega)$ is Kähler and work in the holomorphic category. Then $(M_G,\, \omega_G)$ is also Kähler. The line bundle L (resp. L_G) is now holomorphic over M (resp. M_G).

Let $h : M_G \to \mathbf{R}_+$ be defined by $h(x) = \mathrm{vol}(G_x) = \mathrm{vol}(\pi^{-1}(x))$. Then the Dirac type operator D_Q in Section 4 can be written precisely here as

$$D_Q = \sqrt{2}\,(h^{1/2}\,\bar\partial^{L_G}\,h^{-1/2} + h^{-1/2}\,(\bar\partial^{L_G})^*\,h^{1/2}). \tag{6.1}$$

From (2.2), (6.1), and proceeding as in Sections 2 through 4, one actually gets a $\mathbf{Z}$-graded refined version of Theorem 4.1. This culminates in the following refinement of Theorem 1.1, which is stated for Dolbeault cohomologies, where we use the upperscript G to denote the G-invariant part.

THEOREM 6.1. – *The following Morse type inequalities hold:*

(i) *For any* $0 \le p \le \frac{\dim M}{2}$,

$$\dim H^{0,p}(M,\, L)^G \le \dim H^{0,p}(M_G,\, L_G);$$

(ii) *For any* $0 \le p \le \frac{\dim M}{2}$,

$$\sum_{i=0}^{p} (-1)^i \dim H^{0,p-i}(M,\, L)^G \le \sum_{i=0}^{p} (-1)^i \dim H^{0,p-i}(M_G,\, L_G).$$

Acknowledgements. We are indebted to Professors Jean-Michel Bismut, Kefeng Liu and Siye Wu for their kindness and very helpful suggestions. Part of this work was done while the second author was visiting the Courant Institute of Mathematical Sciences. He would like to thank the Courant Institute for financial support and hospitality. The work of the first author was partially supported by a NSF postdoctoral fellowship. The work of the second author was partially supported by the Chinese National Science Foundation and the Qiu Shi Foundation.

Note remise le 1er août 1996, acceptée le 25 novembre 1996.

References

[1] **Bismut J.-M. and Lebeau G., 1991.** Complex immersions and Quillen metrics, *Pub. Math. IHES.*, Vol. 74.

[2] **Guillemin V. and Sternberg S., 1982.** Geometric quantization and multiplicities of group representations, *Invent. Math.*, 67, pp. 515-538.

[3] **Kostant B., 1970.** Quantization and unitary representations, *in:* Modern Analysis and Applications, *Lecture Notes in Maths.*, Vol. 170, Springer-Verlag, pp. 87-207.

[4] **Lawson H. B. and Michelsohn M.-L., 1989.** Spin Geometry, Princeton Univ. Press.
[5] **Meinrenken E., 1996.** On Riemann-Roch formulas for multiplicities, *J. A. M. S.*, 9, pp. 373-389.
[6] **Meinrenken E.** Symplectic surgery and the Spin^c-Dirac operator. To appear in *Adv. in Math.*
[7] **Tian Y. and Zhang W., 1996.** Symplectic reduction and analytic localization, *Preprint.*
[8] **Vergne M., 1996.** Multiplicity formula for geometric quantization, Part. I. *Duke Math. J.*, 82, pp. 143-179.
[9] **Vergne M., 1996.** Multiplicity formula for geometric quantization, Part II. *Duke Math. J.*, 82, pp. 181-194.
[10] **Witten E., 1992.** Two dimensional gauge theories revisited, *J. Geom. Phys.*, 9, pp. 303-368.