η-invariant and closed geodesics

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- arxiv :1302.2792, 7 pages

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  Springer, Berlin, 1978.
- SC-structure : for every $p \in M$, SC^p-structure holds

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- CROSS = compact symmetric spaces of rank one (namely, the unit spheres, the real projective spaces, the complex projective spaces, the quaternionic projective spaces and the Cayley projective plane, endowed with the corresponding canonical metrics).
- **Natural question :** whether there are manifolds of SC^p-structure not contained in CROSS ?

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- The same question is also raised in the book of Besse

The Milnor fibration (1957)

- For any $h \in \mathbf{Z}$, let ξ_h be the S^3-sphere bundle over S^4 determined by the characteristic map

$$f_h : S^3 \to SO(4)$$

such that for any $u \in S^3$, $v \in \mathbf{R}^4$,

$$f_h(u)v = u^h v u^{1-h}$$

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- M_0 and M_1 are diffeomorphic to S^7 ; while M_2 is an exotic 7-sphere (Milnor's celebrated example)

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$$\sigma : \partial N_h = M_h \to S^7 = \partial D^8$$

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- $X_{h,\sigma}$ denotes the associated attached space $N_h \cup_\sigma D^8$
- When $h = 0$, 1 and $\sigma = \mathrm{Id}$, $X_{h,\sigma}$ is the standard quaternionic projective plane

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- $X_{h,\sigma}$'s are the only closed smooth 8-manifolds admitting a Morse function with 3 critical points
- **Theorem (Krammer-Stolz, 2007)** The diffeomorphism type of $X_{h,\sigma}$ does not depend on σ
- **Theorem (Bérard-Bergery, 1977)** If τ_h is the canonical involution on M_h obtained by the fiberwise antipodal involution on S^3, then $X_{h,\sigma}$ carries an SC^p structure if there is a diffeomorphism σ such that

$$\tau\sigma = \sigma\tau_h,$$

  where τ is the standard involution on S^7

Main results

- **Theorem (Tang-Zhang, 2013)** The involution τ_h on $M_h \equiv S^7$ is equivalent to the standard antipodal involution on S^7 (i.e., M_h/τ_h is diffeomorphic to $\mathbf{R}P^7$)

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- Combining the above three theorems, one gets

- **Corollary (Tang-Zhang, 2013)** Every Eells-Kuiper quaternionic projective plane admits an SC^p structure.

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- The double covering of both is S^7, while

$$\mu(\mathbf{R}P^7) \neq \mu(\mathbf{R}P^7\#14M_2),$$

where μ is the Eells-Kuiper invariant

A result of Mayer

- By a result of Mayer (1970), one knows that M_h/τ_h has only **two** possible diffeomorphism types :

$$\mathbf{R}P^7 \quad \text{and} \quad \mathbf{R}P^7 \# 14 M_2$$

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- Combining with Milnor's result, one need only to show that

$$\mu(M_h/\tau_h) = \mu(\mathbf{R}P^7)$$

The Eells-Kuiper μ-invariant

- M closed spin 7-manifold with $H^4(M, \mathbf{R}) = 0$.
  If N is a spin 8-manifold with boundary M, then the first Pontryajin class $p_1(N)$ is well-defined

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- $\mu(M) \in \mathbf{R}/\mathbf{Z}$ defined by

$$\mu(M) \equiv \frac{p_1^2(N)}{2^7 \cdot 7} - \frac{\mathrm{Sign}(N)}{2^5 \cdot 7} \mod \mathbf{Z}$$

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- Let N_h be the disk bundle associated to M_h. By Milnor and Eells-Kuiper, one has

$$\frac{p_1^2(N_h)}{2^7 \cdot 7} - \frac{\mathrm{Sign}(N_h)}{2^5 \cdot 7} = \frac{h(h-1)}{56} \in \mathbf{Z}$$

Main result

- By Milnor, $\mu(\mathbf{R}P^7) = \pm\frac{1}{32}$, $\mu(\mathbf{R}P^7\#14M_2) = \pm\frac{1}{32} + \frac{1}{2}$

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- Main difficulty : hard to find a spin manifold N' such that $\partial N' = M_h/\tau_h$
- Resolution : using the intrinsic formula due to Donnelly and Kreck-Stolz for μ-invariant, not relying on the bounding manifold (η-invariant appears !)

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- g^{TM} a Riemannian metric on TM, ∇^{TM} the associated Levi-Civita connection
- D_M, B_M the associated Dirac and Signature operators
- $p_1(TM, \nabla^{TM})$ the first Pontryajin form, then there is a 3-form $\widehat{p}_1(TM, \nabla^{TM})$ such that

$$d\,\widehat{p}_1(TM, \nabla^{TM}) = p_1(TM, \nabla^{TM})$$

The formula of Donnely and Kreck-Stolz II

- The formula of Donnelly and Kreck-Stolz :

$$\mu(M) \equiv \overline{\eta}(D_M) + \frac{\eta(B_M)}{2^5 \cdot 7}$$

$$-\frac{1}{2^7 \cdot 7} \int_M p_1(TM, \nabla^{TM}) \wedge \widehat{p}_1(TM, \nabla^{TM}) \mod \mathbf{Z}$$

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- Here for a self-adjoint elliptic operator D, $\eta(D)$ is the η-invariant of D in the sense of Atiyah-Patodi-Singer and $\overline{\eta}(D)$ is the reduced η-invariant defined by

$$\overline{\eta}(D) = \frac{\dim(\ker D) + \eta(D)}{2}$$

The μ-invariant of M_h/τ_h I

- Our purpose is to compute

$$\mu(M_h/\tau_h) \equiv \overline{\eta}(D_{M_h/\tau_h}) + \frac{\eta(B_{M_h/\tau_h})}{2^5 \cdot 7}$$

$$-\frac{1}{2^7 \cdot 7} \int_{M_h/\tau_h} p_1(T(M_h/\tau_h)) \wedge \widehat{p}_1(T(M_h/\tau_h)) \mod \mathbf{Z}$$

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- Lift everything from M_h/τ_h to M_h, one gets

$$\mu(M_h/\tau_h) \equiv \overline{\eta}(P_h D_{M_h}) + \frac{\eta(P_h B_{M_h})}{2^5 \cdot 7}$$

$$-\frac{1}{2^8 \cdot 7}\int_{M_h} p_1(TM_h) \wedge \widehat{p}_1(TM_h) \mod \mathbf{Z},$$

where $P_h = \frac{1+\tau_h}{2}$ denotes the induced projections.

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- One needs to compute the (equivariant) η-invariants in the above formula
- The good thing is that M_h bounds N_h, the disk bundle and the involution τ_h extends on N_h (but not freely)
- One can apply Donnelly's equivariant generalizations of the Atiyah-Patodi-Singer index theorem for manifolds with bounday to compute the η-invariants

The μ-invariant of M_h/τ_h III

- By Donnelly (1978) and Atiyah-Patodi-Singer (1974),

$$\overline{\eta}(P_h D_{M_h}) \equiv \frac{1}{2}\int_{N_h} \widehat{A}(TN_h) + \frac{1}{2}\int_{S^4} A_1 \mod \mathbf{Z},$$

where S^4 is the fixed point set of the τ_h action on N_h ;

$$\eta(P_h B_{M_h}) = \frac{1}{2}\int_{N_h} L(TN_h) + \frac{1}{2}\int_{S^4} A_2$$

$$-\frac{\mathrm{Sign}(N_h) + \mathrm{Sign}(N_h, \tau_h)}{2},$$

where $\mathrm{Sign}(N_h, \tau_h)$ is the notation for the equivariant signature

The μ-invariant of M_h/τ_h IV

- Combining the above three formulas, we get

$$\mu(M_h/\tau_h) \equiv \frac{p_1^2(N_h)}{2^8 \cdot 7} - \frac{\mathrm{Sign}(N_h)}{2^6 \cdot 7} + \frac{1}{2}\int_{S^4} A_1$$

$$+\frac{1}{2}\int_{S^4} A_2 - \frac{\mathrm{Sign}(N_h, \tau_h)}{2} \mod \mathbf{Z}$$

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$$+\frac{1}{2}\int_{S^4} A_2 - \frac{\mathrm{Sign}(N_h,\tau_h)}{2} \mod \mathbf{Z}$$

- But we have seen that

$$\frac{p_1^2(N_h)}{2^7\cdot 7} - \frac{\mathrm{Sign}(N_h)}{2^5\cdot 7} = \frac{h(h-1)}{56}$$

The μ-invariant of M_h/τ_h V

- The fixed point set contributions can also be computed explicitly. In summary, one gets

$$\mu(M_h/\tau_h) = \frac{h(h-1)}{112} \pm \frac{2h-1}{32} \mod \mathbf{Z}$$

The μ-invariant of M_h/τ_h V

- The fixed point set contributions can also be computed explicitly. In summary, one gets

$$\mu(M_h/\tau_h) = \frac{h(h-1)}{112} \pm \frac{2h-1}{32} \mod \mathbf{Z}$$

- Since $\frac{h(h-1)}{56} \in \mathbf{Z}$, one has $h \equiv 0,\ 1,\ 8,\ 49 \mod 56\mathbf{Z}$. By direct verification, one has

$$\mu(M_h/\tau_h) \equiv \pm\frac{1}{32} \mod \mathbf{Z}$$

as required. **Q.E.D.**

Thanks