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Perturbation of sectorial projections of elliptic pseudo-differential operators

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Abstract Over a closed manifold, we consider the sectorial projection of an elliptic pseudo-differential operator A of positive order with two rays of minimal growth. We show that it depends continuously on A when the space of pseudo-differential operators is equipped with a certain topology which we explicitly describe. Our main application deals with a continuous curve of arbitrary first order linear elliptic differential operators over a compact manifold with boundary. Under the additional assumption of the weak inner unique continuation property, we derive the continuity of a related curve of Calderón projections and hence of the Cauchy data spaces of the original operator curve. In the Appendix, we describe a topological obstruction against a verbatim use of R. Seeley's original argument for the complex powers, which was seemingly overlooked in previous studies of the sectorial projection.

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1 Introduction

This note describes how the continuity of a curve of operators (here of sectorial projections) can be derived within the symbolic calculus, supplemented by estimates of some smoothing operators. As usual, the smoothing operators appear as correction terms between pseudo-differential operators and the operators generated by their total symbol.

The power of the symbolic calculus is well established for the investigation of spectral invariants, e.g., derived from asymptotics of the heat kernel. There, perturbations by smoothing operators have no effect. However, the symbolic calculus may appear as having no value for establishing the precise continuity of an operator curve since, a priori, the variation of the operator norm of emergent smoothing operators may be hard to control. This note refutes that view.

1.1 Various definitions of sectorial projections for elliptic pseudo-differential operators of positive order

1.1.1 The bounded and the closed self-adjoint cases

Let $\mathcal{B}(H)$ denote the space of bounded operators in a complex separable Hilbert space H and let $A \in \mathcal{B}(H)$. Assume that there exists a curve $\Gamma_+ \subset \mathbb{C} \setminus \operatorname{spec} A$ that divides $\mathbb{C}$ into two sectors $\Lambda_\pm$ as in Fig. 1a below. Then we can encircle all spectral points in the positive sector Λ_+ by a closed curve Γ_0, as in Fig. 1b, and so get a well-defined projection, the *sectorial projection*, by setting

$$P_{\Gamma_+}(A) := \frac{-1}{2\pi i} \int_{\Gamma_0} (A-\lambda)^{-1} d\lambda. \tag{1.1}$$

From the integral it is clear that

$$\|P_{\Gamma_+}(A+B) - P_{\Gamma_+}(A)\| < C_A \|B\| \text{ for any small bounded perturbation } B, \tag{1.2}$$

i.e., the map $P_{\Gamma_+} : A \mapsto P_{\Gamma_+}(A)$ is continuous in the operator norm of $\mathcal{B}(H)$.

The general functional analytical arguments break down for (graph norm) continuous curves of densely defined closed operators in H. Actually, in our [11, Example 3.13] examples of operators with unbounded sectorial projection were discussed. From the example it becomes clear that additional assumptions will be required.

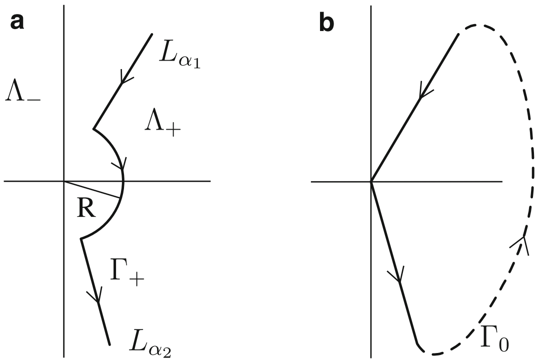

Fig. 1 *Left* two rays of minimal growth and an arc, making the spectral cut curve Γ_+. *Right* specifying a bounded set of eigenvalues by a separating curve Γ_+ made of two rays and capturing it by a closed contour Γ_0.

Most easy is to require that A is self-adjoint: Consider a (graph norm) continuous curve in the space $\mathcal{C}^{\mathrm{sa}}(H)$ of densely defined closed self-adjoint operators in H. The preceding perturbation argument generalizes immediately to this case under the additional condition, that the *Riesz transformation* $F : \mathcal{C}^{\mathrm{sa}} \to \mathcal{B}^{\mathrm{sa}}, A \mapsto F(A) := (I + A^2)^{-1/2}A$ is continuous. It may be worth mentioning that a counter example (a graph-norm convergent sequence of unbounded self-adjoint Fredholm operators with divergent Riesz transforms) was given to us by B. Fuglede several years ago and elaborated in our [10, Example 2.14]. The condition is satisfied, however, for formally self-adjoint elliptic differential operators on closed manifolds: they have a discrete spectrum of finite multiplicity contained in $\mathbb{R}$ and a complete set of eigenvectors. So, the imaginary axis (or a parallel $\{c + ri \mid r \in \mathbb{R}\}$ with $c \notin \operatorname{spec} A$) becomes a suitable separating curve Γ_+ and we obtain $P_{\Gamma_+}(A) = 1_{[c,\infty)}(A) = 1_{[F(c),\infty)}(F(A))$ as a pseudo-differential projection [the Atiyah–Patodi–Singer (APS) projection] by applying the integral representation of (1.1) to the bounded Riesz transform of A. Note that $F(A)$ has its spectrum contained in the interval $(-1, 1)$, but has the same eigenspaces and sectorial projection as A. We refer to our [11, Propositions 7.14-7.15] (see also [9, Thm. 4.8] for a wider purely functional analytic setting) for a proof of the continuity of the Riesz transformation $A \mapsto F(A)$ on the space of self-adjoint elliptic differential operators. That yields the well-known continuous variation of the APS projection under continuous variation of the underlying operator as long no eigenvalue crosses the line Γ_+, i.e., the continuity of the map $A \mapsto P_{\Gamma_+}(A)$, when we take the operator norm $L^2 \to L^2$ for $P_{\Gamma_+}(A)$ and the operator norm $H^m \to L^2$ for A, where m denotes the order of A.

1.1.2 Spectral integrals for elliptic pseudo-differential operators of positive order

It seems that no general functional analysis methods are available to obtain continuous curves of sectorial projections for arbitrary continuous curves of operators with compact resolvent and two rays of minimal growth, if the operators are neither bounded nor self-adjoint.

As explained in our [11, Sect. 3.2], a semigroup $\{Q_+(x, A)\}_{x>0}$ of sectorial operators can be defined by inserting a weight $e^{-\lambda x}$ into the integral (1.1). Then *sectorial projections* can be defined asymptotically in an abstract Hilbert space framework. More precisely, for a closed, not necessarily self-adjoint operator A in separable Hilbert space with compact resolvent and minimal growth of the resolvent in a cone we may take the closure of the densely defined $\lim_{x\to 0+} Q_+(x, A)$. However, such projections are unbounded operators, in general, and do not necessarily vary continuously under perturbation of the underlying operator, see, once again, [11, Example 3.13]. Consequently, one has to exploit the *symbolic calculus* for the investigation of sectorial projections of not necessarily self-adjoint elliptic pseudo-differential operators of positive order with two rays of minimal growth of the resolvent.

Actually, in a slightly different context (namely dealing with well-posed boundary problems), it was already noticed in Burak [13] that sectorial projections are bounded operators. For an elliptic pseudo-differential operator of positive order over a smooth closed (compact and without boundary) manifold, that approach was worked out in Wodzicki [36,37] and the more recent Ponge [31] and Gaarde and Grubb [17]. In some of these papers, the positive sectorial projection plays a prominent role in more refined questions related to spectral asymmetry.

Before preceding, we fix the notation:

Convention 1.1 (a) Let M be an n-dimensional closed Riemannian manifold and $\pi : E \to M$ a Hermitian vector bundle. Let $A : C^\infty(M; E) \to C^\infty(M; E)$ be an elliptic pseudo-differential operator of order $m > 0$.

(b) Let $\operatorname{spec}(A)$ denote the spectrum of A regarded as an operator in $L^2(M; E)$ with the Sobolev space $H^m(M; E)$ as its domain. We recall that $\operatorname{spec}(A)$ is either the whole complex plane or a discrete subset of $\mathbb{C}$. The reason is simply that the resolvent, if it exists, is compact (see Shubin [35, Theorem 8.4], similarly already in Agmon [1, Sect. 2] for well-posed elliptic boundary value problems). Clearly, $\operatorname{ind} A \neq 0$ implies $\operatorname{spec} A = \mathbb{C}$.

(c) Let $L_{\alpha_1} = \{\lambda \in \mathbb{C} \mid \arg\lambda = \alpha_1\}$ and $L_{\alpha_2} = \{\lambda \in \mathbb{C} \mid \arg\lambda = \alpha_2 \equiv \alpha_1 - \theta \mod 2\pi\}$ $(0 < \theta < 2\pi)$ be two rays. We assume that A has only a finite number of eigenvalues on the rays L_{α_j}, $j = 1, 2$.

(d) Let $\Lambda := \{re^{i\alpha} \mid r < 2\rho \text{ or } |\alpha - \alpha_j| < \varepsilon, j = 1, 2\}$ for $\rho, \varepsilon > 0$ and ε sufficiently small. We can choose ρ in such a way that there exists an $R \in [0, \rho]$ such that $A - \lambda$ is invertible for $\lambda \in \Lambda$ with $|\lambda| \geq R$, and there is only a finite number of eigenvalues in the region $\Lambda_R := \{\lambda \in \Lambda \mid |\lambda| < R\}$. For an elaboration of the meaning of such spectral cuttings, also called *rays of minimal growth* (of the resolvent $(A - \lambda)^{-1}$), see Sect. 2.4 below. If A is differential, then $A - \lambda$ is *elliptic with respect to the parameter* $\lambda \in \Lambda$ for sufficiently small $\rho, \varepsilon > 0$ (for that concept c.f. Seeley [34] or Shubin [35]).

(e) Now we choose the curve

$$\Gamma_+ = \{re^{i\alpha_1} \mid \infty > r \geq R\} \cup \{Re^{i(\alpha_1 - t)} \mid 0 \leq t \leq \theta\} \cup \{re^{i\alpha_2} \mid R \leq r < \infty\} \tag{1.3}$$

in the resolvent set of A, see Fig. 1a.

(f) We define an operator in the following form:

$$P_{\Gamma_+}(A) = -\frac{1}{2\pi i} A \int_{\Gamma_+} \lambda^{-1}(A-\lambda)^{-1} d\lambda. \tag{1.4}$$

A priori, the integral (1.4) gives rise to an unbounded operator. A common error in the literature is a *verbatim* use of the arguments of [34, Theorem 3] to prove that $P_{\Gamma_+}(A)$ is a pseudo-differential operator of order 0, see Wojciechowski [38], uncritically reproduced in Nazaikinskii et al. [27], Ponge [31, Proof of Proposition 3.1], and Gaarde and Grubb [17]. In the Appendix, we shall explain the topological obstructions that make the argument defective.

Anyway, in [31, Proof of Proposition 4.1], a beautiful formula is proved, which, according to Ponge was already observed by Wodzicki in 1985: Assume the preceding conventions and, moreover, that A is a classical pseudo-differential operator. Then we have

$$A^s_{\alpha_2} - A^s_{\alpha_1} = (1 - e^{2i\pi s}) P_{\Gamma_+}(A) A^s_{\alpha_2} \quad \text{for all } s \in \mathbb{C}. \tag{1.5}$$

The formula relates the complex powers (which were well established as pseudo-differential operators in [34]) and the sectorial projection. Multiplying by $A^{-s}_{\alpha_2}$ from the right yields that the sectorial projection is a pseudo-differential projection.

1.2 Perturbations of sectorial projections

Unfortunately, the authors of this note were not able to derive a true perturbation result for the sectorial projections from Wodzicki's formula. Clearly, other authors before us have noticed the delicacy of the variation of complex powers of an operator with a ray of minimal growth. For studying variations of trace formulas, Okikiolu [29, Sect. 4], e.g., defines a *symbol-smooth* family of pseudo-differential operators by the smoothness of the (total) symbol in the usual C^∞ Fréchet topology. Then she is able to prove the symbol-smoothness of the complex powers, and the symbol-smoothness of the sectorial projections follows by (1.5). Okikiolu's approximative approach is related to Hörmander's symbolic construction of an "almost" Calderón projection in [24, Theorem 20.1.3]. Admittedly, such approximative constructions can be of great value in some contexts. Actually, Okikiolu's result suffices for proving her trace variation results. More precisely, in [29, Sect. 4], Okikiolu defined a topology on the space of pseudo-differential operators by the smoothness of 'total' symbols in any local coordinates. This topology is sufficient for her purpose to study the variations of trace formulas: it can not miss any smooth operator whose trace is not zero. However, for our aim to study the perturbation of sectorial projections in the operator norms, it seems that the topology defined by Okikiolu is not sufficient. It may miss the smoothing operators with support far away from the diagonal of $M \times M$. Such smoothing operators have zero traces, but may have large operator norms. In this note, we define a locally convex topology, which treats all the lower terms in bulk and does not concern the local charts.

Similarly, we can neglect even compact perturbations when we are interested in index theory. For addressing uniform structures and perturbation results in that direction we refer to Eichhorn [16] who presents a systematic study of compact perturbations of generalized Dirac operators. However, for deciding about the continuous or not continuous variation of the true sectorial projections, we are not allowed to neglect contributions from smoothing operators. We recall that exactly such operators appear as error terms between pseudo-differential operators and their approximative symbolic representation.

To obtain estimates for the precise sectorial projections, we shall therefore not follow the elegant approach by Burak and Wodzicki et al.[1] We choose an approach which does not require any technology beyond symbolic calculus and standard estimates for integral operators. From a technological point of view, our approach may appear less elegant than using logarithms or complex powers, but it is elementary, transparent and self-contained — and it works. The delicacy of Axelsson et al. [6] indicates that there is no easy way through to be expected. They studied the Hodge–Dirac operator D_g defined on a closed Riemannian manifold with metric g. In general, D_g is non-self-adjoint, and its spectrum is contained in an open double sector which includes the real line. They showed — by harmonic analysis methods — that the spectral projections of the Hodge–Dirac operator D_g depend analytically on L_∞ changes in the metric g.

Hence, a priori, the variational properties of the symbol do not suffice for establishing the continuous variation of $P_{\Gamma_+}(A)$ in the topology of the operator norm of a suitable Sobolev space. A smoothing operator may have a large operator norm defined on any Sobolev space. Therefore, our approach is inspired by Seeley's [34, Corollary 2]. We only need slightly sharper estimates than Seeley's original work.

1.3 Main result

1.3.1 Topology and formulation of main theorem

Let $\mathrm{Ell}^m_{\Gamma_+}(M, E)$ denote the space of all elliptic *principally classical* pseudo-differential operators A of order $m > 0$ on M acting on sections of the bundle E such that the leading symbol a_m of A has no eigenvalues on the two rays L_{α_j}, $j = 1, 2$ and A no eigenvalues on Γ_+. Taking for granted that the spaces $\mathrm{CL}^m(M, E)$ and $\mathrm{L}^{m-1}(M, E)$ are well known (we also recall them in Sect. 2.2), we use the notation *principally classical pseudo-differential operators* $\mathrm{L}^m_{\mathrm{pc}}(M, E) := \mathrm{CL}^m(M, E) + \mathrm{L}^{m-1}(M, E)$ for standard pseudo-differential operators with a homogeneous principal symbol, where the principal symbol denotes the class of the operator modulo operators of lower order (for details see below Sect. 2.3). Hence, while we do not restrict our estimates to classical pseudo-differential operators, we must require a homogeneity of the principal symbol.

[1] Having been informed about our results, Prof. Grubb has, perhaps rightly, pointed out to us, that sharper perturbation results might be achievable, as it often happens a posteriori in mathematics, by alternative methods: namely, by exploiting the description of the sectorial projection by a difference of two logarithms in [17, Prop. 4.4]. Unfortunately, no details regarding the perturbation problem were communicated by Prof. Grubb in her subsequent comments [20] to an earlier arXiv version of this note.

We equip the space $\mathrm{Ell}^m_{\Gamma_+}(M, E)$ with the locally convex topology $\mathcal{T}$ described in Sect. 2.3 below. It is not surprising that our topology is stronger than the usual operator topology between Sobolev spaces for pseudo-differential operators of fixed order m on closed manifolds, see Atiyah, Singer [5]. For our applications, continuous or smooth variation of all the symbols does not suffice. In addition, we shall require that all the derivatives of the principal symbol vary continuously. The necessity of rather restrictive requirements for the variation of the highest order coefficients was indicated in our [11, Sect. 7] where we emphasized the elementary character of perturbations of lower order and the delicacy of perturbations of highest order for the continuous variation of the Calderón projection.

In Sect. 5.1, we give an elementary proof for each $A \in \mathrm{Ell}^m_{\Gamma_+}(M, E)$, that the operator $P_{\Gamma_+}(A)$ is well defined by (1.4) as a bounded operator on $H^s(M; E)$, $s \in \mathbb{R}$. The following theorem is our main result:

Theorem 1.2 *With respect to the topology $\mathcal{T}$, the map*

$$P_{\Gamma_+} : \mathrm{Ell}^m_{\Gamma_+}(M, E) \to \mathcal{B}(H^s(M; E)), \quad A \mapsto P_{\Gamma_+}(A) \tag{1.6}$$

is continuous. Here $\mathcal{B}(H^s(M; E))$ denotes the set of bounded linear operators on $H^s(M; E)$, $s \in \mathbb{R}$.

1.3.2 The structure of this note

In Sect. 2 we introduce principally classical symbols and principally classical pseudo-differential operators $\mathrm{L}^m_{\mathrm{pc}}$ on closed manifolds, define a locally convex topology on it, and discuss the natural factorization of sectorial projections. In Sect. 3 we identify the (semi)-norms we need on $\mathrm{L}^m_{\mathrm{pc}}$ to ensure that P_{Γ_+} is continuous. In Sect. 4 we give a technical lemma which is crucial in the proof of our main theorem. We prove some more estimates which possibly are of more general interest, as well. In Sect. 5 we apply our estimates to the perturbation problem for sectorial projections and draw some consequences for index correction formulas and the variation of Cauchy data spaces on manifolds with boundary. In the Appendix A, we give the details of the proof of the technical lemma. In the Appendix B we explain the topological obstructions that are seemingly overlooked in the literature on symbolic calculus.

2 Definitions and notations

To fix the notation, we summarize the basic concepts of symbolic calculus and introduce a space of principally classical pseudo-differential operators on closed manifolds. Moreover, we fix a locally convex topology on it. For elliptic principally classical pseudo-differential operators of positive order and for a fixed contour Γ_+ we define the sectorial projections.

2.1 Classes of symbols

Let $U \subset \mathbb{R}^n$ be an open subset. We denote by $\mathrm{S}^m(U \times \mathbb{R}^n), m \in \mathbb{R}$, the space of (complex valued) symbols (the generalization for matrix valued symbols is straightforward) of Hörmander type $(1, 0)$ (Hörmander [23], Grigis–Sjöstrand [19]). More precisely, $\mathrm{S}^m(U \times \mathbb{R}^n)$ consists of those $a \in \mathrm{C}^\infty(U \times \mathbb{R}^n)$ such that for multi–indices $\alpha, \gamma \in \mathbb{Z}_+^n$ and compact subsets $K \subset U$ we have an estimate

$$|\partial_x^\alpha \partial_\xi^\gamma a(x, \xi)| \le C_{\alpha,\gamma,K}(1 + |\xi|)^{m-|\gamma|}, \quad x \in K. \tag{2.1}$$

The best constants in (2.1) provide a set of semi-norms which endow $\mathrm{S}^\infty(U \times \mathbb{R}^n) := \bigcup_{m\in\mathbb{R}} \mathrm{S}^m(U \times \mathbb{R}^n)$ with the structure of a Fréchet algebra.

The space $\mathrm{CS}^m(U \times \mathbb{R}^n)$ of *classical symbols* consists of all $a \in \mathrm{S}^m(U \times \mathbb{R}^n)$ that admit sequences $a_{m-j} \in \mathrm{C}^\infty(U \times \mathbb{R}^n)$, $j \in \mathbb{Z}_+$ with

$$a_{m-j}(x, r\xi) = r^{m-j} a_{m-j}(x, \xi), \quad r \ge 1, |\xi| \ge 1, \tag{2.2}$$

such that

$$a - \sum_{j=0}^{N-1} a_{m-j} \in \mathrm{S}^{m-N}(U \times \mathbb{R}^n) \quad \text{for all } N \in \mathbb{Z}_+. \tag{2.3}$$

The latter property is usually abbreviated $a \sim \sum_{j=0}^\infty a_{m-j}$.

Homogeneity and smoothness at 0 contradict each other except for monomials. Our convention is that symbols should always be smooth functions, thus the a_{m-j} are smooth everywhere but homogeneous only in the restricted sense of Eq. (2.2).

Furthermore, we denote by $\mathrm{S}^{-\infty}(U \times \mathbb{R}^n) := \bigcap_{a\in\mathbb{R}} \mathrm{S}^a(U \times \mathbb{R}^n)$ the space of *smoothing symbols*.

2.2 (Classical) pseudo-differential operators

Let M be a smooth manifold of dimension n. For convenience and to have an L^2–structure at our disposal, we assume that M is equipped with a Riemannian metric. We denote by $\mathrm{L}^\bullet(M)$ the algebra of *pseudo-differential operators* with symbols of Hörmander type $(1, 0)$ [23,35], see Sect. 2.1. The subalgebra of *classical pseudo-differential operators* is denoted by $\mathrm{CL}^\bullet(M)$. These operator algebras are naturally defined on the manifold M by localizing in coordinate patches in the following way:

Let $U \subset \mathbb{R}^n$ be an open subset. Recall that for a symbol $a \in \mathrm{S}^m(U \times \mathbb{R}^n)$, the *canonical* pseudo-differential operator *associated* to a is defined by

$$(\mathrm{Op}(a)\,u)(x) := \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\, a(x,\xi)\, \hat{u}(\xi)\, đ\xi \tag{2.4}$$
$$= \int_{\mathbb{R}^n}\int_U e^{i\langle x-y,\xi\rangle}\, a(x,\xi)\, u(y) dy đ\xi, \qquad đ\xi := (2\pi)^{-n} d\xi.$$

For a manifold M, elements of $\mathrm{L}^\bullet(M)$ [resp. $\mathrm{CL}^\bullet(M)$] can locally be written as $\mathrm{Op}(\sigma)$ with $\sigma \in \mathrm{S}^\bullet(U \times \mathbb{R}^n)$ [resp. $\mathrm{CS}^\bullet(U \times \mathbb{R}^n)$].

Recall that there is an exact sequence

$$0 \longrightarrow \mathrm{CL}^{m-1}(M) \hookrightarrow \mathrm{CL}^m(M) \xrightarrow{\sigma_m} C^\infty(S^*M) \longrightarrow 0, \tag{2.5}$$

where $\sigma_m(A)$ denotes the *principal* (homogeneous leading) symbol of $A \in \mathrm{CL}^m(M)$. Here, S^*M denotes the cosphere bundle, i.e., the unit sphere bundle $\subset T^*M$. As usual, the principal symbol is locally defined as a map $\sigma_m : \mathrm{S}^m(U \times \mathbb{R}^n) \to C^\infty(U \times S^{n-1})$ by putting

$$\sigma_m(x,\xi) := \lim_{r\to\infty} r^{-m} a(x, r\xi). \tag{2.6}$$

Note that $\sigma_m(A)$ is a homogeneous function on the symplectic cone $T^*M \setminus M$. We will tacitly identify the homogeneous functions on $T^*M \setminus M$ by restriction with $\mathrm{C}^\infty(S^*M)$.

Recall that the principal symbol map is multiplicative in the sense that

$$\sigma_{a+b}(A \circ B) = \sigma_a(A)\sigma_b(B) \tag{2.7}$$

for $A \in \mathrm{CL}^a(M)$, $B \in \mathrm{CL}^b(M)$.

2.3 Principally classical pseudo-differential operators

As mentioned in the Introduction, continuous variation of the operator A by bounded $L^2 \to L^2$ perturbation is sufficient to obtain continuous variation of the Cauchy data space, of the Calderón projection and of the sectorial projection in various cases (see [9, Theorem 3.8], [11, Proposition 7.13]). However, we have a hunch that continuous variation of the operator A in the operator norm, say from $H^m(M)$ to $L^2(M)$ will not always yield continuous variation of the sectorial projection $P_{\Gamma_+}(A)$ in the operator norm from $L^2(M)$ to $L^2(M)$. These are our intuitive arguments:

We know that general functional analysis does not suffice to obtain the boundedness of the sectorial projection. The more refined structure of differential or pseudo-differential operators is required. Apparently, for variation in the highest order, the principal symbol must be singled out. All that indicates that variation in the operator norm hardly will suffice for continuous variation of the sectorial projection.

We use the following convention which will be in effect for the rest of this note:

Convention 2.1 We denote the norm on the space $\mathcal{B}(H^s, H^t)$ of bounded operators from the Sobolev space $H^s(M; E)$ to $H^t(M; E)$ by $\|\cdot\|_{s,t}$.

Let $\mathrm{L}^{m-1}(M, E)$ [resp. $\mathrm{CL}^m(M, E)$] denote the space of $(m-1)$th order pseudo-differential operators on E (resp. mth order classical pseudo-differential operators). Set $\mathrm{L}^m_{\mathrm{pc}}(M, E) := \mathrm{CL}^m(M, E) + \mathrm{L}^{m-1}(M, E)$. We call it the space of *principally classical pseudo-differential operators*. Since $\mathrm{CL}^m(M, E) \cap \mathrm{L}^{m-1}(M, E) = \mathrm{CL}^{m-1}(M, E)$, the principal symbol map

$$\sigma_m : \mathrm{L}^m_{\mathrm{pc}}(M, E) \to C^\infty(S^*M; \mathrm{End}(\pi^* E))$$

is well-defined. Here $\pi : S^*M \to M$ denotes the canonical projection map. We fix a right inverse

$$\mathrm{Op} : C^\infty(S^*M; \mathrm{End}(\pi^* E)) \to \mathrm{L}^m_{\mathrm{pc}}(M, E)$$

of σ_m, obtained by patching together the local Op-maps (2.4) via a partition of unity. Define a map

$$\begin{array}{cccc} C^\infty(S^*M; \mathrm{End}(\pi^* E)) & \oplus\ \mathrm{L}^{m-1}(M, E) & \to & \mathrm{L}^m_{\mathrm{pc}}(M, E) \\ a & \oplus \quad B & \mapsto & \mathrm{Op}(a) + B. \end{array} \tag{2.8}$$

It is a bijection with the inverse

$$\begin{array}{ccccc} \mathrm{L}^m_{\mathrm{pc}}(M, E) & \to & C^\infty(S^*M; \mathrm{End}(\pi^* E)) & \oplus & \mathrm{L}^{m-1}(M, E) \\ T & \mapsto & \mathrm{Op}(\sigma_m(T)) & \oplus & T - \mathrm{Op}(\sigma_m(T)). \end{array} \tag{2.9}$$

We topologize the right hand side of (2.9) as follows:

1. On $\mathrm{L}^{m-1}(M, E)$ we take the countably many semi-norms $\|T\|_{k+m-1,k}$ for $k \in \mathbb{Z}$.
2. The summand $C^\infty(S^*M; \mathrm{End}(\pi^* E))$ is equipped with the C^∞-topology. This is known to be a Fréchet-topology, hence is generated by countably many semi-norms $(p_j)_{j \in \mathbb{Z}_+}$.

Definition 2.2 The locally convex topology on $\mathrm{L}^m_{\mathrm{pc}}(M, E)$ induced by the countably many semi-norms $\|\cdot\|_{k+m-1,k},\ k \in \mathbb{Z}$ and $p_j,\ j \in \mathbb{Z}_+$ is denoted by $\mathcal{T}$.

It follows from complex interpolation that for each *real* s the (semi)-norm $\|\cdot\|_{s+m-1,s}$ is continuous with regard to $\mathcal{T}$. Furthermore, it is straightforward to see that $\mathcal{T}$ is independent of the choice of Op. Moreover, it is worth noting that $\mathcal{T}$ is not complete. By construction, the completion of $\mathrm{L}^m_{\mathrm{pc}}(M, E)$ is a Fréchet space which is of the form

$$\mathrm{CZ}^{m-1}(M, E) \oplus C^\infty(S^*M; \mathrm{End}(\pi^* E)).$$

Here $\mathrm{CZ}^\bullet(M, E)$ is (a variant of) the well-known Calderón–Zygmund graded algebra (cf. [30, Chapter 16]).

Remark 2.3 We record that a sequence $(T_n)_{n \in \mathbb{N}} \subset \mathrm{L}^m_{\mathrm{pc}}(M, E)$ converges to $T \in \mathrm{L}^m_{\mathrm{pc}}(M, E)$ if and only if

(i) $\sigma_m(T_n) \longrightarrow \sigma_m(T)$ in the C^∞-topology of $C^\infty(S^*M; \mathrm{End}(\pi^* E))$, and
(ii) $T_n - \mathrm{Op}(\sigma_m(T_n)) \longrightarrow T - \mathrm{Op}(\sigma_m(T))$ with regard to $\|\cdot\|_{k+m-1,k}$ for all $k \in \mathbb{Z}$.

2.4 The definition of $P_{\Gamma_+}(A)$

We shall give our definition in some detail. These details will be decisive for proving the perturbation results.

2.4.1 Our data

Our data are as in Convention 1.1. More specifically, we shall assume $A \in \mathrm{L}^m_{\mathrm{pc}}(M, E)$ and that the principal symbol $a_m(x, \xi)$ of A has no eigenvalues on the rays L_{α_j}, $j = 1, 2$ for each point $x \in M$ and covector $\xi \in T^*_x M, \xi \neq 0$. For simplicity, denote the principal symbol $\sigma^m_A(x, \xi)$ by $a_m = a_m(x, \xi)$. Note that every ray of minimal growth has a cone-shaped neighborhood Λ such that any ray contained in Λ is also a ray of minimal growth for A. Then there exists $R > 0$ such that Conventions 1.1(c)–(d) are satisfied and $A - \lambda$ is invertible for $\lambda \in \Lambda$, $|\lambda| > R$. Moreover, we have

$$\|(A - \lambda)^{-1}\|_{s,s+p} \leq C|\lambda|^{-1+\frac{p}{m}}, \quad 0 \leq p \leq m,\ s \in \mathbb{R}, \tag{2.10}$$

for any such λ. For the proof of (2.10) see [34, Corollary 1]. For differential operators see also [35, Theorem 9.3]. Equation (2.10) explains the common usage of "ray of minimal growth of the resolvent" for such spectral cutting rays.

2.4.2 Definition of the sectorial projection and our goal

Equation (2.10) explains why we cannot expect convergence of the integral $\int_{\Gamma_+} (A - \lambda)^{-1}\, d\lambda$, which is familiar in the bounded case presented above in (1.1). The common way to get something finite is to guarantee convergence of the integral by inserting a factor λ^{-1} and to compensate by multiplying the integral by A.

Definition 2.4 For the preceding data, we define

$$P_{\Gamma_+}(A) := \frac{-1}{2\pi i} A\, \Phi(A), \qquad \Phi(A) := \int_{\Gamma_+} \lambda^{-1}\, (A - \lambda)^{-1}\, d\lambda. \tag{2.11}$$

Remark 2.5 (a) In view of the estimate (2.10), the composition of A with the integral $\Phi(A)$ a priori gives rise to an unbounded operator on $L^2(M; E)$ with domain $\cup_{s>0} H^s(M; E)$. The nice fact, however, is that $P_{\Gamma_+}(A)$ truly is a bounded operator (see Sect. 5.1).
(b) Our goal is to prove that with respect to the topology $\mathcal{T}$

$$P_{\Gamma_+} : \mathrm{Ell}^m_{\Gamma_+}(M, E) \longrightarrow \mathcal{B}(H^s(M; E)) \quad \text{is continuous for all } s \in \mathbb{R}. \tag{2.12}$$

Here we keep the rays L_{α_j}, $j = 1, 2$ and the contour Γ_+ fixed and set

$$\begin{aligned} \mathrm{Ell}^m_{\Gamma_+}(M, E) := \{A \in \mathrm{L}^m_{\mathrm{pc}}(M, E) \mid A \text{ elliptic, } \operatorname{spec} A \cap \Gamma_+ = \emptyset \\ \text{and } L_{\alpha_j},\ j = 1, 2 \text{ rays of minimal growth}\}. \end{aligned} \tag{2.13}$$

(c) As a side result, we shall show under what conditions $P_{\Gamma_+}(A)$ becomes a pseudo-differential operator. We consider that of minor importance. The proof of (b) will anyway show that $P_{\Gamma_+}(A)$ is of the form $P_{\Gamma_+,0}(A) + K$ with $P_{\Gamma_+,0}(A) \in \mathrm{L}^0_{\mathrm{pc}}(M, E)$ and K a compact operator.

2.5 First reduction

The factorization of $P_{\Gamma_+}(A) = \frac{-1}{2\pi i} A\Phi(A)$ in Eq. (2.11) of Definition 2.4 permits a first reduction of our problem.

Lemma 2.6 *Suppose that the map*

$$\Phi : \mathrm{Ell}^m_{\Gamma_+}(M, E) \ni A \mapsto \int_{\Gamma_+} \lambda^{-1} (A - \lambda)^{-1} \, d\lambda \in \mathcal{B}(H^s, H^{s+m}) \qquad (2.14)$$

is continuous and that $\|\cdot\|_{s,s+m}$ *is a continuous (semi-)norm with respect to* $\mathcal{T}$*. Then our claim* (2.12) *holds.*

Proof Given $A \in \mathrm{Ell}^m_{\Gamma_+}(M, E)$. Then there is a neighborhood U of A such that $\|\cdot\|_{s+m,s}$ is bounded on U. Hence we reach the conclusion from

$$\begin{aligned} &\|P_{\Gamma_+}(A) - P_{\Gamma_+}(B)\|_{s,s} \\ &\quad \le \|A - B\|_{s+m,s} \, \|\Phi(A)\|_{s,s+m} + \|B\|_{s+m,s} \, \|\Phi(A) - \Phi(B)\|_{s,s+m}. \end{aligned}$$

□

This Lemma reduces the problem to the task of considering $\int \lambda^{-1} (A - \lambda)^{-1} \, d\lambda$, which is more convenient.

3 Local considerations

In our Definition 2.2, we specified our topology $\mathcal{T}$, see also Remark 2.5(b). Now we shall successively identify the corresponding (semi-)norms on $\mathrm{L}^m_{\mathrm{pc}}$ which ensure that P_{Γ_+} is continuous.

3.1 Cut-off symbols

In the Appendix B, we explain why we cannot deform and extend the symbol $a_m(x, \xi)$ in a suitable way. However, we can easily deform and extend $(a_m(x, \xi) - \lambda)^{-1}$ in the usual way as a *smoothed resolvent symbol* [35, Sects. 11.3–11.4] (similarly, e.g., Bilyj et al. in the recent [7, Definition 2.5]). Recall that we denote the principal symbol of A by $a_m(x, \xi)$ and that we have assumed that

$$\operatorname{spec} a_m(x, \xi) \cap L_{\alpha_j} = \emptyset \quad \text{for } (x, \xi) \in T^*M, \xi \neq 0, j = 1, 2. \qquad (3.1)$$

Thus, there is a constant $\rho > 0$ such that $a_m(x, \xi) - \lambda$ is invertible for $(x, \xi) \in T^*M$, $|\xi| \geq \rho$ and $\lambda \in \Gamma_+$. Hence, for any cut-off function $\psi \in C^\infty(\mathbb{R}^n)$ with

$$\psi(\xi) = \begin{cases} 0, & \text{for } |\xi| \leq \rho, \\ 1, & \text{for } |\xi| \gg 0, \end{cases} \tag{3.2}$$

(that is, the function $1 - \psi$ is compactly supported) and for each $\lambda \in \Gamma_+$ the symbol

$$(x, \xi) \mapsto \psi(\xi)(a_m(x, \xi) - \lambda)^{-1} \tag{3.3}$$

is a classical symbol of order $-m$.

3.2 Symbol estimates and semi-norms

From now on we shall switch forward and backward between arguing locally (in the open domain $U \subset \mathbb{R}^n$) and globally (on M). With the preceding symbol a_m and cut-off function ψ, we shall write

$$r^\psi(x, \xi, \lambda) := \psi(\xi)(a_m(x, \xi) - \lambda)^{-1}. \tag{3.4}$$

For fixed λ we have $r^\psi(\cdot, \cdot, \lambda) \in \mathrm{CS}^{-m}(U \times \mathbb{R}^n, E)$. Considered as a λ-dependent symbol, it does not necessarily belong to the usual parameter dependent calculus. Actually, the cut-off ψ prevents this.

However, we have the following symbol estimates, which are uniform in $\lambda \in \Gamma_+$:

$$\begin{aligned} &|\partial_x^\alpha \partial_\xi^\beta r^\psi(x, \xi, \lambda)| \\ &\quad \leq \begin{cases} C_{0,0}(1 + |\xi| + |\lambda|^{1/m})^{-m}, & \alpha = \beta = 0, \\ C_{\alpha,\beta}(1 + |\xi|)^{m-|\beta|} (1 + |\xi| + |\lambda|^{1/m})^{-2m}, & (\alpha, \beta) \neq (0, 0), \end{cases} \\ &\quad \leq C_{\alpha,\beta}(1 + |\xi|)^{-|\beta|} (1 + |\xi| + |\lambda|^{1/m})^{-m}. \end{aligned} \tag{3.5}$$

The proof is an exercise in induction and Leibniz rule.

What is important is that the best constants $C_{\alpha,\beta}$ in (3.5), as functions of a_m, are continuous semi-norms on the space $C^k(S^*M; \mathrm{End}(\pi^*E))$ of sections with $k := |\alpha| + |\beta|$. In particular, they are continuous semi-norms on the space $C^\infty(S^*M; \mathrm{End}(\pi^*E))$.

As a consequence, we have the following: for fixed k and fixed symbol $a_m \in C^\infty(S^*M; \mathrm{End}(\pi^*E))$ there is an open neighborhood U of a_m such that $C_{\alpha,\beta}$, $|\alpha| + |\beta| \leq k$, are bounded on U and such that each $b_m \in U$ is "invertible" on Γ_+, that is, it satisfies the same Ell_{Γ_+}-conditions as a_m.

Note. We have to fix k and cannot bound infinitely many semi-norms simultaneously: the intersection of infinitely many open $U_{\alpha,\beta}$ might be non-open.

3.3 A first approximation

The symbolic calculus yields the following first approximation result.

Proposition 3.1 (a) *For a_m and r^ψ as above, the operator*

$$\Phi_0(a_m) := \int_{\Gamma_+} \lambda^{-1} \operatorname{Op}\big(r^\psi(\cdot,\cdot,\lambda)\big)\, d\lambda \tag{3.6}$$

belongs to the class $\mathrm{CL}^{-m}(U, E)$.
(b) $\Phi_0 \circ \sigma_m : \mathrm{Ell}^m_{\Gamma_+}(M, E) \to \mathcal{B}(H^s, H^{s+m})$ *is continuous with regard to* $\mathcal{T}$.

Proof For (a) we see that

$$\psi(\xi) \int_{\Gamma_+} \lambda^{-1} (a_m(x,\xi) - \lambda)^{-1}\, d\lambda = \int_{\Gamma_+} \lambda^{-1} r^\psi(x,\xi,\lambda)\, d\lambda$$

is homogeneous of degree $-m$ outside a compact set, and smooth otherwise. Recall that principal symbols are determined by their values in $\{(x,\xi) \in T^*M \mid |\xi| \geq C\}$ where C is any positive constant. That proves (a).
For (b) we have that in the topology $\mathcal{T}$,

$$\sigma_m : \mathrm{L}^m_{\mathrm{pc}}(M, E) \to C^\infty(S^*M, \mathrm{End}(\pi^*E))$$

is continuous. We denote the space of symbols analogue of the operator space $\mathrm{Ell}^m_{\Gamma_+}$ by $C^\infty_{\Gamma_+}(S^*M, \mathrm{End}(\pi^*E))$. Certainly,

$$\begin{array}{ccc} C^\infty_{\Gamma_+}(S^*M, \mathrm{End}(\pi^*E)) & \longrightarrow & \mathrm{CS}^{-m}(T^*M, \mathrm{End}(\pi^*E)) \\ a_m & \longmapsto & \int_{\Gamma_+} \lambda^{-1}\, \psi(\xi)\, (a_m(x,\xi) - \lambda)^{-1}\, d\lambda \end{array}$$

and

$$\operatorname{Op} : \ \mathrm{CS}^{-m}(T^*M, \mathrm{End}(\pi^*E)) \longrightarrow \mathcal{B}(H^s, H^{s+m})$$

are continuous. That proves (b). □

4 A technical lemma and key estimates

4.1 A technical lemma

In this subsection, we shall give a technical lemma which is crucial in the proof of our main theorem. It is a variant form (with a parameter) of the composition of pseudo-differential operators. As a service to the reader, we give a detailed proof of this lemma in the Appendix A. Our claims and arguments are local for a fixed open coordinate patch $U \subset \mathbb{R}^n$.

Definition 4.1 For a compact subset $K \subset U$ we denote by $\mathrm{S}^m_K(U \times \mathbb{R}^n) \subset \mathrm{S}^m(U \times \mathbb{R}^n)$ those $a \in \mathrm{S}^m(U \times \mathbb{R}^n)$ such that $a(x,\xi) \neq 0$ implies $x \in K$. $\mathrm{CS}^m_K(U \times \mathbb{R}^n)$ is defined accordingly. A typical example is $a(x,\xi) = \theta(x)b(x,\xi)$ with $b \in S^m(U \times \mathbb{R}^n)$ and a cut-off function $\theta \in C_c^\infty(U)$.

Clearly, the preceding definitions carry over to matrix valued symbols and to globally defined symbols with values in bundle endomorphisms.

Lemma 4.2 (Technical Lemma) *Let $m > 0,\ 0 \le r \le m$. Let $f, g \in C^\infty(U \times \mathbb{R}^n \times \Gamma_+)$ such that for $\lambda \in \Gamma_+$*

$$f(\cdot,\cdot,\lambda) \in S_K^r(U \times \mathbb{R}^n), \qquad g(\cdot,\cdot,\lambda) \in S_K^{-m}(U \times \mathbb{R}^n).$$

Assume that

$$\begin{aligned} &|\partial_x^\alpha \partial_\xi^\beta f(x,\xi,\lambda)| \\ &\quad \le \begin{cases} C_{0,0}(f)(1+|\xi|+|\lambda|^{1/m})^r, & \alpha=\beta=0, \\ C_{\alpha,\beta}(f)(1+|\xi|)^{m-|\beta|}(1+|\xi|+|\lambda|^{1/m})^{r-m}, & |\alpha|+|\beta|>0, \end{cases} \end{aligned} \tag{4.1}$$

and

$$|\partial_x^\alpha \partial_\xi^\beta g(x,\xi,\lambda)| \le \widetilde{C}_{\alpha,\beta}(g)(1+|\xi|)^{-|\beta|}(1+|\xi|+|\lambda|^{1/m})^{-m}, \tag{4.2}$$

where $C_{\cdot,\cdot}(\cdot)$, $\widetilde{C}_{\cdot,\cdot}(\cdot)$ are constants depending on certain datas in the dots' positions. Set $C_N(f) = \sum_{|\alpha|,|\beta|\le N} C_{\alpha,\beta}(f)$ and $\widetilde{C}_N(g) = \sum_{|\alpha|,|\beta|\le N} \widetilde{C}_{\alpha,\beta}(g)$. Then for $s \in \mathbb{R}$, there is an $N(s) \in \mathbb{N}$ and $C > 0$ such that

$$\begin{aligned} &\|\mathrm{Op}(g(\cdot,\cdot,\lambda))\mathrm{Op}(f(\cdot,\cdot,\lambda)) - \mathrm{Op}(gf(\cdot,\cdot,\lambda))\|_{s,s+m-r} \\ &\quad \le C C_{N(s)}(f)\widetilde{C}_{N(s)}(g)|\lambda|^{-\min\left(\frac{1}{m},1\right)}. \end{aligned}$$

Remark 4.3 We should notice that $C_N(\cdot)$ and $\widetilde{C}_N(\cdot)$ are semi-norms if we choose the smallest constants $C_{\alpha,\beta}(\cdot)$ and $\widetilde{C}_{\alpha,\beta}(\cdot)$ in (4.1), (4.2). Moreover, $C_N(f)$ and $\widetilde{C}_N(g)$ are dominated by the finitely many constants $C_{\alpha,\beta}(f)$ and $\widetilde{C}_{\alpha,\beta}(g)$, $|\alpha|,\ |\beta| \le N$, respectively.

Now we give some additional examples.

Example 4.4 $g(x,\xi,\lambda) := \psi(\xi)(a_m(x,\xi)-\lambda)^{-1}$ satisfies (4.2). See (3.5).

Example 4.5 $f(x,\xi,\lambda) := a(x,\xi)-\lambda$ satisfies (4.1) with $r=m$. If $b \in CS_K^m(U \times \mathbb{R}^n)$ is a symbol of order m, then $f(x,\xi,\lambda) := \psi(\xi)(a_m(x,\xi)-\lambda)^{-1}b(x,\xi) = r^\psi(x,\xi)b(x,\xi)$ also satisfies (4.1) with $r=0$. Note that in this case

$$\sum_{|\alpha|,|\beta|\le N} C_{\alpha,\beta}(f) \le \left(\sum_{|\alpha|,|\beta|\le N} C_{\alpha,\beta}(r^\psi)\right)\left(\sum_{|\alpha|,|\beta|\le N} C_{\alpha,\beta}(b)\right).$$

Here $C_{\alpha,\beta}(b)$ denotes the best constant in the symbol estimate for $\partial_x^\alpha \partial_\xi^\beta b(x,\xi)$ and $C_{\alpha,\beta}(r^\psi)$ is of similar meaning.

Remark 4.6 Note that in the examples above, $C_{\alpha,\beta}(f)$ and $C_{\alpha,\beta}(g)$ are bounded by a C^k-norm on a_m (and b_m in the preceding example) for sufficiently large k.

4.2 Key estimates

Before proving the main result of this note, we give some more estimates.

Lemma 4.7 *Given $A \in \mathrm{Ell}^m_{\Gamma_+}(M, E)$. Then for $s \in \mathbb{R}$, $0 \le p \le m$, and all $\lambda \in \Gamma_+$ we have*

$$\|\mathrm{Op}(\psi(a_m - \lambda)^{-1})\|_{s,s+p} \le C_s(A)|\lambda|^{-1+\frac{p}{m}}. \tag{4.3}$$

Furthermore, to s there is $N_s \in \mathbb{N}$ such that $C_s(A)$ is bounded by the C^{N_s}-norm of a_m on S^*M.

In other words, to A there is an open neighborhood U of a_m (in the C^{N_s}-topology) such that the map $B \mapsto C_s(B)$ is bounded on the open set $\sigma_m^{-1}(U)$.

Proof Use the standard method of estimating norms of pseudo-differential operators as in Seeley [34, Lemma 2]. Of course it also follows from the method presented in the preceding section. □

Lemma 4.8 *Given $A \in \mathrm{Ell}^m_{\Gamma_+}(M, E)$. Then for $s \in \mathbb{R}$ and all $\lambda \in \Gamma_+$*

$$\|\mathrm{Op}(\psi(a_m - \lambda)^{-1}) - (A - \lambda)^{-1}\|_{s,s+m} \le C_s(A)|\lambda|^{-\min\left(\frac{1}{m},1\right)}. \tag{4.4}$$

$C_s(A)$ has the same property as in Lemma 4.7.

Proof Put $A = \mathrm{Op}(a)$ for the complete symbol a. Write $a = a_m + a_{m-1}$. Then we have

$$\begin{aligned}(A-\lambda)(\mathrm{Op}(\psi(a_m-\lambda)^{-1} - (A-\lambda)^{-1})) &= \mathrm{Op}(a_m-\lambda)\mathrm{Op}(\psi(a_m-\lambda)^{-1}) \\ -\mathrm{Op}(\psi) - \mathrm{Op}(1-\psi) &+ \mathrm{Op}(a_{m-1})\mathrm{Op}(\psi(a_m-\lambda)^{-1}).\end{aligned}$$

Note that $(A - \lambda)(A - \lambda)^{-1} = I = \mathrm{Op}(1)$ and $\mathrm{Op}(1 - \psi)$ is a smoothing operator (because $1 - \psi$ is compactly supported). Hence

$$\begin{aligned}&\|\mathrm{Op}(\psi(a_m-\lambda)^{-1}) - (A-\lambda)^{-1}\|_{s,s+m} \\ &\le \|(A-\lambda)^{-1}\|_{s,s+m}\|\mathrm{Op}(a_m-\lambda)\mathrm{Op}(\psi(a_m-\lambda)^{-1}) - \mathrm{Op}(\psi)\|_{s,s} \\ &\quad + \|(A-\lambda)^{-1}\|_{s+m,s+m}\|\mathrm{Op}(1-\psi)\|_{s,s+m} \\ &\quad + \|(A-\lambda)^{-1}\|_{s,s+m}\|\mathrm{Op}(a_{m-1})\|_{s+m-1,s}\|\mathrm{Op}(\psi(a_m-\lambda)^{-1})\|_{s,s+m-1} \\ &\le C_s(A)|\lambda|^{-\min(\frac{1}{m},1)}\end{aligned}$$

by the Technical Lemma 4.2, applied to $f = a_m - \lambda$, $g = \psi(a_m - \lambda)^{-1}$ and Lemma 4.7. The local boundedness claim on $C_s(A)$ also follows from this lemma. □

5 Applications

As an application of the technical lemma and the preceding estimates, we prove that the sectorial projections depend continuously on the underlying operators in the topology $\mathcal{T}$ to be fixed below. We shall explain in detail how that perturbation result yields the continuous variation of the Calderón projection (and hence of the Cauchy data spaces) of arbitrary linear elliptic differential operators of first order on smooth compact manifolds with boundary under the assumption of the inner unique continuation property.

5.1 The operator type of the sectorial projection

As explained in the Introduction, Wodzicki's Eq. (1.5) yields at once that the sectorial projection is a pseudo-differential operator of order 0, at least for classical pseudo-differential operators.

Our estimates are designed for the perturbation problem and do not give such a sharp result. All we can derive immediately is that the operator $P_{\Gamma_+}(A)$ is bounded $H^s(M;E) \to H^s(M;E)$ for all $s \in \mathbb{R}$. Note that we do not require that A is classical. We only assume that A is principally classical.

As usually, we argue locally. By Proposition 3.1a, $\Phi_0(a_m) \in \mathrm{CL}^{-m}(U,E)$. Furthermore, by Lemma 4.8

$$\|\mathrm{Op}(\psi(a_m-\lambda)^{-1}) - (A-\lambda)^{-1}\|_{s,s+m} \le C_s(A)|\lambda|^{-\min\left(\frac{1}{m},1\right)}$$

for $\lambda \in \Gamma_+$, thus by Definition 2.4 and (3.6)

$$\|P_{\Gamma_+}(A) - A\Phi_0(a_m)\|_{s,s} \le C_s(A)\int_{\Gamma_+} |\lambda|^{-1-\min\left(\frac{1}{m},1\right)}|d\lambda|,$$

and the claim follows.

5.2 Proof of Theorem 1.2

From now on we equip $\mathrm{Ell}^m_{\Gamma_+}(M,E)$ with the topology $\mathcal{T}$. Let $A \in \mathrm{Ell}^m_{\Gamma_+}(M,E)$ and ΔA be in a neighborhood of 0. Since $\Phi(A) = \int_{\Gamma_+} \lambda^{-1}(A-\lambda)^{-1}d\lambda$ and $A \to \Phi_0(A)$ is continuous, it is sufficient to prove an estimate, uniformly on Γ_+, of the form

$$\begin{aligned}\|(A+\Delta A-\lambda)^{-1} - (A-\lambda)^{-1} - \mathrm{Op}(\psi((a_m+\Delta a_m-\lambda)^{-1} - (a_m-\lambda)^{-1}))\|_{s,s+m}\\ \le C_s(A,\Delta A)|\lambda|^{-\min(\frac{1}{m},1)},\end{aligned} \tag{5.1}$$

such that the following holds: given $\epsilon > 0$, there is a neighborhood U of 0 (in the locally convex topology $\mathcal{T}$) such that for all $\Delta A \in U$, $C_s(A,\Delta A) < \epsilon$.

To prove (5.1), we make an elementary algebraic re-ordering of the left side of (5.1) into five summands and invoke the triangle inequality successively:

$$
\begin{aligned}
(5.1),\ \text{left side} &= \|(A-\lambda)^{-1}(-\Delta A)(A+\Delta A-\lambda)^{-1} \\
&\quad - \mathrm{Op}(\psi(a_m-\lambda)^{-1}(-\Delta a_m)(a_m+\Delta a_m-\lambda)^{-1})\|_{s,s+m} \\
&\le \|(A-\lambda)^{-1} - \mathrm{Op}(\chi(a_m-\lambda)^{-1})\|_{s,s+m} \\
&\quad \cdot \|\Delta A(A+\Delta A-\lambda)^{-1}\|_{s,s} \\
&\quad + \|\mathrm{Op}(\chi(a_m-\lambda)^{-1})\|_{s,s+m} \\
&\quad \cdot \|\Delta A - \mathrm{Op}(\chi_1\Delta a_m)\|_{s+m-1,s}\,\|(A+\Delta A-\lambda)^{-1}\|_{s,s+m-1} \\
&\quad + \|\mathrm{Op}(\chi(a_m-\lambda)^{-1})\,\mathrm{Op}(\chi_1\Delta a_m)\|_{s+m,s+m} \\
&\quad \cdot \|A+\Delta A-\lambda)^{-1} - \mathrm{Op}(\chi_2(a_m+\Delta a_m-\lambda)^{-1})\|_{s,s+m} \\
&\quad + \|\{\mathrm{Op}\big(\chi(a_m-\lambda)^{-1}\big)\,\mathrm{Op}(\chi_1\Delta a_m) - \mathrm{Op}(\chi(a_m-\lambda)^{-1}\Delta a_m)\} \\
&\quad \cdot \mathrm{Op}(\chi_2(a_m+\Delta a_m-\lambda)^{-1})\|_{s,s+m} \\
&\quad + \|\mathrm{Op}(\chi(a_m-\lambda)^{-1}\Delta a_m)\,\mathrm{Op}(\chi_2(a_m+\Delta a_m-\lambda)^{-1}) \\
&\quad - \mathrm{Op}(\psi(a_m-\lambda)^{-1}\,\Delta a_m\,(a_m+\Delta a_m-\lambda)^{-1})\|_{s,s+m}.
\end{aligned}
$$

Here we choose χ, χ_1, χ_2 with the same properties like ψ such that $\chi = \chi\chi_1$ and $\chi\chi_2 = \psi$.

Now apply the Technical Lemma 4.2 to the last two summands and the Lemmata 4.7 and 4.8 to the first three summands, and we are done.

Remark 5.1 We note in passing that in the first three lines of the proof of Proposition 3.1 it is decisively used that the integrand is *homogeneous* in λ. This is an obstruction against an immediate generalization of our perturbation result to general holomomorphic functions of A in an H_∞-calculus style (cf., e.g., [7]). To explain this a bit more, let f be a function, which is holomorphic and bounded in a neighborhood of the closure of the sector encircled by Γ_+. This is not quite the situation of the H_∞ calculus, since there are also eigenvalues on the left of the contour. Nevertheless, one might hope that similarly to loc. cit. one can prove that $f(A)$ is a bounded operator. Still there is no immediate analogue of Proposition 3.1 in this case and we leave it as an intriguing open problem whether the Perturbation Theorem 1.2 carries over to $f(A)$ instead of $P_{\Gamma_+}(A)$.

5.3 Index correction formulas

The sectorial projections are significant in the celebrated Atiyah–Patodi–Singer Index Theorem. A common set-up is the following: Let X be a compact smooth Riemannian manifold with boundary M, and E and F be two Hermitian vector bundles over X. Let $D : H^1(X; E) \to L^2(X; F)$ be a first order elliptic differential operator and let $A : H^1(M; E|_M) \to L^2(M; E|_M)$ denote the tangential operator of D on M relative to the fixed metric structures. In the classical works [2–4], Atiyah et al. Singer assumed that D is of Dirac type; all metric structures near M are product; hence, the coefficients of D in normal direction close to M are constant and the tangential operator A is self-adjoint. Imposing a spectral projection condition $P^+(A)u|_{\partial X} = 0$ on the boundary, they proved that the resulting (densely defined) operator $D_{P^+(A)}$ over X is Fredholm. Furthermore, they gave an index formula, comprising topological, spectral and differential terms. The arguments of [4, p. 95] (worked out in [32, Theorem 1.4] and, differently and in detail, in [26, Theorem 7.6]) lead to the index correction formula

$$\operatorname{ind}(D_0)_{P^+(A_0)} - \operatorname{ind}(D_1)_{P^+(A_1)} = \operatorname{sf}\{A_t\}_{t\in[0,1]}, \tag{5.2}$$

where $\{D_t,\ t \in [0,1]\}$ is a smooth homotopy, and $\{A_t\}$ denotes its corresponding family of tangential operators. It is also called the *Spectral Flow Theorem*. The continuous dependence of $P^+(A_t)$ on A_t [in the sense that $P^+(A_t)$ has the same jumps as $1_{(-\varepsilon,\varepsilon)}(A_t)$, if $\pm\varepsilon \notin \operatorname{spec} A_t$] is important in this theorem. When A_t is self-adjoint, it can be proved by standard techniques of functional analysis (cf. [12, Chapter 17] or above Sect. 1.1.1 of this note).

It is natural to consider a more general case. In [33], Savin et al. gave a similar formula for the case that the tangential family A_t is non-self-adjoint. It seems very satisfactory that we now have a proof of the continuous dependence of $P^+(A_t)$ on A_t when A_t has no spectral points on the imaginary axis for all $t \in [0, 1]$.

5.4 Continuous dependence of the Calderón projection on the data

In [11, Sect. 7] we discussed the problem of continuous dependence on the input data of the Calderón projection associated to a first order elliptic differential operator on a compact manifold with boundary. On the one hand this can be viewed as the rather classical problem of showing that the space of solutions of a PDE (here the equation $Du = 0$ in the interior) depends continuously on the data. The question arises naturally in connection with the Spectral Flow Theorem of *symplectic geometry* and has been proved in various special cases (see e.g. [9,14,15,22,25,28]).

Since our main motivation for writing the current note comes from this problem (see the recent [8]), let us briefly describe the set-up and the main result of [11, Sect. 7] as well as the improvement provided by Theorem 1.2.

Let X be a compact connected manifold with boundary M and E, F vector bundles over X. We fix a Riemannian metric and Hermitian metrics on the vector bundles to have Hilbert space structures on the sections of E, F. We choose the metrics in such a way that all structures are product in a collar neighborhood $U = [0, \varepsilon) \times M$ of the boundary. We emphasize that this is not a loss of generality since we will consider *variable* coefficient differential operators, see the detailed discussion in [11, Sect. 2.1].

For a first order elliptic differential operator $D \in \operatorname{Diff}^1(X; E, F)$ we write (cf. [11, (2.18), (2.19), (5.11)–(5.16)]) in the collar U:

$$
\begin{aligned}
D &= J_x\left(\frac{d}{dx}+B_x\right)\\
&=: J_0\left(\frac{d}{dx}+B_0\right)+C_1x-J_0'\\
D^t &= \left(-\frac{d}{dx}+B_0^t\right)J_0^t+\widetilde{C}_1x. \qquad (5.3)
\end{aligned}
$$

where $J_x \in \mathrm{Hom}(E_M, F_M), 0 \le x \le \varepsilon$, is a smooth family of bundle homomorphisms and $(B_x)_{0\le x\le\varepsilon}$ is a smooth family of first order elliptic differential operators between sections of E_M. $C_1, \widetilde{C}_1$ are first order differential operators and, by slight abuse of notation, x will also denote the operator of multiplication by the function $x \mapsto x$. We note that x is intentionally on the right of $C_1, \widetilde{C}_1$. D^t denotes the formal adjoint of D with respect to the L^2-structure. We consider $J_x, B_x, C_1, \widetilde{C}_1$ as functions of D.

Next we denote by $\mathcal{E}(X; E, F)$ the set of pairs $(D, T) \in \mathrm{Diff}^1(X; E, F) \times \mathrm{Diff}^0(M; E_M, F_M)$ such that

(1) D is elliptic
(2) $J_0^t T$ is positive definite (in particular self-adjoint) and the commutator $[J_0^t T, B_0]$ is a differential operator of order 0.

Recall from [11, Sect. 4] the invertible double construction associated to a pair $(D, T) \in \mathcal{E}(X; E, F)$: Put

$$
\widetilde{D} := D \oplus (-D^t), \qquad (5.4)
$$

acting on sections of $E \oplus F$, and impose the boundary condition

$$
\begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in \mathrm{dom}(\widetilde{D}_T) :\Leftrightarrow f_-|_{\partial X} = T f_+|_{\partial X} \Leftrightarrow (f_+|_{\partial X}, f_-|_{\partial X}) \in \ker(\,-T\ \mathrm{Id}\,). \qquad (5.5)
$$

Our [11, Theorem 4.7] states that $\widetilde{D}_T$ is a realization of a local elliptic boundary value problem (in the classical Šapiro–Lopatinskiĭ sense) and that the kernel and cokernel of $\widetilde{D}_T$ are isomorphic to the direct sum of the spaces of ghost solutions $Z_0(D) = \{u \in L^2(X, E) \mid Du = 0, u|_{\partial X} = 0\}$ and $Z_0(D^t)$ for D and D^t. In particular if D and D^t satisfy the weak inner UCP (i.e. $Z_0(D) = 0 = Z_0(D^t)$) then $\widetilde{D}_T$ is indeed invertible. This *canonical* invertible double construction is the natural generalization of the geometric invertible double construction for Dirac type operators in the product situation (cf. e.g. [12]) to general first order elliptic differential operators.

Furthermore, in [11, Sect. 5] we showed that from $\widetilde{D}_T$ one obtains a projection (the Calderón projection) onto the Cauchy data space $N^0(D) := \{u|_{\partial X} \in L^2(\partial X; E|_{\partial X}) \mid Du = 0\}$ of D by the formula [11, (5.31)]

$$
C_+(D, T) = (P_+ - \varrho_+ \widetilde{G} S(D, T))(P_+ + P_-^*)^{-1}. \qquad (5.6)
$$

Note that the range of $C_+(D,T)$ equals $N^0(D)$ and is independent of T. However, $C_+(D,T)$ is in general not an orthogonal projection and may depend on T. $C_+(D,J_0^t)$ is indeed the orthogonal projection onto $N^0(D)$.

Now denote by $\mathcal{E}_{\mathrm{UCP}}(X;E,F)$ the set of $(D,T)\in\mathcal{E}(X;E,F)$ such that D and D^t satisfy weak inner UCP.

Let $\mathcal{V}(X;E,F)$ be the linear subspace of $\mathrm{Diff}^1(X;E,F)\times\mathrm{Diff}^0(M;E_M,F_M)$ consisting of those (D,T) such that $[B_0^t, J_0^t T]$ is of order 0; and introduce the following two norms on $\mathcal{V}(X;E,F)$:

$$N_0(D,T) := \|D\|_{1,0} + \|D^t\|_{1,0} + \|T\|_{1/2,1/2}, \tag{5.7}$$

and

$$\begin{aligned} N_1(D,T) := {}& \|B_0\|_{1,0} + \|B_0^t\|_{1,0} + \|[B_0^t, J_0^t T]\|_0 + \|T\|_0 \\ &+ \|J_0\|_0 + \|C_1\|_{1,0} + \|\widetilde{C}_1\|_{1,0}. \end{aligned} \tag{5.8}$$

Compared to [11, (7.1), (7.2)] we have omitted a few redundant terms. We obtain a metric on $\mathcal{V}(X;E,F)$ and hence on its subsets by putting

$$d_{\mathrm{str}}((D,T),(D',T')) := N_0(D-D',T-T') + N_1(D-D',T-T'). \tag{5.9}$$

Finally, let Γ be a contour as in (1.3) and let $\mathcal{E}_{\mathrm{UCP},\Gamma}(X;E,F)$ be the set of those $(D,T)\in\mathcal{E}_{\mathrm{UCP}}(X;E,F)$ such that the leading symbol of B_0 has no eigenvalues on the two rays L_{α_j} of Γ and B_0 no eigenvalues on Γ.

[11, Theorem 7.2 (b)] can now be phrased as follows:

Theorem 5.2 *Let $s\in[-1/2,1/2]$ and let $\mathcal{T}_\Gamma$ be the coarsest topology on $\mathcal{E}_{\mathrm{UCP},\Gamma}(X;E,F)$ such that*

(1) *d_{str} is continuous on $\mathcal{E}_{\mathrm{UCP},\Gamma}(X;E,F)\times\mathcal{E}_{\mathrm{UCP},\Gamma}(X;E,F)$,*
(2) *$(D,T)\mapsto P_\Gamma(B_0)\in\mathcal{B}(H^s(M;E_M))$ is continuous.*

Then the map $\mathcal{E}_{\mathrm{UCP},\Gamma}(X;E,F)\ni(D,T)\mapsto C_+(D,T)\in\mathcal{B}(H^s(M;E_M))$ is continuous.

As pointed out in [11, Remark 7.3] the obvious weakness of this result is that the continuous dependence of $P_\Gamma(B_0)$ has to be assumed.

Combining Theorem 5.2 with Theorem 1.2 we obtain a much more satisfactory formulation of the continuous dependence of $C_+(D,T)$ without reference to a positive sectorial projection:

Theorem 5.3 *Let $s\in[-1/2,1/2]$ and let $\mathcal{T}$ be the coarsest topology on $\mathcal{E}_{\mathrm{UCP}}(X;E,F)$ such that*

(1) *d_{str} is continuous on $\mathcal{E}_{\mathrm{UCP}}(X;E,F)\times\mathcal{E}_{\mathrm{UCP}}(X;E,F)$,*
(2) *The leading symbol map for the tangential operator $\sigma:\mathcal{E}_{\mathrm{UCP}}(X;E,F)\mapsto\Gamma^\infty(S^*M,\pi^*(\mathrm{End}E_M))$, $(D,T)\mapsto\sigma_1(B_0)$ ($\pi:S^*M\to M$ the projection map) is continuous when $\Gamma^\infty(S^*M,\pi^*(\mathrm{End}(E_M))$ is equipped with the C^∞–topology.*

Then $C_+ : (\mathcal{E}_{\mathrm{UCP}}(X; E, F), \mathcal{T}) \longrightarrow \mathcal{B}(H^s(M, E_M))$ *is continuous.*

Remark 5.4 1. The restrictions $s \in [-1/2, 1/2]$ in Theorems 5.2 and 5.3 are not serious. For other values of s the metric d_{str} has to be modified in a fairly straightforward way.
2. Let $\mathcal{T}^\infty$ be the coarsest topology on $\mathcal{E}_{\mathrm{UCP}}(X; E, F)$ such that in each chart the coefficient functions of a coordinate representation of $D \in \mathcal{E}_{\mathrm{UCP}}(X; E, F)$ vary continuously in the C^∞-topology (cf. e.g. [22, Theorem 3.16]).

Then it is a routine matter to check that $\mathcal{T}^\infty$ is finer than the topology $\mathcal{T}$ of Theorem 5.3. In fact it is fine enough to guarantee the continuity $C_+ : (\mathcal{E}_{\mathrm{UCP}}(X; E, F), \mathcal{T}^\infty) \longrightarrow \mathcal{B}(H^s(M, E_M))$ for *all* real s (cf. item 1. of this Remark).

This version of the continuous dependence of C_+, although strictly speaking somewhat weaker than Theorem 5.3, is probably the most satisfactory way of summarizing its content.

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Appendix A

In this appendix, we provide the details of the proof of our Technical Lemma 4.2. We first translate the wanted estimates into statements about integral operators.

A.1. L^2-estimates for integral operators and other estimates

We recall the well-known and very useful Schur's Test for integral operators (see, e.g., Halmos and Sunder [21, Theorem 5.2]):

Lemma A.1 (Schur's Test) *Let* K *be an integral operator with measurable kernel* $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$. *Assume that*

$$\sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |k(x, y)| dy \le C_1 < +\infty \text{ and } \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} |k(x, y)| dx \le C_2 < +\infty.$$

Then K *is bounded* $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *and* $\|K\|_{L^2\to L^2} \le \sqrt{C_1 C_2}$.

In particular, if for some $p > n$

$$|k(x, y)| \le C_3(1 + |x - y|)^{-p},$$

then the criterion is fulfilled with

$$C_1 = C_2 = C_3 \int_{\mathbb{R}^n} (1 + |\xi|)^{-p} d\xi.$$

Now fix $U \subset \mathbb{R}^n$ open, $K \subset U$ compact and $a \in \mathrm{S}^m_K(U \times \mathbb{R}^n)$. Then Schur's test yields an effective estimate for $\|\operatorname{Op}(a)\|_{s,s-m}$.

To explain that, we introduce some notations. For the Fourier transform, we shall follow Hörmander's convention

$$(\mathcal{F}f)(\xi) := \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)dx, \quad (\mathcal{F}^{-1}u)(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} u(\xi)d\xi$$
$$đ\xi := (2\pi)^{-n} d\xi.$$

Then we have

$$\begin{aligned}(\mathcal{F}\operatorname{Op}(a)u)(\eta) &:= \int_{\mathbb{R}^n} e^{-i\langle x,\eta\rangle} \big(\operatorname{Op}(a)u\big)(x)dx \\ &= \int_{\mathbb{R}^n} \left[\int_U e^{i\langle x,\xi-\eta\rangle} a(x,\xi) đx \right] \widehat{u}(\xi) d\xi.\end{aligned}$$

We set $q_a(\xi-\eta,\xi) := [\cdots]$ in the preceding formula and define

Definition A.2 For $a \in \mathrm{S}^m_K(U \times \mathbb{R}^n)$, we set

$$q_a(\zeta,\xi) := (\mathcal{F}^{-1}_{x\to\zeta} a(x,\xi))(\zeta) = \int_K e^{i\langle\zeta,x\rangle} a(x,\xi) đx.$$

Since $a(x,\xi)$ is nonzero at most if $x \in K$ the integral certainly exists. The method we are going to employ is adapted from [18, Lemma 1.2.1]. Lemma 1.2.1 (b) of loc. cit. shows that $q_a(\zeta,\xi)$ decays to arbitrarily high powers (reproved below) in ζ (as $\zeta \to \infty$) and is polynomially bounded in ξ. Hence all integrals below converge in the usual sense.

Consequently, the kernel of the integral operator $\mathcal{F}\operatorname{Op}(a)\mathcal{F}^{-1}$ is given by $k_a(\tau,\xi) := q_a(\xi-\tau,\xi)$. To estimate the operator norm $\|\cdot\|_{s,s-m}$ of $\operatorname{Op}(a)$ it suffices therefore to estimate the norm of the operator $\mathcal{F}\operatorname{Op}(a)\mathcal{F}^{-1}$ as a map from the weighted L^2-space $L^2(\mathbb{R}^n, (1+\|\xi\|^2)^s)$ into $L^2(\mathbb{R}^n, (1+\|\xi\|^2)^{s-m})$. By Schur's test an estimate of the form

$$|(1+|\tau|)^{s-m} k_a(\tau,\xi)(1+|\xi|)^{-s}| \le C(a)C(p)(1+|\tau-\xi|)^{-p} \quad \text{for some } p > n \tag{A.1}$$

implies

$$\|\operatorname{Op}(a)\|_{s,s-m} \le C(a)\widetilde{C}(p) \quad \text{with } \widetilde{C}(p) := C(p) \int_{\mathbb{R}^n} (1+|x|)^{-p} dx.$$

Proof of the Main Technical Lemma. Let $f(\cdot,\cdot,\lambda) \in \mathrm{S}^r_K(U\times\mathbb{R}^n)$, $g(\cdot,\cdot,\lambda) \in \mathrm{S}^m_K(U\times\mathbb{R}^n)$ satisfying (4.1), (4.2) be given. In the sequel we will suppress the argument λ from the notation for simplicity. We should be aware that all expressions will depend on λ unless otherwise stated. The kernel of the operator $\mathcal{F}\,\mathrm{Op}(f)\,\mathrm{Op}(g)\mathcal{F}^{-1}$ is given by

$$k_{f\cdot g}(\tau,\xi) = \int_{\mathbb{R}^n} k_f(\tau,\eta)k_g(\eta,\xi)d\eta = \int_{\mathbb{R}^n} q_f(\eta-\tau,\eta)q_g(\xi-\eta,\xi)d\eta.$$

On the other hand

$$\begin{aligned} q_{f\cdot g}(\zeta,\xi) &= \int_{\mathbb{R}^n} e^{i\langle\zeta,x\rangle} f(x,\xi)g(x,\xi)đx \\ &= \mathcal{F}^{-1}\big(f(\cdot,\xi)g(\cdot,\xi)\big)(\zeta) \\ &= \int_{\mathbb{R}^n} q_f(\zeta-\eta,\xi)q_g(\eta,\xi)d\eta, \end{aligned}$$

respectively,

$$\begin{aligned} k_{f\cdot g}(\tau,\xi) &= q_{f\cdot g}(\xi-\tau,\xi) \\ &= \int_{\mathbb{R}^n} q_f(\xi-\tau-\eta,\xi)q_g(\eta,\xi)d\eta; \quad \xi-\eta \rightsquigarrow \eta \\ &= \int_{\mathbb{R}^n} q_f(\eta-\tau,\xi)q_g(\xi-\eta,\xi)d\eta. \end{aligned}$$

Thus the kernel of $\mathcal{F}\{\mathrm{Op}(f)\,\mathrm{Op}(g) - \mathrm{Op}(f\cdot g)\}\mathcal{F}^{-1}$ is given by

$$k(\tau,\xi,\lambda) := \int_{\mathbb{R}^n} \{q_f(\eta-\tau,\eta) - q_f(\eta-\tau,\xi)\}q_g(\xi-\eta,\xi)d\eta. \tag{A.2}$$

We are now going to estimate this kernel. The estimate of q_g is standard: for any multiindex α, $\zeta\in\mathbb{R}^n$ we have (for $D^\alpha_x := -i\partial^{\alpha_1+\cdots\alpha_n}/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$, as usual):

$$|\zeta^\alpha q_g(\zeta,\xi,\lambda)| = \left|\int_K e^{i\langle\zeta,x\rangle} D^\alpha_x g(x,\xi,\lambda)đx\right| \le \mathrm{vol}(K)C_\alpha(g)(1+|\xi|+|\lambda|^{1/m})^{-m}.$$

Since α is arbitrary, we see that for any $N\in\mathbb{N}$

$$|q_g(\zeta,\xi,\lambda)| \le \widetilde{C}_N(g)(1+|\zeta|)^{-N}(1+|\xi|+|\lambda|^{1/m})^{-m}. \tag{A.3}$$

Next we discuss the difference $q_f(\zeta,\eta,\lambda) - q_f(\zeta,\xi,\lambda)$. Again for a multiindex α we have

$$
\begin{aligned}
|\zeta^\alpha(q_f(\zeta,\eta,\lambda) - q_f(\zeta,\xi,\lambda))| &= \left|\int_K e^{i\langle\zeta,x\rangle}\{D_x^\alpha(f(x,\eta,\lambda) - f(x,\xi,\lambda))\} đx\right| \\
&\le \int_K \sup_{t\in[0,1],\,|\beta|=1} |D_x^\alpha \partial_\xi^\beta f(x,\xi+t(\eta-\xi),\lambda)| đx\, |\xi-\eta| \\
&\le C \sup_{t\in[0,1]} (1+|\xi+t(\eta-\xi)|)^{m-1} \\
&\quad \times (1+|\xi+t(\eta-\xi)|+|\lambda|^{\frac{1}{m}})^{r-m}\, |\xi-\eta|
\end{aligned}
$$

with $C := \mathrm{vol}(K) C_N(f)$ and $N := \max(|\alpha|, 1)$, that is,

$$
\begin{aligned}
&|q_f(\zeta,\eta,\lambda) - q_f(\zeta,\xi,\lambda)| \le \mathrm{vol}(K) C_N(f)(1+|\zeta|)^{-N}|\xi-\eta| \cdot \\
&\quad \sup_{t\in[0,1]} (1+|\xi+t(\eta-\xi)|)^{m-1}(1+|\xi+t(\eta-\xi)|+|\lambda|^{\frac{1}{m}})^{r-m}. \qquad \text{(A.4)}
\end{aligned}
$$

To estimate the norm of $\mathrm{Op}(f)\,\mathrm{Op}(g) - \mathrm{Op}(f\cdot g)$ as an operator from H^s to H^{s+m-r} we need to estimate the norm of the integral operator in $L^2(\mathbb{R}^n)$ whose kernel is given by [see (A.1)]

$$
\widetilde{k}(\tau,\xi,\lambda) = (1+|\tau|)^{s+m-r} k(\tau,\xi,\lambda)(1+|\xi|)^{-s},
$$

where $k(\tau,\xi,\lambda)$ is defined in (A.2). From (A.2), (A.3) and (A.4) we infer

$$
\begin{aligned}
|\widetilde{k}(\tau,\xi,\lambda)| \le C_N(f)\widetilde{C}_N(g) &\int (1+|\eta-\tau|)^{-N}|\xi-\eta|\,(1+|\xi-\eta|)^{-N} \\
&\times (1+|\xi|+|\lambda|^{\frac{1}{m}})^{-m}(1+|\tau|)^{s+m-r} \\
&\times (1+|\xi|)^{-s} \sup_{t\in[0,1]} (1+|\xi+t(\eta-\xi)|)^{m-1} \\
&\times (1+|\xi+t(\eta-\xi)|+|\lambda|^{\frac{1}{m}})^{r-m} đ\eta. \qquad \text{(A.5)}
\end{aligned}
$$

Note that we may choose N as large as we please. We now distinguish two cases. *Case I*: $|\eta-\xi| \le \frac{1}{2}|\xi|$. Then for $0 \le t \le 1$, $\frac{1}{2}|\xi| \le |\xi+t(\eta-\xi)| \le \frac{3}{2}|\xi|$, and thus the integrand of the right hand side of (A.5) can be estimated (absorbing another constant into $C_N(f)\widetilde{C}_N(g)$) by

$$
\begin{aligned}
&\le C_N(f)\widetilde{C}_N(g)(1+|\eta-\tau|)^{-N}(1+|\xi-\eta|)^{1-N}(1+|\tau|)^{s+m-r} \\
&\quad \times (1+|\xi|)^{-s+m-1}(1+|\xi|+|\lambda|^{\frac{1}{m}})^{r-2m}. \qquad \text{(A.6)}
\end{aligned}
$$

Using Peetre's Inequality (we suppress the constant), we have

$$
(1+|\tau|)^{s+m-r}(1+|\xi|)^{-s+m-1} \le (1+|\tau-\xi|)^{|s+m-r|}(1+|\xi|)^{2m-r-1}.
$$

Then (A.6)

$$\leq C_N(f)\widetilde{C}_N(g)(1+|\eta-\tau|)^{-N}(1+|\xi-\eta|)^{1-N}(1+|\tau-\xi|)^{|s+m-r|} \\ \times(1+|\xi|)^{2m-r-1}(1+|\xi|+|\lambda|^{\frac{1}{m}})^{r-2m}. \tag{A.7}$$

For $0 \leq r \leq m$,

$$(1+|\xi|)^{2m-r-1}(1+|\xi|+|\lambda|^{\frac{1}{m}})^{r-2m} \leq \begin{cases} \left(1+|\lambda|^{\frac{1}{m}}\right)^{-1}, & 2m-r-1 \leq 0, \\ \left(1+|\lambda|^{\frac{1}{m}}\right)^{-m}, & 2m-r-1 > 0. \end{cases}$$

Thus (A.7)

$$\leq C_N(f)\widetilde{C}_N(g)(1+|\eta-\tau|)^{-N}(1+|\xi-\eta|)^{1-N}(1+|\tau-\xi|)^{|s+m-r|} \\ \times(1+|\lambda|)^{-\min\left(\frac{1}{m},1\right)}. \tag{A.8}$$

Again Peetre's Inequality (once again suppressing the constant) gives that for $N > n+1$,

$$\int_{\mathbb{R}^n}(1+|\eta-\tau|)^{-N}(1+|\xi-\eta|)^{1-N}\bar{d}\eta \\ \leq \int_{\mathbb{R}^n}(1+|\eta|)^{-N}(1+|\xi-\eta-\tau|)^{1-N+n}\bar{d}\eta \\ \leq \int_{\mathbb{R}^n}(1+|\eta|)^{-n-1}(1+|\xi-\tau|)^{1-N+n}\bar{d}\eta.$$

Taking this into account and integrating the right side of (A.8) over η yields

$$\int_{|\eta-\xi|\leq\frac{1}{2}|\xi|}\cdots\bar{d}\eta \leq C_N(f)\widetilde{C}_N(g)\int(1+|\eta|)^{-n-1}\bar{d}\eta \\ \times(1+|\xi-\tau|)^{1+n+|s+m-r|-N}(1+|\lambda|)^{-\min\left(\frac{1}{m},1\right)}. \tag{A.9}$$

Here we choose N large enough such that $N > n+1+|s+m-r|$.
Case II: $|\eta-\xi| > \frac{1}{2}|\xi|$. Then the integrand of the right hand side of (A.5) is estimated by

$$\leq C_N(f)\widetilde{C}_N(g)(1+|\eta-\tau|)^{-N}(1+|\xi-\eta|)^{m-N}(1+|\tau|)^{s+m-r} \\ \times(1+|\xi|)^{-s+m-1}(1+|\lambda|^{\frac{1}{m}})^{r-2m}.$$

Since $\frac{1}{2}|\xi| < |\eta - \xi|$, we estimate

$$(1+|\xi|)^{-s+m-1} \leq \begin{cases} 1, & -s+m-1 \leq 0, \\ C_{s,m}(1+|\xi-\eta|)^{-s+m-1}, & -s+m-1 > 0. \end{cases}$$

Now we proceed as in Case I.

In sum we have proved that for N large enough,

$$|\widetilde{k}(\tau,\xi,\lambda)| \leq C_N(f)\widetilde{C}_N(g)(1+|\xi-\tau|)^{-n-1}(1+|\lambda|)^{-\min\left(\frac{1}{m},1\right)}.$$

The lemma follows from Schur's test finally. □

Appendix B

We shall explain a topological obstruction which excludes repeating Seeley's construction literally and which was overlooked by various authors (see, for example, [27,31,38]).

Given the two rays of minimal growth L_{α_j}, $j = 1, 2$ with $\operatorname{spec} a_m(x,\xi) \cap L_{\alpha_j} = \emptyset$ for $x \in M, \xi \in T_x^*M, \xi \neq 0, j = 1, 2$, we are guaranteed a symbol "ingredient" $(a_m(x,\xi) - \lambda)^{-1}$ of order $-m$ for each $\lambda \in L_{\alpha_1} \cup L_{\alpha_2}$ and for $\xi \neq 0$. Moreover, we can find a small arc of radius R connecting the two rays such that the resulting curve Γ_+ belongs to the resolvent set of A, as explained above.

B.1. The problem

It might be tempting to look for a smooth deformation and extension $\widetilde{a}$ of $a(x,\xi)$ to $\xi = 0$ in such a way that for *all* $(x,\xi) \in T^*M$ one has

$$\operatorname{spec} \widetilde{a}(x,\xi) \cap \Gamma_+ = \emptyset.$$

Actually, we may choose $R > 0$ such that $\operatorname{spec} a(x,\xi) \cap \Gamma_+ = \emptyset$ for, say, $|\xi| = 1$. Then the problem arises whether such map

$$a(x,\cdot) : S^{n-1} \to \mathcal{M}(N,\Gamma_+), \tag{B.1}$$

$$\mathcal{M}(N,V) := \{a \in \mathcal{M}(N) \mid \operatorname{spec} a \cap V = \emptyset\}, V \subset \mathbb{C} \tag{B.2}$$

can be extended over the whole n-dimensional ball to a map $\widetilde{a} : B^n \to \mathcal{M}(N,\Gamma_+)$ in a continuous way. In the preceding, $x \in M$ is fixed, $\dim M = n$, the fibre dimension of the Hermitian bundle is $\dim E_x = N$, $\mathcal{M}(N)$ denotes the space of $N \times N$ matrices with complex entries, and the matrix spaces inherit the topology of $\mathbb{C}^{N^2}$. We assume that we are given a trivialization of the cotangent bundle $T_x^*M = \mathbb{R}^n$ and of the fibre $E_x = \mathbb{C}^N$.

B.2. A one-dimensional counter example

The most simple one-dimensional example $A := -i\frac{d}{d\theta}$ on $M = S^1, N = 1$ refutes that naive hope. A is the tangential operator for the Cauchy–Riemann operator on the 2-ball $\{|z| \leq 1\}$. We have $a(\theta, \xi) = \xi$ with $\operatorname{spec} a(\theta, \xi) = \{\xi\}$, $\operatorname{spec} A = \mathbb{Z}$, and the imaginary line $i\mathbb{R} = L_{\pi/2} \cup L_{3\pi/2}$ as spectral cut for $a(\theta, \xi), \xi \neq 0$. Clearly, we cannot get anything useful, if we multiply a just by a cut-off function leading to

$$\widetilde{a}(\theta, \xi) = \begin{cases} \xi & \text{for } |\xi| \geq 1, \\ 0 & \text{for } |\xi| \leq \varepsilon. \end{cases}$$

By the Intermediate Value Theorem, for each $R \in (0, 1)$ there will always be a $\hat{\xi} \in (\varepsilon, 1)$ such that $\widetilde{a}(\theta, \hat{\xi}) = R$. However, if we exempt only one ray, say $L_{\pi/2}$ instead of the whole imaginary line, we *can* deform the given $a(\cdot, \cdot) : S^1 \times (\mathbb{R} \setminus (-1, 1)) \to \mathcal{M}(1, L_{\pi/2})$ into

$$\begin{aligned} \widetilde{a}(\cdot, \cdot) : S^1 \times \mathbb{R} &\longrightarrow \mathcal{M}(1, L_{\pi/2}), \\ (\theta, \xi) &\mapsto \begin{cases} \xi, & \text{for } |\xi| \geq 1, \\ e^{-i(1-\xi)\frac{\pi}{2}}, & \text{for } 0 \leq |\xi| < 1. \end{cases} \end{aligned} \tag{B.3}$$

Here the point is that we only require that $\widetilde{a}(\theta, \xi)$ has no purely non-negative eigenvalues. What we did was a spectral deformation of the original matrices (here complex numbers) into the point $\{-i\}$. Clearly, that deformation breaks down, if we have two rays of minimal growth forming a separating curve in $\mathbb{C}$: There is no continuous path connecting $\{1\}$ and $\{-1\}$ that is not crossing the imaginary line. The topological obstruction for $n = 1$ is simply that the space $\mathcal{M}(1, i\mathbb{R})$ has two connected components, $(-\infty, 0), (0, \infty)$ and that $a(\theta, 1), a(\theta, -1)$ belong to different components.

B.3. The essence of the topological obstruction

Let us muse upon the cases $n, N > 1$. Shortly, the essence of the topological difficulties overlooked by our predecessors is the following: Without loss of generality, let Γ_+ be the imaginary line $i\mathbb{R}$. Fix a *non-trivial* smooth complex vector bundle G on the sphere S^{n-1} (or on the sphere cotangent bundle S^*M over the n-dimensional manifold M–for simplicity, however, we shall ignore the spatial variables). Next, we embed G into a trivial bundle $S^{n-1} \times \mathbb{C}^k$ for k sufficiently large. Let $\{P_\xi\}_{\xi \in S^{n-1}}$ denote the smooth family of self-adjoint projections of $\mathbb{C}^k$ onto the fibers $G_\xi, \xi \in S^{n-1}$.

Set $a(\xi) := 2P_\xi - I : \mathbb{C}^k \to \mathbb{C}^k$ and extend it, say by homogeneity 1 to $\mathbb{R}^n$ and smooth it out in 0. Then this is an elliptic symbol with the two imaginary half-axes being rays of minimal growth. More precisely, we have $\operatorname{spec} a(\xi) = \{-1, 1\}, \xi \in S^{n-1}$, and $E_{1,\xi} = G_\xi$ and $E_{-1,\xi} = G_\xi^\perp$, where $E_{\lambda,\xi}$ denotes the linear span of the eigenvectors of $a(\xi)$ for $\lambda \in \operatorname{spec} a(\xi)$.

Then it is impossible to find a $k \times k$ matrix valued function $\widetilde{a}$ on the whole $\mathbb{R}^n$ which coincides with a outside a large ball such that $\operatorname{spec} \widetilde{a}(\xi) \cap \Gamma_+ = \emptyset$ for all $\xi \in \mathbb{R}^n$: Let us assume we could. Let $\widetilde{E}_{\Lambda_+,\xi} = \operatorname{im} \widetilde{P}_+(\xi)$ denote the linear span of all root vectors of $\widetilde{a}(\xi)$ for eigenvalues in the positive half plane $\Lambda_+ \subset \mathbb{C}$. The family of vector subspaces of $\mathbb{C}^k$ is continuous and forms a vector bundle over the unit ball B^n. It is trivial because the base space is contractible, but its restriction on the $n-1$ sphere is G which is by assumption non-trivial. That is a contradiction. So, we have a necessary condition for the construction to work.

Since Seeley only dealt with one ray of minimal growth, this problem did not occur there.

Therefore, we cannot expect to be able to make the wanted extension, respectively deformation in general. Instead of the direct (and futile) search for a suitable modification of the principal symbol to get a well-defined resolvent for A along the spectral cut Γ_+ we shall apply the symbolic calculus solely to obtain a parametrix for $A - \lambda$.

B.4. The topology of the underlying space of hyperbolic matrices

As a service to the reader we determine the precise homotopy type of the matrix space $\mathcal{M}(N, \Gamma_+)$. By deformation, we may assume that the imaginary line is the given spectral cut for all matrices $a(x, \xi)$ for $\xi \neq 0$. In $\mathbb{C} \setminus \Gamma_+$, we denote the two complementary sectors by $\Lambda_\pm$. Then the space $\mathcal{M}(N, i\mathbb{R})$ of $N \times N$ matrices with no purely imaginary (generalized) eigenvalues decomposes into $N + 1$ connected components

$$\mathcal{M}_k(N, i\mathbb{R}) := \{a \in \mathcal{M}(N, i\mathbb{R}) \mid \dim \operatorname{im} P^+(a) = k\}, \quad k = 0, 1, \dots N, \tag{B.4}$$

where

$$\begin{array}{rcl} P^+ : \mathcal{M}(N, i\mathbb{R}) =: \mathcal{E} & \longrightarrow & \mathcal{P}(N) \\ a & \longmapsto & -\frac{1}{2\pi i} \displaystyle\int_{\Gamma_+} (a - \lambda I)^{-1} \, d\lambda. \end{array} \tag{B.5}$$

Here $\mathcal{P}(N) = \cup_{k=0}^N \mathcal{P}_k(N)$ denotes the space of projections (idempotent $N \times N$ matrices, fibred according to the dimension of their ranges) and $P^+(a)$ denotes the projection onto the generalized eigenspaces of a for generalized eigenvalues in the positive sector Λ_+.

For $k = 0$ and $k = N$, the spaces $\mathcal{M}_k(N, i\mathbb{R})$ are homeomorphic to the full space $\mathcal{M}(N)$ of all square matrices and hence contractible. That explains why Seeley's deformation is always possible for one ray of minimal growth, dividing $\mathbb{C}$ into one sector without spectrum and one sector with all the eigenvalues, see once again Fig. 1b.

To investigate the homotopy type of $\mathcal{M}_k(N, i\mathbb{R})$ for $k = 1, \dots N - 1$, we restrict the map (B.5) to a single component $\mathcal{M}_k(N, i\mathbb{R})$. We obtain a fibration of the total space $\mathcal{M}_k(N, i\mathbb{R})$ as a fibre bundle over the base $\mathcal{P}$ with contractible fibre

$$(P^+)^{-1}\{P_0\} = \{a \in \mathcal{M}(\operatorname{im} P_0) \mid \operatorname{spec} a \subset \Lambda_+\} \times \{a \in \mathcal{M}(\ker P_0) \mid \operatorname{spec} a \subset \Lambda_-\}$$

for any $P_0 \in \mathcal{P}_k(N)$. Hence, the topological spaces, the base $\mathcal{P}_k(N)$ and the total space $\mathcal{M}_k(N, i\mathbb{R})$ have the same homotopy type. By orthogonalization, it suffices to consider a projection space made of orthogonal projections which easily can be identified with the subspaces of $\mathbb{C}^N$ of dimension k. So we arrive at the complex Grassmannian $\mathrm{Gr}_{\mathbb{C}}(N, k)$, which is known for non-trivial homotopy, if $0 < k < N$.

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