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Differential Geometry

Superconnection and family Bergman kernels

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Abstract

We establish an asymptotic expansion for families of Bergman kernels. The key idea is to use the superconnection formalism as in the local family index theorem. ***To cite this article: X. Ma, W. Zhang, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Superconnexion et noyaux de Bergman en famille. Nous annonçons des résultats sur le développement asymptotique du noyau de Bergman en famille. L'idée principale est d'utiliser le formalisme des superconnexions comme dans la preuve du théorème de l'indice local en famille. ***Pour citer cet article : X. Ma, W. Zhang, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Soit $\pi : W \to S$ une submersion holomorphe de variétés compactes de fibre X, et soit L un fibré en droites holomorphes sur W qui est positif le long de la fibre X. Soit E un fibré vectoriel holomorphe sur W. Pour p assez grand, la première classe de Chern du fibré vectoriel holomorphe $H^0(X, L^p \otimes E)$ est calculé par le théorème de Grothendieck–Riemann–Roch (G.R.R.).

En considérant la courbure de $H^0(X, L^p \otimes E)$ comme un opérateur agissant le long de la fibre, nous étudions dans cette Note le développement asymptotique de son noyau quand la puissance p tend vers $+\infty$. Nos résultats raffinent le développement asymptotique de la première classe de Chern donnée par le théorème de G.R.R. au niveau des formes différentielles comme dans la version locale du théorème de l'indice en famille. L'idée principale est d'utiliser « un morceau » de la superconnexion introduite par Bismut dans la preuve du théorème de l'indice local en famille.

Les résultats annoncés dans cette Note sont démontrés dans [12].

1. Introduction

Let W, S be smooth compact complex manifolds. Let $\pi : W \to S$ be a holomorphic submersion with compact fiber X and $\dim_{\mathbb{C}} X = n$. Let E be a holomorphic vector bundle on W. Let L be a holomorphic line bundle on W.

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We suppose that L is positive along the fiber X.

We will add a subscript $\mathbb{R}$ for the corresponding real objects. Thus TX is the holomorphic relative tangent bundle of π, and $T_{\mathbb{R}}X$ is the corresponding real vector bundle. Let $J^{T_{\mathbb{R}}X}$ be the complex structure on $T_{\mathbb{R}}X$.

By the Kodaira vanishing theorem, there exists $p_0 \in \mathbb{N}$ such that the higher fiberwise cohomologies vanish and that $H^0(X, (L^p \otimes E)|_X)$ forms a vector bundle, denoted by $H^0(X, L^p \otimes E)$, on S for $p > p_0$. From now on, we always assume $p > p_0$.

By the Grothendieck–Riemann–Roch Theorem, in $H^2(S, \mathbb{R})$, as $p \to +\infty$, we have

$$c_1\big(H^0\big(X, L^p \otimes E\big)\big) = \mathrm{rk}(E) \int_X \frac{c_1(L)^{n+1}}{(n+1)!} p^{n+1} + \int_X \bigg(c_1(E) + \frac{\mathrm{rk}(E)}{2} c_1(TX)\bigg) \frac{c_1(L)^n}{n!} p^n + \mathscr{O}\big(p^{n-1}\big). \tag{1}$$

Now, in view of the Bismut local family index theorem [2], it is natural to ask whether a local version of (1) still holds which involves the curvature of the vector bundle $H^0(X, L^p \otimes E)$.

Let us introduce our geometric data now. Let h^L be a Hermitian metric on L such that the restriction of $\sqrt{-1}\, R^L$ along the fiber X is a positive $(1,1)$-form, here R^L is the curvature of the holomorphic Hermitian connection ∇^L on (L, h^L). Let $\omega := c_1(L, h^L)$ be the Chern–Weil representative of the first Chern class $c_1(L)$ of (L, h^L), then

$$\omega = c_1\big(L, h^L\big) = \frac{\sqrt{-1}}{2\pi} R^L. \tag{2}$$

Thus ω is a smooth real 2-form of complex type $(1,1)$ on W. Moreover, ω defines a Kähler form along the fiber X, i.e.

$$g^{T_{\mathbb{R}}X}(u, v) = \omega\big(u, J^{T_{\mathbb{R}}X} v\big) \tag{3}$$

defines a Riemannian metric on $T_{\mathbb{R}}X$. We denote by h^{TX} the corresponding Hermitian metric on TX.

Let dv_X be the Riemannian volume form on $(X, g^{T_{\mathbb{R}}X})$.

Let h^E be a Hermitian metric on E. Let ∇^E be the holomorphic Hermitian connection on (E, h^E) with its curvature R^E.

Let $h^{H^0(X, L^p\otimes E)}$ be the L^2-metric on $H^0(X, L^p \otimes E)$ induced by h^L, h^E and $g^{T_{\mathbb{R}}X}$. Let $\nabla^{H^0(X, L^p\otimes E)}$ be the holomorphic Hermitian connection on $(H^0(X, L^p \otimes E), h^{H^0(X, L^p\otimes E)})$. Let $R^{H^0(X, L^p\otimes E)} = (\nabla^{H^0(X, L^p\otimes E)})^2$ be the curvature of $\nabla^{H^0(X, L^p\otimes E)}$. Then

$$R^{H^0(X, L^p\otimes E)} \in \Lambda^2\big(T_{\mathbb{R}}^* S\big) \otimes \mathrm{End}\big(H^0\big(X, L^p \otimes E\big)\big).$$

For $s \in S$, let $P_{p,s}$ be the orthogonal projection from $\mathscr{C}^\infty(X_s, (L^p \otimes E)|_{X_s})$ onto $H^0(X_s, (L^p \otimes E)|_{X_s})$. In the sequel, we write instead P_p.

We will identify $R^{H^0(X, L^p\otimes E)}$ to

$$P_p R^{H^0(X, L^p\otimes E)} P_p \in \Lambda^2\big(T_{\mathbb{R}}^* S\big) \otimes \mathrm{End}\big(\mathscr{C}^\infty\big(X, \big(L^p \otimes E\big)\big|_X\big)\big).$$

Let $R^{H^0(X, L^p\otimes E)}(x, x')$ $(x, x' \in X_s, s \in S)$ be the smooth kernel of the operator $R^{H^0(X, L^p\otimes E)}$ with respect to $dv_{X_s}(x')$. Then

$$R^{H^0(X, L^p\otimes E)}(x, x) \in \pi^*\big(\Lambda^2\big(T_{\mathbb{R}}^* S\big)\big) \otimes \mathrm{End}(E_x). \tag{4}$$

The purpose of this Note is to evaluate the asymptotics as $p \to \infty$ of the kernel of $R^{H^0(X, L^p\otimes E)}$.

2. Main result

Let $T^H W$ be the orthogonal bundle to TX with respect to ω. Then $T^H W$ is a sub-bundle of TW such that

$$TW = T^H W \oplus TX. \tag{5}$$

Let P^{TX} be the projection from $TW = T^H W \oplus TX$ onto TX. For $U \in T_{\mathbb{R}}S$, let $U^H \in T_{\mathbb{R}}^H W$ be the horizontal lift of U.

Let $T \in \Lambda^2(T^*_{\mathbb{R}}W) \otimes T_{\mathbb{R}}X$ be the tensor defined in the following way: for $U, V \in T_{\mathbb{R}}S$, $X, Y \in T_{\mathbb{R}}X$,

$$\begin{aligned} &T\big(U^H, V^H\big) := -P^{TX}\big[U^H, V^H\big], \quad T(X,Y) := 0, \\ &T\big(U^H, X\big) := \frac{1}{2}\big(g^{T_{\mathbb{R}}X}\big)^{-1}\big(\mathcal{L}_{U^H} g^{T_{\mathbb{R}}X}\big)X. \end{aligned} \tag{6}$$

Let R^{TX} be the curvature of the holomorphic Hermitian connection ∇^{TX} on (TX, h^{TX}). Then the Chern–Weil representative of the first Chern class of (TX, h^{TX}) is $c_1(TX, h^{TX}) = \frac{\sqrt{-1}}{2\pi}\operatorname{Tr}[R^{TX}]$.

Let $\{g_\alpha\}$ be a frame of TS and $\{g^\alpha\}$ its dual frame.

Clearly, (5) induces canonically a decomposition $\Lambda(T^*_{\mathbb{R}}W) = \pi^*(\Lambda(T^*_{\mathbb{R}}S))\widehat{\otimes}\Lambda(T^*_{\mathbb{R}}X)$. We will denote by $A^{(i)}$ the component in $\pi^*(\Lambda^i(T^*_{\mathbb{R}}S))\widehat{\otimes}\Lambda(T^*_{\mathbb{R}}X)$, of a differential form A on W. Then $dv_X = (\omega^n)^{(0)}/n!$.

Theorem 2.1. *There exist smooth sections $b_{2,r}(x) \in \mathscr{C}^\infty(W, \Lambda^2(T^*_{\mathbb{R}}S) \otimes \operatorname{End}(E_x))$ which are polynomials in R^{TX}, T, R^E (and R^L), and their derivatives of order $\leqslant 2r-1$ (resp. $2r$) along the fiber X such that for any $k, l \in \mathbb{N}$, there exists $C_{k,l} > 0$ such that for any $p \in \mathbb{N}$, $p > p_0$,*

$$\left| R^{H^0(X, L^p\otimes E)}(x,x) - \sum_{r=0}^{k} b_{2,r}(x)p^{n-r+1} \right|_{\mathscr{C}^l(W)} \leqslant C_{k,l}\, p^{n-k}, \tag{7}$$

with

$$\begin{aligned} &\frac{\sqrt{-1}}{2\pi} b_{2,0} = \frac{(\omega^{n+1})^{(2)}}{(n+1)(\omega^n)^{(0)}}\operatorname{Id}_E = g^\alpha \wedge \bar g^\beta \omega\big(g^H_\alpha, \bar g^H_\beta\big)\operatorname{Id}_E, \\ &\frac{\sqrt{-1}}{2\pi} b_{2,1} = \left(\left(\frac{1}{2}c_1\big(TX, h^{TX}\big) + \frac{\sqrt{-1}}{2\pi}R^E - \frac{1}{8\pi} g^\alpha \wedge \bar g^\beta \Delta_X\big(\omega\big(g^H_\alpha, \bar g^H_\beta\big)\big)\right)\omega^n\right)^{(2)} \Big/ \big(\omega^n\big)^{(0)}, \end{aligned} \tag{8}$$

where Δ_X is the (positive) Laplace operator of the fiber X_s.

If we take the trace of this asymptotic (7) on E and integrate along X, we get a refinement of (1) on the level of differential forms, in the spirit of the local family index theorem.

3. Idea of the proof

Proof. By using the full off-diagonal asymptotic expansion of the Bergman kernel [6] with the parameter $s \in S$, it is not hard to prove the existence of an expansion with leading term p^{n+2}, but further work is needed to get the vanishing of the first coefficient, and it is difficult to compute the other coefficients this way.

Our main idea here is to use the superconnection formalism to prove Theorem 2.1. This gives us a conceptually clear way to get our result: an important feature of our superconnection is that its curvature is a second order differential operator along the fiber X, while the superconnection itself involves derivatives along the horizontal direction. Just as in the Bismut local family index theorem [2], this property of our superconnection plays an important role in our proof.

We now explain briefly the superconnection formalism.

Let $\bar\partial^{L^p\otimes E,*}$ be the formal adjoint of the fiberwise Dolbeault operator $\bar\partial^{L^p\otimes E}$ on the Dolbeault complex $\Omega^{0,\bullet}(X, L^p \otimes E)$. Set

$$D_p = \sqrt{2}\big(\bar\partial^{L^p\otimes E} + \bar\partial^{L^p\otimes E,*}\big). \tag{9}$$

Let ∇^{E_p} be the connection on $E_p := \Lambda(T^{*(0,1)}X) \otimes L^p \otimes E$ induced by the holomorphic Hermitian connections $\nabla^{TX}, \nabla^L, \nabla^E$ on $TX,\ L,\ E$ respectively.

For $U \in T_{\mathbb{R}}S$, if σ is a smooth section of $\Omega^{0,\bullet}(X, L^p \otimes E)$ over S, i.e. $\sigma \in \mathscr{C}^\infty(W, E_p)$, set

$$\nabla^\Omega_U \sigma = \nabla^{E_p}_{U^H}\sigma. \tag{10}$$

Then ∇^Ω is a Hermitian connection on $\Omega^{0,\bullet}(X, L^p \otimes E)$ over S. Let B_p be the superconnection on $\Lambda(T^*_{\mathbb{R}}S) \,\widehat{\otimes}\, \Omega^{0,\bullet}(X, L^p \otimes E)$ defined by

$$B_p = D_p + \nabla^{\Omega}. \tag{11}$$

We now describe the explicit geometric construction of $\nabla^{H^0(X,L^p\otimes E)}$ given in [4, Theorem 3.4] (cf. [3, Theorem 3.11]). Let $\nabla^{L^p\otimes E}$ be the connection on $L^p \otimes E$ induced by ∇^L, ∇^E. For $U \in T_{\mathbb{R}}S$, $\sigma \in \mathscr{C}^{\infty}(S, H^0(X, L^p \otimes E))$, then

$$\nabla_U^{H^0(X,L^p\otimes E)}\sigma = P_p \nabla_{U^H}^{L^p\otimes E}\sigma, \tag{12}$$

where σ is considered as a section of $L^p \otimes E$ on W.

From (11), (12) and the spectral gap property of D_p^2 (cf. [5,9]), for p large enough, we have

$$R^{H^0(X,L^p\otimes E)} = \frac{1}{2\pi\sqrt{-1}}\left[\int_{|\lambda|=2\pi p} \left(\lambda - B_p^2\right)^{-1}\lambda\, d\lambda\right]^{(2)}. \tag{13}$$

Now, by using the formal power series trick developed in [10], we get a general and algorithmic way to compute the coefficients in the expansion. More details will appear in [12]. □

Remark 3.1. In this Note, we have only formulated our results in the case of holomorphic line bundles which are fiberwise positive. Actually, the results hold also for symplectic line bundles. In [12], we also prove the existence of an off-diagonal asymptotic expansion which implies, for example, that $R^{H^0(X,L^p\otimes E)}$ is a Toeplitz operator with values in $\Lambda^2(T_{\mathbb{R}}^*S)$ in the sense of [11, Chapter 7].

By (7), (8), the curvatures $R^{H^0(X,L^p\otimes E)}(x,x)$ provide a natural approximation of the Monge–Ampère operator on the space of Kähler metrics. It should have relations with the existence problem of geodesics on the space of Kähler metrics (cf. [7,8,14,13]).

From Eq. (8), for large p, we can obtain more precise positivity estimates for $H^0(X, L^p \otimes E)$ than in [1, §6].

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