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C. R. Acad. Sci. Paris, Ser. I 347 (2009) 389–394
COMPTES RENDUS MATHEMATIQUE

Differential Geometry

Geometric quantization for proper moment maps

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Received 2 December 2008; accepted 28 January 2009

Available online 6 March 2009

Presented by Jean-Michel Bismut

Abstract

We establish a geometric quantization formula for Hamiltonian actions of a compact Lie group acting on a non-compact symplectic manifold such that the associated moment map is proper. In particular, we give a solution to a conjecture of Michèle Vergne. ***To cite this article: X. Ma, W. Zhang, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Quantification géométrique pour les applications moment propres. Nous établissons une formule de quantification géométrique pour les actions hamiltoniennes d'un groupe de Lie compact agissant sur une variété symplectique non-compacte dont l'application moment est propre. En particulier, nous résolvons une conjecture formulée par Michèle Vergne dans son exposé à l'ICM 2006. ***Pour citer cet article : X. Ma, W. Zhang, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Soit G un groupe de Lie compact connexe agissant sur une variété symplectique non-compacte (M,ω) par une action hamiltonienne. Soit (L,∇^L) un fibré en droites hermitien muni d'une connexion hermitienne et G-invariante, et tel que $(\nabla^L)^2=\frac{2\pi}{\sqrt{-1}}\omega$. Soit J une structure presque complexe G-invariante sur TM, telle que $g^{TM}(u,v)=\omega(u,Jv)$ est une métrique riemannienne sur TM. Soit $\mathfrak{g}$ l'algèbre de Lie de G. On munit $\mathfrak{g}^*$ d'une métrique Ad_G-invariante.

On suppose que l'application moment associée $\mu: M\to\mathfrak{g}^*$ est propre.

Soit $\mathcal{H}=|\mu|^2$. Soit $X^{\mathcal{H}}=-J(d\mathcal{H})^*$ le champ de vecteurs hamiltonien associé à $\mathcal{H}$.

Pour $a>0$, on pose $U_a:=\{x\in M\colon \mathcal{H}(x)\leqslant a\}$. Pour une valeur régulière a de $\mathcal{H}$, d'après Atiyah [1], le symbole $\sqrt{-1}\,c(\cdot+X^{\mathcal{H}})\otimes \mathrm{Id}_L$ ($c(\cdot)$ est l'action de Clifford) définit un symbole transversalement elliptique sur U_a, et son indice transversal $Q(L)_a$ est une distribution sur G. Tian et Zhang [13] ont introduit l'opérateur de Dirac correspondant dans leur approche de la quantification géométrique quand M est compacte. Le symbole associé a été utilisé par Paradan [10,11].

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.02.003

Soit Λ_+^* l'ensemble des poids dominants. Pour $\gamma \in \Lambda_+^*$, soit V_γ^G la représentation G-irréductible de plus haut poids γ, soit $Q(L)_a^\gamma \in \mathbb{Z}$ la multiplicité de la représentation V_γ^G dans $Q(L)_a$. Pour $\gamma \in \Lambda_+^*$, on note $Q(L_\gamma)$ l'indice de l'opérateur de Dirac Spin^c sur la réduction symplectique de M en γ.

Théorème 0.1. *Pour $\gamma \in \Lambda_+^*$, il existe $a_\gamma > 0$ telle que $Q(L)_a^\gamma \in \mathbb{Z}$ ne dépend pas de $a \geqslant a_\gamma$, pour a une valeur régulière de $\mathcal{H}$. On le note comme $Q(L)^\gamma$.*

Théorème 0.2. *Pour $\gamma \in \Lambda_+^*$, on a $Q(L)^\gamma = Q(L_\gamma)$.*

Si $\{X^{\mathcal{H}} = 0\}$ est compact, le Théorème 0.2 a été conjecturé par Michèle Vergne dans [15, §4.3], et des cas particuliers de cette conjecture ont été démontrés par Paradan [11,12], dans le cadre des séries discrètes. Les résultats annoncés dans cette note sont démontrés dans [8].

1. Transversal index and the APS index

In this Note, we present a solution of an extended version of the conjecture of Michèle Vergne in her ICM 2006 lecture. Details will appear in [8].

Let M be a compact manifold with non-empty boundary ∂M. Let J be an almost complex structure on TM. Let g^{TM} be a J-invariant Riemannian metric on the tangent vector bundle $\pi : TM \to M$. Let (E, h^E) be a Hermitian vector bundle over M with Hermitian connection ∇^E.

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. We assume that G acts on M and that this action lifts to an action on E, and preserves J, g^{TM}, h^E and ∇^E.

For any $W \in TM$ such that $W = w + \overline{w} \in T^{(1,0)}M \oplus T^{(0,1)}M = TM \otimes_{\mathbb{R}} \mathbb{C}$, the Clifford action $c(W)$ on $\Lambda(T^{*(0,1)}M)$ is defined by $c(W) = \sqrt{2}\,\overline{w}^* \wedge - \sqrt{2}\, i_{\overline{w}}$, where $\overline{w}^* \in T^{*(0,1)}M$ is the metric dual of w. Recall that the spin^c structure associated to the Clifford module $\Lambda(T^{*(0,1)}M)$ is induced by the line bundle $\det(T^{(1,0)}M)$.

Let $\nabla^{T^{(1,0)}M}$ be the Hermitian connection on $T^{(1,0)}M$ induced by projection by the Levi-Civita connection ∇^{TM} on (TM, g^{TM}). Let $\nabla^{\det}$ be the Hermitian connection on $\det(T^{(1,0)}M)$ induced by $\nabla^{T^{(1,0)}M}$. Let $\nabla^{\Lambda^{0,\bullet}\otimes E}$ be the Clifford connection on $\Lambda(T^{*(0,1)}M) \otimes E$ induced by ∇^{TM}, $\nabla^{\det}$, and ∇^E. Let $\{e_i\}$ be an orthonormal basis of TM.

Then one can construct canonically the Spin^c-Dirac operator (twisted by E), $D^E = \sum_{i=1}^{\dim M} c(e_i)\nabla_{e_i}^{\Lambda^{0,\bullet}\otimes E}$: $\Omega^{0,\bullet}(M, E)$, with $\Omega^{0,\bullet}(M, E)$ the space of smooth sections of $\Lambda(T^{*(0,1)}M) \otimes E$ on M.

Let $e_{\mathfrak{n}}$ be the inward unit normal vector field perpendicular to ∂M. Let $e_1, \ldots, e_{\dim M-1}$ be an orthonormal basis of $T\partial M$. Let $\pi_{ij} = \langle \nabla_{e_i}^{TM} e_j, e_{\mathfrak{n}} \rangle$ be the second fundamental form of the isometric embedding $\iota_{\partial M} : \partial M \hookrightarrow M$. Let $D_{\partial M}^E : \Omega^{0,\bullet}(M, E)|_{\partial M} \to \Omega^{0,\bullet}(M, E)|_{\partial M}$ be the induced (by D^E) Dirac operator on ∂M defined by (cf. [6, p. 142])

$$D_{\partial M}^E = -\sum_{i=1}^{\dim M-1} c(e_{\mathfrak{n}})c(e_i)\nabla_{e_i}^{\Lambda^{0,\bullet}\otimes E} + \frac{1}{2}\sum_{i=1}^{\dim M-1} \pi_{ii}. \tag{1}$$

Let $\Psi : M \to \mathfrak{g}$ be a G-equivariant map with Ad_G-action on $\mathfrak{g}$. Let Ψ^M denote the vector field over M such that $\Psi^M(x)$ equals to the value at x of the vector field generated by $\Psi(x) \in \mathfrak{g}$ over M.

We suppose that $\Psi^M|_{\partial M} \in T\partial M$ is nowhere zero on ∂M.

Following [6, Lemma 2.2] and [14, §1c)], set for $T \in \mathbb{R}$,

$$\begin{aligned} D_T^E &= D^E + \frac{\sqrt{-1}\,T}{2} c\big(\Psi^M\big), \qquad D_{\pm,T}^E = D_T^E\big|_{\Omega^{0,\frac{\text{even}}{\text{odd}}}(M,E)}, \\ D_{\partial M,T}^E &= D_{\partial M}^E - \frac{\sqrt{-1}\,T}{2} c(e_{\mathfrak{n}})c\big(\Psi^M\big), \qquad D_{\partial M,\pm,T}^E = D_{\partial M,T}^E\big|_{\Omega^{0,\frac{\text{even}}{\text{odd}}}(M,E)|_{\partial M}}. \end{aligned} \tag{2}$$

Then $D_{\partial M,T}^E$ preserves the $\mathbb{Z}_2$-grading on $\Omega^{0,\bullet}(M, E)|_{\partial M}$. For any $\lambda \in \mathrm{Spec}\{D_{\partial M,\pm,T}^E\}$, let $E_{\lambda,\pm,T}$ be the corresponding eigenspace. Let $P_{\geqslant 0,\pm,T}$ (resp. $P_{>0,\pm,T}$) be the orthogonal projections from the L^2-completions of $\Omega^{0,\frac{\text{even}}{\text{odd}}}(M, E)|_{\partial M}$ onto $\bigoplus_{\lambda\geqslant 0} E_{\lambda,\pm,T}$ (resp. $\bigoplus_{\lambda>0} E_{\lambda,\pm,T}$).

For any $T \in \mathbb{R}$, let $(D^E_{+,T}, P_{\geqslant 0,+,T})$ (resp. $(D^E_{-,T}, P_{>0,-,T})$) denote the corresponding Atiyah–Patodi–Singer type boundary valued problem [2], [6, Theorem 2.3]. Then both $(D^E_{+,T}, P_{\geqslant 0,+,T})$ and $(D^E_{-,T}, P_{>0,-,T})$ are elliptic and G-equivariant.

Let T be a maximal torus of G, and let $\mathfrak{t}$ be its Lie algebra and $\mathfrak{t}^*$ its dual. Then the set of the finite dimensional G-irreducible representations is parameterized by the set of dominant weights $\Lambda^*_+ \subset \mathfrak{t}^*$. For $\gamma \in \Lambda^*_+$, we denote by V^G_γ the irreducible G-representation with highest weight γ. Then V^G_γ, $\gamma \in \Lambda^*_+$ is a $\mathbb{Z}$-basis of the representation ring $R(G)$.

Recall that $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{r}$ with $\mathfrak{r} = [\mathfrak{t}, \mathfrak{g}]$, and so $\mathfrak{g}^* = \mathfrak{t}^* \oplus \mathfrak{r}^*$. Thus we identify naturally Λ^*_+ as a subset of $\mathfrak{g}^*$.

Let $Q^M_{\mathrm{APS},T}(E, \Psi^M)^\gamma \in \mathbb{Z}$, $\gamma \in \Lambda^*_+$, be defined by

$$\bigoplus_{\gamma \in \Lambda^*_+} Q^M_{\mathrm{APS},T}\big(E, \Psi^M\big)^\gamma \cdot V^G_\gamma := \mathrm{Ker}\big(D^E_{+,T}, P_{\geqslant 0,+,T}\big) - \mathrm{Ker}\big(D^E_{-,T}, P_{>0,-,T}\big). \tag{3}$$

Proposition 1.1. *For any $\gamma \in \Lambda^*_+$, there exists $T_\gamma \geqslant 0$ such that $Q^M_{\mathrm{APS},T}(E, \Psi^M)^\gamma$ does not depend on $T \geqslant T_\gamma$.*

Denote by $Q^M_{\mathrm{APS}}(E, \Psi^M)^\gamma$ the quantization number $Q^M_{\mathrm{APS},T}(E, \Psi^M)^\gamma$ for $T \geqslant T_\gamma$. Then $Q^M_{\mathrm{APS}}(E, \Psi^M)^\gamma$ does not depend on g^{TM}, h^E, ∇^E and depends only on the homotopy classes of J, Ψ^M.

Let $\widehat{M} = M \setminus \partial M$ be the interior of M. One identifies TM and T^*M via g^{TM}. Let $\sigma^M_{E,\Psi^M} \in \mathrm{Hom}(\pi^*(\Lambda^{\mathrm{even}}(T^{*(0,1)}M) \otimes E), \pi^*(\Lambda^{\mathrm{odd}}(T^{*(0,1)}M) \otimes E))$ denote the symbol defined by $\sigma^M_{E,\Psi^M}(x, v(x)) = \sqrt{-1}\,\pi^*(c(v + \Psi^M) \otimes \mathrm{Id}_E)_{(x,v(x))}$.[1] The symbol σ^M_{E,Ψ^M} defines a G-transversally elliptic symbol on $T_G\widehat{M}$ in the sense of Atiyah [1, §3], which in turn determines a transversal index (cf. also [10, §3], [11, §3])

$$\mathrm{Ind}\big(\sigma^M_{E,\Psi^M}\big) = \bigoplus_{\gamma \in \Lambda^*_+} \mathrm{Ind}_\gamma\big(\sigma^M_{E,\Psi^M}\big) \cdot V^G_\gamma, \quad \text{with each } \mathrm{Ind}_\gamma\big(\sigma^M_{E,\Psi^M}\big) \in \mathbb{Z}. \tag{4}$$

The following result can be proved by using the main theorem of Braverman in [4]:

Theorem 1.2. *For any $\gamma \in \Lambda^*_+$, one has* $\mathrm{Ind}_\gamma(\sigma^M_{E,\Psi^M}) = Q^M_{\mathrm{APS}}(E, \Psi^M)^\gamma$.

2. Geometric quantization for proper moment maps

Let (M, ω) be a non-compact symplectic manifold with symplectic form ω. We assume that there exists a Hermitian line bundle (L, h^L) (called a prequantized line bundle) carrying a Hermitian connection ∇^L such that $(\nabla^L)^2 = \frac{2\pi}{\sqrt{-1}}\omega$. Let J be an almost complex structure on TM such that $g^{TM}(u, v) = \omega(u, Jv)$ defines a J-invariant Riemannian metric on TM.

We assume that the compact connected Lie group G acts on M and this action can be lifted to an action on L. We assume also that G preserves g^{TM}, J, h^L and ∇^L. Then the moment map $\mu : M \to \mathfrak{g}^*$ is defined by $2\pi\sqrt{-1}\mu(K) := \nabla^L_{K^M} - L_K$, for any $K \in \mathfrak{g}$, which verifies $d\mu(K) = i_{K^M}\omega$.

Basic Assumption. The corresponding moment map $\mu : M \to \mathfrak{g}^*$ is proper.

Take any $\gamma \in \Lambda^*_+$. If γ is a regular value of the moment map μ, then one can construct the Marsden–Weinstein symplectic reduction $(M_\gamma, \omega_\gamma)$, where $M_\gamma = \mu^{-1}(G \cdot \gamma)/G$ is a *compact* orbifold. Moreover, L (resp. J) induces a prequantized line bundle L_γ (resp. an almost complex structure J_γ) over $(M_\gamma, \omega_\gamma)$. One can then construct the associated Spin^c-Dirac operator (twisted by L_γ) on M_γ whose index $Q(L_\gamma)$ is well-defined. If $\gamma \in \Lambda^*_+$ is not a regular value of μ, then by proceeding as in [9], one still gets a well-defined quantization number $Q(L_\gamma)$ extending the above definition.

[1] This kind of deformation (by Ψ^M) has been used by Tian–Zhang [13,14] and Paradan [10,11] in their approaches to the Guillemin–Sternberg geometric quantization conjecture [7].

We equip $\mathfrak{g}$ with a Ad_G-invariant metric, and we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by the metric. Set $\mathcal{H} = |\mu|^2$. Let $X^{\mathcal{H}} = -J(d\mathcal{H})^* = 2\mu^M$ be the Hamiltonian vector field associated to $\mathcal{H}$.

For any $a > 0$, $U_a := \{x \in M\colon\ \mathcal{H}(x) \leqslant a\}$ is a compact subset of M. For any regular value $a > 0$ of $\mathcal{H}$, $X^{\mathcal{H}}$ is nowhere zero on $\partial U_a = \mathcal{H}^{-1}(a)$.

Theorem 2.1. *For any $\gamma \in \Lambda^*_+$, there exists $a_\gamma > 0$ such that $Q^{U_a}_{\mathrm{APS}}(L, \mu^M)^\gamma \in \mathbb{Z}$ does not depend on $a \geqslant a_\gamma$, with a a regular value of $\mathcal{H}$.*

For any $\gamma \in \Lambda^*_+$, we denote by $Q(L)^\gamma$ the well-defined integer $Q^{U_a}_{\mathrm{APS}}(L, \mu^M)^\gamma$ not depending on the regular value $a \gg 0$.

Let (N, ω^N) be a compact symplectic manifold and (F, h^F, ∇^F) the prequantized line bundle over N, and G acts on N, F and preserves J^N, h^F, ∇^F. Let $\eta : N \to \mathfrak{g}^*$ be the corresponding moment map.

We will use the same notation for the natural extension of the objects on M, N to $(M \times N, \omega \oplus \omega^N)$. In particular, $L \otimes F$ is the prequantized line bundle over $M \times N$ obtained by the tensor product of the natural liftings of L and F to $M \times N$. Then the induced moment map $\theta : M \times N \to \mathfrak{g}^*$ is given by $\theta(x, y) = \mu(x) + \eta(y)$.

Theorem 2.2. *For the induced action of G on $(M \times N, \omega \oplus \omega^N)$ and $L \otimes F$, we have,*

$$Q\big((L \otimes F)_{\gamma=0}\big) = \sum_{\gamma \in \Lambda^*_+} Q(L)^\gamma Q(F)^{-\gamma}. \tag{5}$$

The proof of Theorem 2.2 is deferred to Section 4.

By taking N in (5) to be the orbits of the co-adjoint action of G on $\mathfrak{g}^*$, we get:

Theorem 2.3. *For any $\gamma \in \Lambda^*_+$, one has $Q(L)^\gamma = Q(L_\gamma)$.*

Remark 2.4. (i) If M is compact, then Theorem 2.1 holds tautologically and Theorem 2.3 is the Guillemin–Sternberg geometric quantization conjecture [7] proved in [9].

(ii) In view of Theorem 1.2, Theorem 2.3 was conjectured by Michèle Vergne in [15, §4.3] in the case when the zero set of $X^{\mathcal{H}}$ is compact. Special cases of this conjecture, related to the discrete series of semi-simple Lie groups, have been proved by Paradan [11,12].

Theorem 2.3 provides a proof of this conjecture even when the zero set of $X^{\mathcal{H}}$ is non-compact.

3. A vanishing result

Our proof of Theorem 2.2 relies essentially on a vanishing result, Theorem 3.1, which is established in this section.

For $A > 0$ large enough which is a regular value of the functions $|\mu|^2$ and $\frac{1}{2}|\theta|^2$ on $M \times N$, we define $\mathcal{M} = \{(x, y) \in M \times N; |\mu|^2 \geqslant A,\ |\theta|^2 \leqslant 2A\}$, $\mathcal{M}_1 = \{(x, y) \in M \times N;\ |\mu|^2 = A\}$, $M_A = \{x \in M; |\mu|^2 \leqslant A\}$ and $\mathcal{M}_2 = \{(x, y) \in M \times N;\ |\theta|^2 = 2A\}$. Let $\overline{\mathcal{M}}_A = \{(x, y) \in M \times N, |\mu(x)|^2 \leqslant A\}$, $\overline{\mathcal{M}} = \mathcal{M} \cup \overline{\mathcal{M}}_A = \{|\theta|^2 \leqslant 2A\}$.

Then $\mathcal{M}$ is a smooth manifold with boundary $\partial\mathcal{M}$ and $\partial\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$, $\mathcal{M}_1 = \partial M_A \times N$.

Set $\tilde\alpha, \tilde\phi \in \mathscr{C}^\infty(\mathbb{R})$ verify the following conditions,

$$\tilde\alpha(t) = \begin{cases} t^2, & \text{for } t \leqslant \frac{1}{3}, \\ 1, & \text{for } t \geqslant \frac{2}{3}, \end{cases} \qquad \tilde\phi(t) = \begin{cases} 1 - t^3, & \text{for } t \leqslant \frac{1}{3}, \\ 2(1 - t), & \text{for } t \geqslant \frac{2}{3}, \end{cases}$$
$$\tilde\alpha(t) + \tilde\phi(t) \geqslant \frac{29}{27}, \quad \tilde\phi'(t) < 0, \quad \text{for } \frac{1}{3} \leqslant t \leqslant \frac{2}{3}. \tag{6}$$

For $A > 0$, set $\alpha(t) = \tilde\alpha(\frac{t}{A} - 1)$, $\phi(t) = \tilde\phi(\frac{t}{A} - 1)$.

We define $\beta \in \mathscr{C}^\infty(M \times N)$, $\rho : M \times N \to \mathfrak{g}^* \simeq \mathfrak{g}$ by $\beta = |\mu|^2 + \alpha(|\mu|^2)(|\theta|^2 - |\mu|^2)$, $\rho = \theta - \phi(\beta)\eta$.

Let $V_1, \dots, V_{\dim G}$ be an orthonormal basis of $\mathfrak{g}$. Denote by V_i^M, V_i^N the Killing vector fields on M, N induced by V_i. For any function Q with values in $\mathfrak{g}$, we will denote Q_i its i-component with respect to the basis $\{V_i\}$, and when a subscript index appears two times in a formula, we sum up with this index. Set

$$\gamma_j = 2\big(1+\alpha'\big(|\mu|^2\big)\big(|\theta|^2-|\mu|^2\big)\big)\mu_j + 2\alpha\big(|\mu|^2\big)\eta_j,$$
$$\psi_j = 2\rho_j - 2\phi'(\beta)\rho_i\eta_i\gamma_j. \tag{7}$$

Let $\psi := \psi_j V_j : \mathcal{M} \to \mathfrak{g}$ be the induced G-equivariant map and let Y be the vector field on $\mathcal{M}$ induced by ψ, then $Y := \psi^{\mathcal{M}} = \psi_j(V_j^M + V_j^N)$. We also have $\psi|_{\mathcal{M}_1} = 2\mu$, and $\psi|_{\mathcal{M}_2} = (2 + \frac{8}{A}\rho_i\eta_i)\theta$.

Theorem 3.1. *When $A > 0$ is large enough, we have $Q_{\mathrm{APS}}^{\mathcal{M}}(L\otimes F, Y)^{\gamma=0} = 0$.*

Outline of the proof of Theorem 3.1. For $T \in \mathbb{R}$, let $D_T^{\mathcal{M}}$ be the operator defined by $D_T^{\mathcal{M}} = D^{L\otimes F} + \frac{\sqrt{-1}T}{2}c(Y) : \Omega^{0,\bullet}(\mathcal{M}, L\otimes F) \to \Omega^{0,\bullet}(\mathcal{M}, L\otimes F)$. For any $T \in \mathbb{R}$, let $F_T^{\mathcal{M}} : \Omega^{0,\bullet}(\mathcal{M}, L\otimes F) \to \Omega^{0,\bullet}(\mathcal{M}, L\otimes F)$ be defined by $F_T^{\mathcal{M}} = D_T^{\mathcal{M},2} + \sqrt{-1}T\psi_j L_{V_j}$.

Let $\{e_k\}_{k=1}^{\dim M}$ (resp. $\{f_i\}_{i=1}^{\dim N}$) be an orthonormal frame of TM (resp. TN). Let $\nabla^{T^{(1,0)}M}$, $\nabla^{T^{(1,0)}N}$ be the connections on $T^{(1,0)}M$, $T^{(1,0)}N$ induced by the Levi-Civita connections. Set

$$I_1 = \frac{1}{4}c\big((d^M\psi_j)^*\big)c\big(V_j^M + 2V_j^N\big) + \frac{1}{2}c\big((d^N\psi_j)^*\big)c\big(V_j^M + V_j^N\big),$$
$$I_2 = \frac{1}{4}\Big\langle\Big(1 + \frac{J}{\sqrt{-1}}\Big)V_j^M, \big(d^M\psi_j\big)^*\Big\rangle,$$
$$R_j = \frac{\sqrt{-1}}{4}\sum_{i=1}^{\dim N} c(f_i)c\big(\nabla_{f_i}^{TN}V_j^N\big) - \frac{\sqrt{-1}}{2}\mathrm{Tr}\big[\nabla^{T^{(1,0)}N}V_j^N\big|_{T^{(1,0)}N}\big]. \tag{8}$$

Proposition 3.2. *The following Bochner type identity holds on $\mathcal{M}$,*

$$F_T^{\mathcal{M}} = D^{L\otimes F,2} + T\big(2\pi\psi_j\theta_j + \psi_j R_j + \sqrt{-1}(I_1+I_2)\big) + \frac{T^2}{4}|Y|^2$$
$$+ \frac{\sqrt{-1}T}{4}\sum_{k=1}^{\dim M} c(e_k)c\big(\nabla_{e_k}^{TM}\big(\psi_j V_j^M\big)\big) - \frac{\sqrt{-1}T}{2}\mathrm{Tr}\big[\nabla^{T^{(1,0)}M}\big(\psi_j V_j^M\big)\big|_{T^{(1,0)}M}\big]. \tag{9}$$

Proposition 3.3. *There exists $A_0 > 0$ such that for any $A > A_0$, $z \in \mathcal{M}\setminus\partial\mathcal{M}$ there exists an open neighborhood U_z of z; $C_z, C_{1,z} > 0$ such that for any $T \geqslant 1$ and $s \in \Omega^{0,\bullet}(\mathcal{M}, L\otimes F)$ with* $\mathrm{supp}(s) \subset U_z$*, one has*

$$\mathrm{Re}\big\langle F_T^{\mathcal{M}}s, s\big\rangle \geqslant C_z\big(\big\|D^{L\otimes F}s\big\|_0^2 + (T - C_{1,z})\|s\|_0^2\big). \tag{10}$$

Proof. If $Y(z) \neq 0$, then by (9), we get easily (10). If $Y(z) = 0$, from our choice of ψ_j, we have the following crucial estimate: there exist $C > 0$, $A_0 > 0$ such that for any $A > A_0$ and $(x, y) \in \{Y = 0\} \subset \mathcal{M}$, we have

$$\sqrt{-1}(I_1 + I_2)_{(x,y)} \geqslant -C\,\mathrm{Id}_{\Lambda(T^{*(0,1)}(M\times N))\otimes L\otimes F}. \tag{11}$$

Moreover, $\psi_j\theta_j = |\mu|^2 + \mathcal{O}(A^{1/2})$, $|R_j|$'s are bounded, and $\psi_j V_j^M = -J(d^M|\rho|^2)^*$. We then apply the arguments in [13, §2.4] to get (10). □

From Proposition 3.2, as Y is nowhere zero on $\partial\mathcal{M}$, by proceeding as in the proof of [14, Proposition 2.4], we get an estimate similar to [14, Proposition 2.4] for $D_T^{\mathcal{M}}$ on an open neighborhood of $\partial\mathcal{M}$. Now from Proposition 3.3, the estimate near $\partial\mathcal{M}$ for $D_T^{\mathcal{M}}$, and the gluing arguments in [3, p. 115–116], we get finally:

Theorem 3.4. *For A as in Proposition 3.3, there exist $C, C_1 > 0$ such that for any $T \geqslant 1$ and $s \in \Omega^{0,\bullet}(\mathcal{M}, L\otimes F)^G$ with $P_{\geqslant 0,\pm,T}(s|_{\partial\mathcal{M}}) = 0$, one has*

$$\big\|D_T^{\mathcal{M}}s\big\|_0^2 \geqslant C\big(\big\|D^{L\otimes F}s\big\|_0^2 + (T - C_1)\|s\|_0^2\big). \tag{12}$$

In particular, for $T > 0$ large enough, Theorem 3.1 holds.

Remark 3.5. There are other choices of α and ϕ for which Theorem 3.1 is still valid. A possible simpler choice for Y would be to use $\alpha \equiv 0$ or $\alpha \equiv 1$ in (7), and then choose a suitable ϕ. However, we could not get the corresponding estimate (11) for these choices, thus we could not eliminate the potential contributions caused by the possible zero set of Y. This explains why we introduce the non-linear deformation β in Theorem 3.1.

4. Proof of Theorem 2.2

Let $\psi^{\overline{\mathcal{M}}}$ be a smooth G-invariant vector field on $\overline{\mathcal{M}}$ induced by a G-equivariant map $\psi : \overline{\mathcal{M}} \to \mathfrak{g}$, such that $\psi^{\overline{\mathcal{M}}} = Y$ on $\mathcal{M}$. Then for $A > 0$ large enough, we have

$$\begin{aligned} Q_{\mathrm{APS}}^{\overline{\mathcal{M}}}\big(L \otimes F, \theta^{\overline{\mathcal{M}}}\big)^{\gamma=0} &= Q_{\mathrm{APS}}^{\overline{\mathcal{M}}}\big(L \otimes F, \psi^{\overline{\mathcal{M}}}\big)^{\gamma=0} \\ &= Q_{\mathrm{APS}}^{\overline{\mathcal{M}}_A}\big(L \otimes F, \psi^{\overline{\mathcal{M}}}\big)^{\gamma=0} + Q_{\mathrm{APS}}^{\mathcal{M}}(L \otimes F, Y)^{\gamma=0} \\ &= Q_{\mathrm{APS}}^{\overline{\mathcal{M}}_A}\big(L \otimes F, \psi^{\overline{\mathcal{M}}}\big)^{\gamma=0}, \end{aligned} \tag{13}$$

where in the first equation, we use the deformation $t\theta^{\overline{\mathcal{M}}} + (1-t)\psi^{\overline{\mathcal{M}}}$, $0 \leqslant t \leqslant 1$, which is nowhere zero on $\mathcal{M}_2$; while in the second equation, we make use of the splitting property of the APS type index (cf. [5, Theorem 1.1] for the product metrics case, and one deduces the general case by a deformation argument); while in the third equation, we use Theorem 3.1. We then use the deformation $tY + (1-t)\mu^M$, $0 \leqslant t \leqslant 1$, which is μ^M along M_A and thus nowhere zero on $\mathcal{M}_1$, to complete the proof of Theorem 2.2.

Acknowledgements

We thank the referee for very helpful suggestions. The work of the second author was partially supported by MOEC and NSFC. Part of the Note was written while the second author was visiting the School of Mathematics of Fudan University during November and December of 2008. He would like to thank Professor Jiaxing Hong and other members of the School for hospitality.

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