A family quantization formula for symplectic manifolds with boundary

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Abstract This paper generalizes the family quantization formula of Zhang to the case of manifolds with boundary. As an application, Tian-Zhang's analytic version of the Guillemin-Kalkman-Martin residue formula is generalized to the family case.

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In a series of papers[1−3], analytic approaches of the Guillemin-Sternberg geometric quantization conjecture[4] as well as generalizations to the cases of symplectic manifolds with boundary and a family of symplectic manifolds without boundary were developed. In this paper, we generalize the work of Tian and Zhang to prove a family quantization formula for symplectic manifolds with boundary. As an application, we prove a family residue formula which generalizes the corresponding formulas of Guillemin-Kalkman-Martin[5] and Tian-Zhang[2] to the family case.

1 A family quantization formula for symplectic manifolds with boundary

Let $Z \to M \overset{\pi}{\to} B$ be a smooth fibration with compact connected fibres and the compact base B. Let ∂Z denote the boundary of Z. Then $\partial Z \to \partial M \to B$ is also a smooth fibration, where $\partial M = \bigcup_{b\in B} \partial Z_b$ is the boundary of M. For any $b \in B$, we denote by $i_b: Z_b = \pi^{-1}(b) \to M$ the canonical embedding. Let TZ be the associated vertical tangent bundle. We make the basic assumption that there exists a smooth 2-form $\omega \in \Gamma(\Lambda^2(T^*Z))$ such that, for any $b \in B$, $\omega_b = i_b^*\omega$ is a symplectic form over Z_b.

Let J be an almost complex structure on TZ such that

$$g(u,v) = \omega(u, Jv),\quad u,v \in \Gamma(TZ) \tag{1.1}$$

defines a smooth Euclidean metric on TZ. The existence of J is clear. Then with respect to J (cf. ref. [1]), we have the canonical splitting

$$TZ \otimes \mathbb{C} = T^{(1,0)}Z \oplus T^{(0,1)}Z. \tag{1.2}$$

We assume that there exists a Hermitian line bundle L over M with a Hermitian connection ∇^L satisfying $\frac{\sqrt{-1}}{2\pi}(\nabla^L)^2 = \omega$. Then for any $b \in B$, we get an induced Hermitian connection $\nabla^{L_b} = i_b^*\nabla^L$ on $L_b = i_b^*L$.

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For any $b\in B$, let $\Omega^{0,*}(Z_b,L_b)$ denote the space of smooth sections of $\Lambda^*(T^{(0,1)*}Z_b)\otimes L_b$. As in ref. [2, Definitions 1.1, 1.2], for each $b\in B$, we define the Spin^c-Dirac operator:

$$D_+^{L_b}:\Omega^{0,\mathrm{even}}(Z_b,L_b)\to\Omega^{0,\mathrm{odd}}(Z_b,L_b) \tag{1.3}$$

from ω_b, $J_b=J|_{TZ_b}$ and (L_b,∇^{L_b}), and the canonically induced formally self-adjoint Spin^c-Dirac operator on the boundary ∂Z_b:

$$D_{\partial Z_b,+}^{L_b}:\Omega^{0,\mathrm{even}}(Z_b,L_b)|_{\partial Z_b}\to\Omega^{0,\mathrm{even}}(Z_b,L_b)|_{\partial Z_b}. \tag{1.4}$$

Let G be a compact connected Lie group with Lie algebra $\boldsymbol{g}$. We assume that the total manifold M admits a smooth G-action such that G preserves each Z_b and thus each boundary ∂Z_b. We also assume that there exists a G-equivariant smooth map $\mu:M\to\boldsymbol{g}^*$ such that the G-action on each fiber Z_b, $b\in B$, is Hamiltonian with the moment map given by $\mu_b=\mu|_{Z_b}:Z_b\to\boldsymbol{g}^*$. Furthermore, we make the assumption that $0\in\boldsymbol{g}^*$ is a regular value of μ as well as μ_b for each $b\in B$, and that G acts freely on $\mu^{-1}(0)$. We make the assumption in this paper that $\mu^{-1}(0)\cap\partial M=\emptyset$. Then on each fibre Z_b, one has as in ref. [2] the Marsden-Weinstein reduction $(Z_{G,b}=\mu_b^{-1}(0)/G,\omega_{G,b})$. As in ref. [3, (1.3)], these reduction spaces together form a smooth fibration of closed symplectic manifolds

$$(Z_G,\omega_G)\to M_G=\mu^{-1}(0)/G\xrightarrow{\pi_G}B. \tag{1.5}$$

We also assume that the G-action preserves the almost complex structure J and the Hermitian line bundle (L,∇^L), which induce canonically the almost complex structure J_G on TZ_G and the Hermitian line bundle (L_G,∇^{L_G}) over M_G, respectively. As in refs. [1,2], one defines the Spin^c-Dirac operator for each $b\in B$:

$$D_+^{L_{G,b}}:\Omega^{0,\mathrm{even}}(Z_{G,b},L_{G,b})\to\Omega^{0,\mathrm{odd}}(Z_{G,b},L_{G,b}). \tag{1.6}$$

Thus one has a smooth family of operators $D_{M_G/B,+}^{L_G}=\{D_+^{L_{G,b}}\}_{b\in B}$ which admits a well-defined index bundle[6]

$$\mathrm{ind}(D_{M_G/B,+}^{L_G})\in K(B). \tag{1.7}$$

Let $\boldsymbol{g}$ (and thus $\boldsymbol{g}^*$) be equipped with an AdG-invariant metric. Let $\mathscr{H}=|\mu|^2$ be the norm square of μ. Let $X^{\mathscr{H}}\in\Gamma(TZ)$ be such that for any $b\in B$, $X^{\mathscr{H}}|_{Z_b}$ is the Hamiltonian vector field associated with $\mathscr{H}_b=\mathscr{H}|_{Z_b}$. Clearly, $X^{\mathscr{H}}|_{Z_b}\in\Gamma(TZ_b)$ and $X^{\mathscr{H}}|_{\partial Z_b}\in\Gamma(T\partial Z_b)$, which we denote by $X^{\mathscr{H}_b}$ and $X_{\partial Z_b}^{\mathscr{H}_b}$, respectively. Set

$$B(X^{\mathscr{H}})=\{x\in M: X^{\mathscr{H}}(x)=0\}.$$

We assume that $B(X^{\mathscr{H}})\cap\partial M=\emptyset$.

For any $b\in B$ and $v\in TZ_b$, as in refs. [1,2], let $c(v)$ be the Clifford action of v on $\Omega^{0,*}(Z_b,L_b)$ which interchanges $\Omega^{0,\mathrm{even}}(Z_b,L_b)$ and $\Omega^{0,\mathrm{odd}}(Z_b,L_b)$. Let $e_{\dim Z}$ denote the unit inward normal vector field on ∂M. Then for each $b\in B$, the restriction $e_{b,\dim Z}=e_{\dim Z}\big|_{Z_b}$ is the unit inward normal vector field on ∂Z_b. As in ref. [2, (3.5)], for any $b\in B$ and $v\in TZ_b\big|_{\partial Z_b}$, set $\tilde{c}(v)=-c(e_{b,\dim Z})c(v)$. Following ref. [2, (1.19), (1.20)], we define the deformed operators for any $b\in B$ and $T\in\mathbb{R}$:

$$D_{+,T}^{L_b} = D_{+}^{L_b} + \frac{\sqrt{-1}T}{2} c(X^{\mathscr{H}_b}) : \Omega^{0,\text{even}}(Z_b, L_b) \to \Omega^{0,\text{odd}}(Z_b, L_b), \tag{1.8}$$

$$D_{\partial Z_b,+,T}^{L_b} = D_{\partial Z_b,+}^{L_b} + \frac{\sqrt{-1}T}{2} \tilde{c}(X_{\partial Z_b}^{\mathscr{H}_b}) : \Omega^{0,\text{even}}(Z_b, L_b)\,|_{\partial Z_b} \to \Omega^{0,\text{even}}(Z_b, L_b)\,|_{\partial Z_b}. \tag{1.9}$$

We denote the family $\{D_{+,T}^{L_b}\}_{b\in B}$ (resp. $\{D_{\partial Z_b,+,T}^{L_b}\}_{b\in B}$) by $D_{M/B,+,T}^{L}$ (resp. $D_{\partial M/B,+,T}^{L}$).

Since the G-action preserves everything, it commutes with the operators $D_{+,T}^{L_b}$ and $D_{\partial Z_b,+,T}^{L_b}$ respectively. For each $b\in B$, when restricted to the G-invariant part $\Omega^{0,*}(Z_b, L_b)^G$ of $\Omega^{0,*}(Z_b, L_b)$ (resp. $\Omega^{0,*}(Z_b, L_b)^G|_{\partial Z_b}$ of $\Omega^{0,*}(Z_b, L_b)|_{\partial Z_b}$), we get the corresponding G-invariant operator $D_{G,+,T}^{L_b}$ (resp. $D_{\partial Z_b,G,+,T}^{L_b}$). We denote $\{D_{G,+,T}^{L_b}\}_{b\in B}$ and $\{D_{\partial Z_b,G,+,T}^{L_b}\}_{b\in B}$ by $D_{M/B,G,+,T}^{L}$ and $D_{\partial M/B,G,+,T}^{L}$, respectively. In sec. 2 below, we will prove that, for T large enough, the operators $D_{\partial Z_b,G,+,T}^{L_b}$, $b\in B$, are invertible. Hence for sufficiently large T, we can and will choose the Atiyah-Patodi-Singer boundary condition[7] $P_{b,G,+,T,\geqslant 0}$ associated with $D_{\partial Z_b,G,+,T}^{L_b}$ for each operator $D_{G,+,T}^{L_b}$, $b\in B$. Then the family $\{P_{b,G,+,T,\geqslant 0}\}_{b\in B}$ is continuous and provides a spectral section for $D_{\partial M/B,G,+,T}^{L}$ in the sense of ref. [8], which we denote by $P_{G,+,T,\geqslant 0}$. Denote $\{(D_{G,+,T}^{L_b}, P_{b,G,+,T,\geqslant 0})\}_{b\in B}$ by $D_{M/B,G,+,T,APS}^{L}$, which is a continuous family of Fredholm operators and determines an index bundle[8]

$$\operatorname{ind}(D_{M/B,G,+,T,APS}^{L}) \in K(B). \tag{1.10}$$

Now we can state the main result of this section as follows:

Theorem 1.1. There exists $T_0 > 0$ such that, for any $T \geqslant T_0$, $\operatorname{ind}(D_{M/B,G,+,T,APS}^{L}) \in K(B)$ does not depend on T. Moreover, the following identity holds in $K(B)$,

$$\operatorname{ind}(D_{M_G/B,+}^{L_G}) = \operatorname{ind}(D_{M/B,G,+,T,APS}^{L}). \tag{1.11}$$

2 Proof of Theorem 1.1

This section is organized as follows. In subsec. 2.1, we prove the invertibility of the operator $D_{\partial Z_b,G,+,T}^{L_b}$ for sufficiently large $T>0$ and any $b\in B$. In subsec. 2.2, we prove Theorem 1.1 by combining the arguments in refs. [2,3].

2.1 The invertibility of $D_{\partial Z_b,G,+,T}^{L_b}$

As in ref. [2], for each $b\in B$, there exists $\varepsilon_0 > 0$ such that $\partial Z_b \times [0, \varepsilon_0)$ can be identified with some neighborhood of the boundary ∂Z_b by the inward geodesic flow which is perpendicular to ∂Z_b. Let $\{e_{b,1}, \cdots, e_{b,\dim Z-1}, e_{b,\dim Z}\}$ be an orthonormal basis of TZ_b on $\partial Z_b \times [0, \varepsilon_0)$ such that, when restricted to ∂Z_b, $e_{b,\dim Z}$ is the unit inward normal vector field on ∂Z_b and $\{e_{b,1}, \cdots, e_{b,\dim Z-1}\}$ is an oriented orthonormal basis of $T\partial Z_b$. Here ε_0 can be chosen uniformly on B by the compactness of B.

Let $h_1, \cdots, h_{\dim G}$ be an orthonormal basis of $\boldsymbol{g}^*$. Then μ has the expression $\mu = \sum_{i=1}^{\dim G}\mu_i h_i$, where each μ_i is a real function on M. For any $b \in B$, set $\mu_{b,i} = \mu_i|_{Z_b}$ and denote by $V_{b,i}$ the Killing vector field on Z_b induced by the dual of h_i.

Now we recall a Bochner type formula from ref. [2, Theorem 3.3][1)], to which we also refer for the corresponding notation.

Theorem 2.1. For each $b \in B$, the following identity on ∂Z_b holds:

$$(D^{L_b}_{\partial Z_b,+,T})^* D^{L_b}_{\partial Z_b,+,T} = (D^{L_b}_{\partial Z_b,+})^* D^{L_b}_{\partial Z_b,+} + \frac{\sqrt{-1}T}{2}\sum_{j=1}^{\dim Z-1}\sum_{i=1}^{\dim G}\tilde{c}(e_{b,j})\tilde{c}(\nabla^{TZ_b}_{e_{b,j}}(\tilde{\mu}_{b,i}V_{b,i}))$$
$$+ 4\pi T\mathscr{H}_b - \frac{\sqrt{-1}T}{2}\sum_{i=1}^{\dim G}\tilde{c}(e_{b,\dim Z})\tilde{c}(\tilde{\mu}_{b,i}\nabla^{TZ_b}_{e_{b,\dim Z}}V_{b,i}) - \sqrt{-1}T\sum_{i=1}^{\dim G}\mathrm{Tr}[\nabla^{T^{(1,0)}Z_b}(\tilde{\mu}_{b,i}V_{b,i})]$$
$$+ \frac{T}{2}\sum_{i=1}^{\dim G}(\sqrt{-1}\tilde{c}(d^{\partial Z_b}\mu_{b,i})\tilde{c}(V_{b,i}) + |d^{\partial Z_b}\mu_{b,i}|^2) - 2\sqrt{-1}T\sum_{i=1}^{\dim G}\mu_{b,i}L_{V_{b,i}} + \frac{T^2}{4}|X^{\mathscr{H}_b}_{\partial Z_b}|^2. \tag{2.1}$$

From Theorem 2.1, one can proceed as in ref. [2] to get:

Theorem 2.2. There exist $C > 0$, $c > 0$ and $T_0 > 0$ such that for any $b \in B$, $T \geqslant T_0$ and any $s \in \Omega^{0,\mathrm{even}}(Z_b, L_b)^G|_{\partial Z_b}$, the following Sobolev norm estimate holds:

$$\| D^{L_b}_{\partial Z_b,+,T}s\|^2_{\partial Z_b,0} \geqslant C(\|s\|^2_{\partial Z_b,1} + (T-c)\|s\|^2_{\partial Z_b,0}). \tag{2.2}$$

Proof. In ref. [2], (2.2) is proved fiberwisely. As B is compact, one sees easily that the constants C, c and T_0 can be chosen not depending on $b \in B$.

2.2 Proof of Theorem 1.1

For each $b \in B$, we define for $q \geqslant 0$ that $E^q_{b,\pm}$ (resp. $F^q_{b,\pm}$) is the set of sections of $\Lambda^{\mathrm{even/odd}}(T^{(0,1)*}Z_b) \otimes L_b$ (resp. $\Lambda^{\mathrm{even/odd}}(T^{(0,1)*}Z_{G,b}) \otimes L_{G,b}$) over Z_b (resp. over $Z_{G,b}$) which lie in the q-th Sobolev space with the Sobolev q-norm $\|\cdot\|_q$, respectively. We denote the G-invariant parts of $E^q_{b,\pm}$ by $E^{q,G}_{b,\pm}$, respectively. For any $T \geqslant 0$, set

$$E^{1,G}_b(T) = \{s \in E^{1,G}_{b,+} : P_{b,G,+,T,\geqslant 0}s|_{\partial Z_b} = 0\}. \tag{2.3}$$

Then $E^{1,G}_b(T)$ is a Hilbert space with respect to $\|\cdot\|_1$. Set

$$\begin{aligned} D^{L_b}_{G,+,T,P_b \geqslant 0} &= (D^{L_b}_{G,+,T}, P_{b,G,+,T,\geqslant 0}), \\ D^{L}_{M/B,G,+,T,APS} &= \{D^{L_b}_{G,+,T,P_b\geqslant 0}\}_{b\in B}. \end{aligned} \tag{2.4}$$

For any $b \in B$, $D^{L_b}_{G,+,T,P_b\geqslant 0}: E^{1,G}_b(T) \to E^{0,G}_{b,-}$ is a Fredholm operator[7]. By Theorem 2.2, when T is large enough, $D^L_{M/B,G,+,T,APS}$ is a continuous family of Fredholm operators[8].

Recall that we have assumed $\mu^{-1}(0) \cap \partial M = \emptyset$.

Let $U \subset M$ be a sufficiently small G-invariant open neighborhood of $\mu^{-1}(0)$ with $\bar{U} \cap \partial M = \emptyset$. Denote $U_b = U \cap Z_b$ for each $b \in B$. Then as in ref. [2, sec. 4b)], there is a linear map $J_{T,b}: F^q_{b,\pm} \to E^{q,G}_{b,\pm}$ such that for any $u \in \Omega^{0,*}(Z_{G,b}, L_{G,b})$, one has $J_{T,b}u \in \Omega^{0,*}(Z_b,$

1) Note that the extra term $\frac{\sqrt{-1}T}{2}(\sum_{j=1}^{m-1}\pi_{jj})\tilde{c}(X^{\mathscr{H}})$ in ref. [2, Theorem 3.3] has been eliminated here.

$L_b)^G$ with $\mathrm{Supp}\,(J_{T,b}u) \subset U_b$. Let $E^{q,G}_{T,b,\pm}$ be the image of $F^q_{b,\pm}$ in $E^{q,G}_{b,\pm}$ by $J_{T,b}$. Then $J_{T,b}$: $F^0_{b,\pm} \to E^{0,G}_{T,b,\pm}$ is an isometry. Clearly, $E^{q,G}_{T,b,+} \subset E^{q,G}_b(T)$. Let $E^{0,G}_{T,b,\pm,\perp}$ be the orthogonal space to $E^{0,G}_{T,b,\pm}$ in $E^{0,G}_{b,\pm}$. For any $q \geqslant 0$, set

$$E^{q,G}_{b,+,\perp}(T) = E^{q,G}_b(T) \cap E^{0,G}_{T,b,+,\perp}. \tag{2.5}$$

Let $p_{T,b,\pm}, p_{T,b,\pm,\perp}$ be the orthogonal projection operators from $E^{0,G}_{b,\pm}$ to $E^{0,G}_{T,b,\pm}$, $E^{0,G}_{T,b,\pm,\perp}$, respectively. Then we have the following decomposition of $D^{L_b}_{G,+,T,P_b\geqslant 0}$:

$$D^{L_b}_{G,+,T,P_b\geqslant 0} = \sum_{i=1}^{4} D_{b,T,i}, \tag{2.6}$$

where

$$D_{b,T,1} = p_{T,b,-} D^{L_b}_{G,+,T,P_b\geqslant 0} p_{T,b,+}, \qquad D_{b,T,2} = p_{T,b,-} D^{L_b}_{G,+,T,P_b\geqslant 0} p_{T,b,+,\perp},$$
$$D_{b,T,3} = p_{T,b,-,\perp} D^{L_b}_{G,+,T,P_b\geqslant 0} p_{T,b,+}, \qquad D_{b,T,4} = p_{T,b,-,\perp} D^{L_b}_{G,+,T,P_b\geqslant 0} p_{T,b,+,\perp}. \tag{2.7}$$

By ref. [2, Proposition 4.5], we have the following fibrewise estimates:

Proposition 2.3. (i) For any $b \in B$, as $T \to +\infty$,

$$J^{-1}_{T,b} D_{b,T,1} J_{T,b} = D^{L_{G,b}}_{Q,+} + O\left(\frac{1}{\sqrt{T}}\right) : \Omega^{0,\mathrm{even}}(Z_{G,b}, L_{G,b}) \to \Omega^{0,\mathrm{odd}}(Z_{G,b}, L_{G,b}), \tag{2.8}$$

where $D^{L_{G,b}}_{Q,+}$ is the Dirac type operator on $Z_{G,b}$ defined in ref. [1, Definition 3.12].

(ii) There exist $C_{b,1} > 0$, $C_{b,2} > 0$ and $T_{b,0} > 0$ such that for any $T \geqslant T_{b,0}$, any $s \in E^{1,G}_{b,+,\perp}(T)$, $s' \in E^{1,G}_{b,+,T}$, then

$$\| D_{b,T,2} s \|_0 \leqslant C_{b,1}\left(\frac{\| s \|_1}{\sqrt{T}} + \| s \|_0\right), \tag{2.9}$$

$$\| D_{b,T,3} s' \|_0 \leqslant C_{b,1}\left(\frac{\| s' \|_1}{\sqrt{T}} + \| s' \|_0\right), \tag{2.10}$$

and

$$\| D_{b,T,4} s \|_0 \geqslant C_{b,2}\left(\| s \|_1 + \sqrt{T} \| s \|_0\right). \tag{2.11}$$

By the compactness of B, one can construct the $J_{T,b}$'s in such a way that $J_{T,b}$ depends smoothly on $b \in B$. Moreover, one can choose the positive constants $T_{b,0}$, $C_{b,1}$, $C_{b,2}$ in Proposition 2.3 so that they do not depend on $b \in B$.

Now following ref. [3, (2.8)], for any $u \in \mathbb{R}$ and $b \in B$, set

$$D^{L_b}_{G,+,T,P_b\geqslant 0}(u) = D_{b,T,1} + D_{b,T,4} + u(D_{b,T,2} + D_{b,T,3}). \tag{2.12}$$

Since $\overline{U} \cap \partial M = \emptyset$, one verifies easily that $\{D^{L_b}_{G,+,T,P_b\geqslant 0}(u)\}_{b\in B}$ form a continuous B-family of operators which is also continuous with respect to $u \in [0,1]$.

The following lemma plays a key role in our proof of Theorem 1.1.

Lemma 2.4. There exists $T_1 > 0$ such that for any $T \geqslant T_1$, $D^{L_b}_{G,+,T,P_b\geqslant 0}(u)$ is a Fredholm operator for each $b \in B$ and $u \in [0,1]$. Hence, $\{D^{L_b}_{G,+,T,P_b\geqslant 0}(u)\}_{b\in B}$, $0 \leqslant u \leqslant 1$, forms a continuous curve of continuous B-families of Fredholm operators.

Proof. Since B is compact, we only need to show the Fredholm property on each fibre. From the second part of Proposition 2.3, one sees that there exists $C_3>0$ such that for any $T\geqslant T_0$, $u\in[0,1]$ and $s\in E_b^{1,G}(T)$, one has

$$\| D^{L_b}_{G,+,T,P_b\geqslant 0}s - D^{L_b}_{G,+,T,P_b\geqslant 0}(u)s\|_0 \leqslant C_3\left(\frac{\|s\|_1}{\sqrt{T}}+\|s\|_0\right). \tag{2.13}$$

On the other hand, by the Bochner type formula in ref. [2, (2.6)], one deduces easily that there exist $C_4, C_5>0$ such that for any $T\geqslant T_0$,

$$\| D^{L_b}_{G,+,T,P_b\geqslant 0}s\|_0 \geqslant C_4\|s\|_1 - C_5\sqrt{T}\|s\|_0. \tag{2.14}$$

From (2.13) and (2.14), we get

$$\| D^{L_b}_{G,+,T,P_b\geqslant 0}s - D^{L_b}_{G,+,T,P_b\geqslant 0}(u)s\|_0 \leqslant \frac{C_3}{C_4\sqrt{T}}\| D^{L_b}_{G,+,T,P_b\geqslant 0}s\|_0 + (C_3+C_5)\|s\|_0. \tag{2.15}$$

From (2.15) and the Fredholm property of $D^{L_b}_{G,+,T,P_b\geqslant 0}$, we obtain the Fredholm property of $D^{L_b}_{G,+,T,P_b\geqslant 0}(u)$ for sufficiently large T.

Clearly, the index bundle construction in ref. [6] applies well to our continuous families of Fredholm operators. Moreover, the homotopy invariance property for index bundle still holds. Thus by Lemma 2.4, we have the following identity of index bundles:

$$\begin{aligned}\operatorname{ind}\{D^{L_b}_{G,+,T,P_b\geqslant 0}\}_{b\in B} &= \operatorname{ind}\{D^{L_b}_{G,+,T,P_b\geqslant 0}(0)\}_{b\in B} = \operatorname{ind}\{D_{b,T,1}+D_{b,T,4}\}_{b\in B}\\ &= \operatorname{ind}\{D_{b,T,1}\}_{b\in B} + \operatorname{ind}\{D_{b,T,4}\}_{b\in B} \quad \text{in} \quad K(B),\end{aligned} \tag{2.16}$$

where in the last line, each $D_{b,T,1}$ (resp. $D_{b,T,4}$), $b\in B$, is now regarded as a Fredholm operator mapping from $E^{1,G}_{T,b,+}$ (resp. $E^{1,G}_{b,+,\perp}(T)$) to $E^{0,G}_{T,b,-}$ (resp. $E^{0,G}_{T,b,-,\perp}$).

Now by an obvious analogue of the first part of ref. [9, Prop. 9.16], which follows from the second part of Proposition 2.3 and its adjoint analogue, one finds

$$\operatorname{ind}\{D_{b,T,4}\}_{b\in B} = 0 \quad \text{in} \quad K(B). \tag{2.17}$$

From (2.16), (2.17), we have

$$\operatorname{ind}(D^{L}_{M/B,G,+,T,APS}) = \operatorname{ind}\{D^{L_b}_{G,+,T,P_b\geqslant 0}\}_{b\in B} = \operatorname{ind}\{D_{b,T,1}\}_{b\in B} \quad \text{in} \quad K(B). \tag{2.18}$$

By the first part of Proposition 2.3, one also knows that when T is large enough,

$$\operatorname{ind}\{J^{-1}_{T,b}D_{b,T,1}J_{T,b}\}_{b\in B} = \operatorname{ind}\{D^{L_{G,b}}_{Q,+}\}_{b\in B} \quad \text{in} \quad K(B). \tag{2.19}$$

By (2.18), (2.19) and again the homotopy invariance property of the index bundle[6], one obtains for T sufficiently large that

$$\operatorname{ind}(D^{L}_{M_G/B,+}) = \operatorname{ind}\{D^{L_{G,b}}_{Q,+}\}_{b\in B} = \operatorname{ind}(D^{L}_{M/B,G,+,T,APS}) \quad \text{in} \quad K(B), \tag{2.20}$$

which is exactly (2.11). The proof of Theorem 1.1 is completed.

3 Applications to circle actions

In ref. [2, sec.5], an analytic version of a residue formula of Guillemin-Kalkman-Martin[5] is given. In this section, we will generalize it to the family case.

This section is organized in three parts. In subsec. 3.1, we prove a special family quantization formula which holds for any auxiliary bundle. In subsec. 3.2, we apply the trick introduced in ref. [2, sec. 5b)] and also the concept of higher spectral flow in ref. [10] to reduce the calculation of the index bundle on the family of symplectic quotients to a certain kind of invariant higher spectral flow which no longer involves the symplectic conditions. In subsec. 3.3, we identify the above invariant higher spectral flow in a specific situation which implies our family extension of the Guillemin-Kalkman-Martin residue formula.

3.1 A simple family quantization formula for a family of symplectic manifolds with boundary

In this subsection, we replace the line bundle L by a G-equivariant Hermitian vector bundle E over M with a G-equivariant Hermitian connection ∇^E and get the following family extension of Theorem 5.1 in ref. [2].

Theorem 3.1. If $B(X^{\mathscr{H}})\cap(M\setminus\mu^{-1}(0))=\emptyset$, then there exists $T_0\geqslant 0$ such that for any $T\geqslant T_0$,

$$\operatorname{ind}(D^{E_G}_{M_G/B,+})=\operatorname{ind}(D^{E}_{M/B,G,+,T,APS})\qquad\text{in}\qquad K(B).\tag{3.1}$$

Proof. Since now $X^{\mathscr{H}}$ is never zero on $M\setminus\mu^{-1}(0)$, one proceeds fiberwisely as in ref. [2, sec. 5a)] and uses the compactness of B to show that the problem can be localized directly to arbitrary sufficiently small neighborhoods of $\mu^{-1}(0)$. One then proceeds as in sec. 2 to complete the proof of (3.1).

3.2 Specialization to $G=S^1$ case

We now consider the special case where $G=S^1$. Note that μ is now a real function on M. Let V denote the Killing vector field generated by the unit base of the Lie algebra of S^1. Let V_b and $V_{\partial Z_b}$ be the restrictions of V on Z_b and ∂Z_b for each $b\in B$, respectively. Clearly, $V_b\in TZ_b$ and $V_{\partial Z_b}\in T\partial Z_b$.

For any $b\in B$ and $T\in\mathbb{R}$, set

$$\begin{aligned}\widetilde{D}^{E_b}_{G,+,T}&=D^{E_b}_{G,+}+\sqrt{-1}\,Tc(V_b):\Omega^{0,\text{even}}(Z_b,E_b)^G\to\Omega^{0,\text{odd}}(Z_b,E_b)^G,\\ \widetilde{D}^{E_b}_{\partial Z_b,G,+,T}&=D^{E_b}_{\partial Z_b,G,+}+\sqrt{-1}\,T\tilde{c}(V_{\partial Z_b}):\Omega^{0,\text{even}}(Z_b,E_b)^G|_{\partial Z_b}\to\Omega^{0,\text{even}}(Z_b,E_b)^G|_{\partial Z_b}.\end{aligned}\tag{3.2}$$

By proceeding as in subsec. 3.1, when $|T|$ is large enough, we get the invertibility of $\widetilde{D}^{E_b}_{\partial Z_b,G,+,T}$ for each $b\in B$. Thus $\{\widetilde{D}^{E_b}_{\partial Z_b,G,+,T}\}_{b\in B}$ form a continuous family of invertible operators which we denote by $\widetilde{D}^{E}_{\partial M/B,G,+,T}$. We will use the notation $\widetilde{P}_{b,G,+,T,\geqslant 0}$ to denote the Atiyah-Patodi-Singer projection[7] associated with $\widetilde{D}^{E_b}_{\partial Z_b,G,+,T}$ for each $b\in B$. Then the continuous family $\{\widetilde{P}_{b,G,+,T,\geqslant 0}\}_{b\in B}$ is a spectral section in the sense of ref. [8], which we denote by $\widetilde{P}_{G,+,T,\geqslant 0}$. Now we have a continuous family of Fredholm operators

$$\widetilde{D}^{E}_{M/B,G,+,T,APS}=\{(\widetilde{D}^{E_b}_{G,+,T},\widetilde{P}_{b,G,+,T,\geqslant 0})\}_{b\in B}.\tag{3.3}$$

The following family extension of Proposition 5.2 in ref. [2] follows obviously from the compactness of B.

Proposition 3.2. Under the same conditions as in Theorem 3.1 for the $G=S^1$ case, there exists $T_0>0$ such that for any $T\geqslant T_0$,

$$\operatorname{ind}(\widetilde{D}^{E}_{M/B,G,+,\pm T,APS}) = 0 \quad \text{in} \quad K(B). \tag{3.4}$$

Set

$$\hat{D}^{E_b}_{\partial Z_b,G}(u) = (1-u)\widetilde{D}^{E_b}_{\partial Z_b,G,+,T} + uD^{E_b}_{\partial Z_b,G,+,T}, \tag{3.5}$$

$$\check{D}^{E_b}_{\partial Z_b,G}(u) = (1-u)\widetilde{D}^{E_b}_{\partial Z_b,G,+,-T} + uD^{E_b}_{\partial Z_b,G,+,T}, \tag{3.6}$$

$$\hat{D}^{E}_{\partial M/B,G}(u) = \{\hat{D}^{E_b}_{\partial Z_b,G}(u)\}_{b\in B},\ \check{D}^{E}_{\partial M/B,G}(u) = \{\check{D}^{E_b}_{\partial Z_b,G}(u)\}_{b\in B}. \tag{3.7}$$

By a G-invariant version of Theorem 5.2 in ref. [10] and Theorem 3.1, Proposition 3.2, we get the following important consequence which is a family version of Corollary 5.3 in ref. [2].

Corollary 3.3. Under the assumption of Theorem 3.1 and that $G = S^1$, there exists $T_0 > 0$ such that for any $T \geqslant T_0$, one has the following identities in $K(B)$,

$$\begin{aligned}\operatorname{ind}(D^{E_G}_{M_G/B,+}) &= -\operatorname{sf}((\widetilde{D}^{E}_{\partial M/B,G,+,-T},\widetilde{P}_{G,+,-T,\geqslant 0}),(D^{E}_{\partial M/B,G,+,T}, P_{G,+,T,\geqslant 0}))\\ &= -\operatorname{sf}((\widetilde{D}^{E}_{\partial M/B,G,+,T},\widetilde{P}_{G,+,T,\geqslant 0}),(D^{E}_{\partial M/B,G,+,T},P_{G,+,T,\geqslant 0})),\end{aligned} \tag{3.8}$$

where "sf" is the notation of higher spectral flow in the sense of ref. [10].

We now decompose each ∂Z_b into two disjoint parts: $\partial Z_b = (\partial Z_b)_+ \cup (\partial Z_b)_-$, such that $\mu_b|_{(\partial Z_b)_+} > 0$ and $\mu_b|_{(\partial Z_b)_-} < 0$. Set $(\partial M)_\pm = \bigcup_{b\in B}(\partial Z_b)_\pm$.

By Lemma 5.4 of ref. [2] and the compactness of B, one gets:

Lemma 3.4. There exists $T_0 > 0$ such that for any $T \geqslant T_0$ and $u \in [0,1]$, the operator $\hat{D}^{E_b}_{\partial Z_b,G}(u)$ (resp. $\check{D}^{E}_{\partial Z_b,G}(u)$) is invertible on $(\partial Z_b)_+$ (resp. $(\partial Z_b)_-$) for each $b \in B$.

From Corollary 3.3, Lemma 3.4 and the additivity of higher spectral flow, we get the following main result of this subsection which extends Theorem 5.5 in ref. [2].

Theorem 3.5. Under the same condition as Theorem 3.1 and that $G = S^1$, there exists $T_0 > 0$ such that for any $T \geqslant T_0$,

$$\begin{aligned}\operatorname{ind}(D^{E_G}_{M_G/B,+}) &= -\operatorname{sf}_{(\partial M)_+}((\widetilde{D}^{E}_{\partial M/B,G,+,-T},\widetilde{P}_{G,+,-T,\geqslant 0}), (\widetilde{D}^{E}_{\partial M/B,G,+,T},\widetilde{P}_{G,+,T,\geqslant 0}))\\ &= -\operatorname{sf}_{(\partial M)_-}((\widetilde{D}^{E}_{\partial M/B,G,+,T},\widetilde{P}_{G,+,T,\geqslant 0}),(\widetilde{D}^{E}_{\partial M/B,G,+,-T},\widetilde{P}_{G,+,-T,\geqslant 0})).\end{aligned} \tag{3.9}$$

3.3 A relative family index theorem for symplectic quotients of circle actions

In this subsection, we assume again that $G = S^1$ but we no longer assume that the fibration $Z \to M \xrightarrow{\pi} B$ has compact fibres. However, we will assume that the smooth map $\mu: M \to \mathbb{R}$ is proper.

As in ref. [2, sec. 5c)], for any regular value $c \in \mathbb{R}$ of μ as well as μ_b for each $b \in B$, we can construct fibrewise the symplectic quotient $(M_c = \mu_b^{-1}(c)/S^1,\ \omega_{b,c})$. Here for simplicity we also make the assumption that S^1 acts on $\mu^{-1}(c)$ freely.

Let $c_1 < c_2$ be two regular values of μ as well as μ_b for each $b \in B$ and assume that S^1 acts on both $\mu^{-1}(c_1)$ and $\mu^{-1}(c_2)$ freely. Let E_{c_i} be the induced Hermitian vector bundles over M_{c_i}, $i = 1, 2$. In what follows, we will express the difference $\operatorname{ind}(D^{E_{c_2}}_{M_{c_2}/B,G,+}) - \operatorname{ind}(D^{E_{c_1}}_{M_{c_1}/B,G,+})$ through quantities on the S^1-fixed point set in $\mu^{-1}([c_1, c_2])$.

Since μ is proper, one knows that $\mu^{-1}([c_1, c_2])$ is compact. One also knows that $\mu_b^{-1}(c_i)$ $(i=1,2)$ are connected for each $b\in B$ (cf. ref. [11]). Let $F_1(c_1,c_2),\cdots,F_q(c_1,c_2)$ be the connected components of the fixed point set $F(c_1,c_2)$ of the S^1-action in $\mu^{-1}([c_1,c_2])$. In other words, $F_j(c_1,c_2), 1\leqslant j\leqslant q$, are the connected components of the zero set $F(c_1,c_2)$ of the Killing vector field V. Then for each $j=1,2,\cdots,q$, the projection $\pi: M\to B$ induces a fibration $\pi_j: F_j(c_1,c_2)\to B$, with each fibre $F_{j,b}(c_1,c_2)=\pi_j^{-1}(b)$ being one of the connected components of the fixed point set of the S^1-action on Z_b. As in ref. [2, Appendix], to any $F_{j,b}(c_1,c_2)$ one has two natural Dirac-type operators $D^{E_b}_{F_{j,b}(c_1,c_2),+}(V_b)$ and $D^{E_b}_{F_{j,b}(c_1,c_2),+}(-V_b)$. Then for each j, we get two continuous families of Fredholm operators

$$\begin{aligned} D^{E}_{F_j(c_1,c_2),+}(V) &= \{D^{E_b}_{F_{j,b}(c_1,c_2),+}(V_b)\}_{b\in B},\\ D^{E}_{F_j(c_1,c_2),+}(-V) &= \{D^{E_b}_{F_{j,b}(c_1,c_2),+}(-V_b)\}_{b\in B}, \end{aligned} \tag{3.10}$$

which admit the index bundles $\operatorname{ind}(D^{E}_{F_j(c_1,c_2),+}(V))$ and $\operatorname{ind}(D^{E}_{F_j(c_1,c_2),+}(-V))$ in $K(B)$ respectively. These index bundles give the local contributions of $F_j(c_1,c_2)$ to an S^1-invariant index bundle.

For each $j=1,2,\cdots,q$, set $v_{j,\pm}=\dim_{\mathbb{C}} N^{(1,0)}_{j,b,\pm}$ where $N^{(1,0)}_{j,b,\pm}$ is a fibrewise analogue of what in ref. [2, Appendix]. Clearly, $v_{j,\pm}$ are well-defined.

We can now state the main result of this subsection which is the family generalization of Theorem 5.7 in ref. [2].

Theorem 3.6. The following identity holds in $K(B)$,

$$\begin{aligned} &\operatorname{ind}(D^{E_{c_2}}_{M_{c_2}/B,G,+})-\operatorname{ind}(D^{E_{c_1}}_{M_{c_1}/B,G,+})\\ &=\sum_{j=1}^{q}\left((-1)^{v_{j,+}}\operatorname{ind}(D^{E}_{F_j(c_1,c_2),+}(V))-(-1)^{v_{j,-}}\operatorname{ind}(D^{E}_{F_j(c_1,c_2),+}(-V))\right). \end{aligned} \tag{3.11}$$

Proof. As in the proof of ref. [2, Theorem 5.7], we assume that c_1 (resp. c_2) is not the minimal (resp. the maximal) value of each μ_b on Z_b, $b\in B$. Thus there exists a sufficiently small $\varepsilon>0$ such that no critical value of μ as well as μ_b lies in $[c_1-\varepsilon, c_1+\varepsilon]$ (resp. $[c_2-\varepsilon, c_2+\varepsilon]$), and that $\mu^{-1}(c_1-\varepsilon)$ (resp. $\mu^{-1}(c_2+\varepsilon)$) is not empty. By applying Theorem 3.5 to $\mu^{-1}[c_1-\varepsilon, c_1+\varepsilon]$ and $\mu^{-1}[c_2-\varepsilon, c_2+\varepsilon]$, one derives easily the following equalities in $K(B)$ for any $T\geqslant T_0$ with $T_0>0$ large enough,

$$\begin{aligned} \operatorname{ind}(D^{E_{c_1}}_{M_{c_1}/B,G,+}) &= -\operatorname{sf}_{\mu^{-1}(c_1-\varepsilon)}\big((\widetilde{D}^{E}_{\mu^{-1}(c_1-\varepsilon),G,+,T},\widetilde{P}_{G,+,T,\geqslant 0}),\\ &\qquad (\widetilde{D}^{E}_{\mu^{-1}(c_1-\varepsilon),G,+,-T},\widetilde{P}_{G,+,-T,\geqslant 0})\big),\\ \operatorname{ind}(D^{E_{c_2}}_{M_{c_2}/B,G,+}) &= -\operatorname{sf}_{\mu^{-1}(c_2+\varepsilon)}\big((\widetilde{D}^{E}_{\mu^{-1}(c_2+\varepsilon),G,+,-T},\widetilde{P}_{G,+,-T,\geqslant 0}),\\ &\qquad (\widetilde{D}^{E}_{\mu^{-1}(c_2+\varepsilon),G,+,T},\widetilde{P}_{G,+,T,\geqslant 0})\big). \end{aligned} \tag{3.12}$$

Thus one gets

$$\operatorname{ind}(D^{E_{c_2}}_{M_{c_2}/B,G,+})-\operatorname{ind}(D^{E_{c_1}}_{M_{c_1}/B,G,+})$$

$$= - \mathrm{sf}_{\mu^{-1}(c_1-\varepsilon)}\left(\left(\widetilde{D}^E_{\mu^{-1}(c_1-\varepsilon),G,+,-T},\widetilde{P}_{G,+,-T,\geqslant 0}\right),\left(\widetilde{D}^E_{\mu^{-1}(c_1-\varepsilon),G,+,T},\widetilde{P}_{G,+,T,\geqslant 0}\right)\right)$$
$$- \mathrm{sf}_{\mu^{-1}(c_2+\varepsilon)}\left(\left(\widetilde{D}^E_{\mu^{-1}(c_2+\varepsilon),G,+,-T},\widetilde{P}_{G,+,-T,\geqslant 0}\right),\left(\widetilde{D}^E_{\mu^{-1}(c_2+\varepsilon),G,+,T},\widetilde{P}_{G,+,T,\geqslant 0}\right)\right). \quad (3.13)$$

Considering the operator in (3.2) on $\mu_b^{-1}([c_1-\varepsilon, c_2+\varepsilon])$, we have the deformed operator

$$\widetilde{D}^{E_b}_{G,+,T} = D^{E_b}_{G,+} + \sqrt{-1}\,Tc(V_b) : \Omega^{0,\mathrm{even}}(\mu_b^{-1}([c_1-\varepsilon, c_2+\varepsilon]),E_b)^G \to \Omega^{0,\mathrm{odd}}(\mu_b^{-1}([c_1-\varepsilon, c_2+\varepsilon]),E_b)^G, \quad (3.14)$$

and its associated boundary operator

$$\widetilde{D}^{E_b}_{\partial\mu_b^{-1}([c_1-\varepsilon,c_2+\varepsilon]),G,+,T} : \Omega^{0,\mathrm{even}}(\mu_b^{-1}([c_1-\varepsilon, c_2+\varepsilon]),E_b)^G|_{\partial\mu_b^{-1}([c_1-\varepsilon,c_2+\varepsilon])} \to \Omega^{0,\mathrm{even}}(\mu_b^{-1}([c_1-\varepsilon, c_2+\varepsilon]),E_b)^G|_{\partial\mu_b^{-1}([c_1-\varepsilon,c_2+\varepsilon])}. \quad (3.15)$$

One verifies similarly that $\widetilde{D}^{E_b}_{\partial\mu_b^{-1}([c_1-\varepsilon, c_2+\varepsilon]),G,+,T}$ is invertible when $|T|$ is large enough and thus one gets the corresponding continuous family of Fredholm operators[8]

$$\widetilde{D}^E_{G,+,T,APS} = \left\{\left(\widetilde{D}^{E_b}_{G,+,T},\widetilde{P}_{b,G,+,T,\geqslant 0}\right)\right\}_{b\in B}. \quad (3.16)$$

On the other hand, one also has the localization principle corresponding to $\widetilde{D}^E_{G,+,T,APS}$. Then applying the results in ref. [2, sec. 5c)] to the family case, one deduces easily that there exists $T_0>0$ such that for any $T\geqslant T_0$, the following family analogue of (5.21) and (5.22) in ref. [2] hold in $K(B)$ 1):

$$\mathrm{ind}\left(\widetilde{D}^E_{G,+,T,APS}\right) = \sum_{j=1}^q(-1)^{v_{j,+}}\mathrm{ind}\left(D^E_{F_j(c_1,c_2),+}(V)\right), \quad (3.17)$$

$$\mathrm{ind}\left(\widetilde{D}^E_{G,+,-T,APS}\right) = \sum_{j=1}^q(-1)^{v_{j,-}}\mathrm{ind}\left(D^E_{F_j(c_1,c_2),+}(-V)\right). \quad (3.18)$$

From (3.17), (3.18) and Theorem 5.2 in ref. [10], one deduces that

$$\sum_{j=1}^q\left((-1)^{v_{j,+}}\mathrm{ind}\left(D^E_{F_j(c_1,c_2),+}(V)\right) - (-1)^{v_{j,-}}\mathrm{ind}\left(D^E_{F_j(c_1,c_2),+}(-V)\right)\right)$$
$$= - \mathrm{sf}_{\mu^{-1}(c_2-\varepsilon)}\left(\left(\widetilde{D}^E_{\mu^{-1}(c_1-\varepsilon),G,+,-T},\widetilde{P}_{G,+,-T,\geqslant 0}\right),\left(\widetilde{D}^E_{\mu^{-1}(c_1-\varepsilon),G,+,T},\widetilde{P}_{G,+,T,\geqslant 0}\right)\right)$$
$$- \mathrm{sf}_{\mu^{-1}(c_2+\varepsilon)}\left(\left(\widetilde{D}^E_{\mu^{-1}(c_2+\varepsilon),G,+,-T},\widetilde{P}_{G,+,-T,\geqslant 0}\right),\left(\widetilde{D}^E_{\mu^{-1}(c_2+\varepsilon),G,+,T},\widetilde{P}_{G,+,T,\geqslant 0}\right)\right). \quad (3.19)$$

From (3.13) and (3.19), we get (3.11).

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1) Note the corrected factors $(-1)^{v_{j,\pm}}$ which were missing in ref. [2].

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