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Differential Geometry

Witten genus and vanishing results on complete intersections

Genre de Witten et résultats d'annulation sur les intersections completes

Qingtao Chen [a], Fei Han [b,c], Weiping Zhang [d]

[a] *Department of Mathematics, University of Southern California, Los Angeles, CA 90089, USA*
[b] *Department of Mathematics, National University of Singapore, 2, Science Drive 2, Singapore 117543, Singapore*
[c] *Department of Mathematics, Stanford University, Stanford, CA, 94305, USA*
[d] *Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, PR China*

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ABSTRACT

We construct a mod 2 analogue of the Witten genus for $8k+2$ dimensional spin manifolds, as well as modular characteristic numbers for a class of spin^c manifolds which we call string^c manifolds. When these spin^c manifolds are actually spin, one recovers the original Witten genus on string manifolds. These genera vanish on string and string^c complete intersections respectively in complex projective spaces.

RÉSUMÉ

Nous construisons un analogue du genre de Witten pour les variétés spins de dimension $8k+2$. Nous construisons aussi des nombres caractéristiques modulaires sur une classe de variétés spin^c, qu'on appelle variétés cordes^c. Si les variétés cordes^c sont spin, on retrouve le genre de Witten sur les variétés cordes. Ces genres sont nuls sur les intersections complètes correspondantes dans les espaces projectives complexes.

For any closed oriented spin manifold X, there exists a cohomology class $\frac{p_1(TX)}{2}$ in $H^4(X,\mathbf{Z})$ determined by the spin structure on X, twice of which is the first integral Pontryagin class $p_1(TX)$. A closed oriented spin manifold X with $\frac{p_1(TX)}{2}=0$ is called a string manifold. The famous Witten genus $W(X)$ defined in [9, (17)] is an integral modular form over $SL(2,\mathbf{Z})$ on a $4k$ dimensional string manifold X. A well-known result due to Landweber–Stong states that $W(X)=0$ if X is a string complete intersection in a complex projective space (cf. [6, pp. 87–88]).

We present in this Note two extensions of this Landweber–Stong vanishing theorem. The first one concerns a mod 2 analogue of the Witten genus on $8k+2$ dimensional spin manifolds, while the second one concerns an extension of the Witten genus to certain spin^c manifolds (which we call string^c manifolds) such that this generalized Witten genus is an integral modular form over $SL(2,\mathbf{Z})$. The main results in this Note show that these generalized Witten genera vanish on string and string^c complete intersections respectively, extending the Landweber–Stong theorem to the corresponding situations. This indicates that the genera we introduce here are in some sense the "right" extensions of the Witten genus.

Details and further generalizations will appear in [4].

E-mail addresses: qingtaoc@usc.edu (Q. Chen), mathanf@nus.edu.sg, fhan@math.stanford.edu (F. Han), weiping@nankai.edu.cn (W. Zhang).

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1. A mod 2 vanishing theorem for $8k+2$ dimensional string complete intersections

Let $\mathbf{H}$ be the upper half plane. For $\tau\in\mathbf{H}$, set $q=e^{\pi\sqrt{-1}\tau}$.

If E is a real (resp. complex) vector bundle over a manifold, we write $\widetilde{E}=E-\mathbf{R}^{\mathrm{rk}(E)}$ (resp. $\widetilde{E}=E-\mathbf{C}^{\mathrm{rk}(E)}$). Let $S_t(E)$, $\Lambda_t(E)$, with $t\in\mathbf{C}$, denote the total symmetric and exterior powers of E respectively.

Let B be a closed oriented $8k+2$ dimensional spin manifold carrying a fixed spin structure. Let $\Theta(TB)=\bigotimes_{m=1}^{\infty}S_{q^{2m}}(\widetilde{TB})$ be the Witten bundle constructed in [9, (17)]. Set $\phi(B)=\mathrm{ind}_2(\Theta(TB))\in\mathbf{Z}_2[[q]]$, where ind_2 is the mod 2 index in the sense of Atiyah and Singer [2].

Theorem 1.1. *If B is an $8k+2$ $(k\geqslant 1)$ dimensional string complete intersection in a complex projective space, then $\phi(B)=0$.*

Proof. Let B be the complete intersection of r hypersurfaces of degrees $d_1,d_2,\dots,d_r$ in the complex projective space $\mathbf{C}P^{4k+1+r}$. Since B is string and $k\geqslant 1$, one finds

$$\sum_{\alpha=1}^{r}\left(d_\alpha^2-1\right)=4k+2. \tag{1}$$

If each d_α, $1\leqslant\alpha\leqslant r$, is an odd number, then the left-hand side of (1) is divisible by 8, which results in a contradiction. Without loss of generality, we assume that d_r is even.

Let V_r be the complete intersection of $r-1$ hypersurfaces of degrees $d_1,d_2,\dots,d_{r-1}$. Let $i:V_r\hookrightarrow\mathbf{C}P^{4k+r+1}$ denote the corresponding embeddings. Let $\xi=i^*\mathcal{O}(d_r)$. By (1), one finds that $[B]\in H_{8k+2}(V_r,\mathbf{Z})$ is dual to $c_1(\xi)$ and $c_1(\xi)\equiv w_2(TV_r)$ mod 2. One then applies the higher dimensional Rokhlin congruence formula in [10,11] to get

$$\phi(B)=\mathrm{ind}_2\big(\Theta(TB)\big)\equiv\left\langle\widehat{A}(TV_r)\cosh\left(\frac{d_ri^*x}{2}\right)\mathrm{ch}\big(\Theta(T_{\mathbf{C}}V_r-\xi_{\mathbf{R}}\otimes\mathbf{C})\big),[V_r]\right\rangle \mod 2\mathbf{Z}[[q]], \tag{2}$$

where $\xi_{\mathbf{R}}$ denotes the real plane bundle underlying ξ and $x\in H^2(\mathbf{C}P^{4k+r+1},\mathbf{Z})$ is the generator.

On the other hand, one can show that

$$\Theta(T_{\mathbf{C}}V_r-\xi_{\mathbf{R}}\otimes\mathbf{C})\equiv\Theta(T_{\mathbf{C}}V_r)\otimes\left(\bigotimes_{n=1}^{\infty}\Lambda_{q^{2n}}(\widetilde{\xi_{\mathbf{R}}\otimes\mathbf{C}})\right) \mod 2(\widetilde{\xi_{\mathbf{R}}\otimes\mathbf{C}})\cdot K(V_r)[[q]]. \tag{3}$$

By (2) and (3), we have

$$\begin{aligned}\phi(B)&\equiv\left\langle\widehat{A}(TV_r)\cosh\left(\frac{d_ri^*x}{2}\right)\mathrm{ch}\left(\bigotimes_{n=1}^{\infty}\Lambda_{q^{2n}}(\widetilde{\xi_{\mathbf{R}}\otimes\mathbf{C}})\right)\mathrm{ch}\big(\Theta(T_{\mathbf{C}}V_r)\big),[V_r]\right\rangle\\&=\left\langle\left[\frac{x}{2\pi\mathrm{i}\frac{\theta(\frac{x}{2\pi\mathrm{i}},\tau)}{\theta'(0,\tau)}}\right]^{4k+r+2}\left(\prod_{\alpha=1}^{r-1}\frac{2\pi\mathrm{i}\theta\left(\frac{d_\alpha x}{2\pi\mathrm{i}},\tau\right)}{\theta'(0,\tau)}\right)\frac{\theta_1\left(\frac{d_rx}{2\pi\mathrm{i}},\tau\right)}{\theta_1(0,\tau)},\left[\mathbf{C}P^{4k+1+r}\right]\right\rangle \mod 2\mathbf{Z}[[q]],\end{aligned} \tag{4}$$

where θ and θ_1 are the Jacobi theta functions (cf. [3]), and we have used Poincaré duality to deduce the second equality.

Set $g(x)=\prod_{\alpha=1}^{r-1}\big(\frac{2\pi\mathrm{i}\theta(\frac{d_\alpha x}{2\pi\mathrm{i}},\tau)}{\theta'(0,\tau)}\big)\frac{\theta_1(\frac{d_rx}{2\pi\mathrm{i}},\tau)}{\theta_1(0,\tau)}$, $f(x)=\big[\frac{2\pi\mathrm{i}\theta(\frac{x}{2\pi\mathrm{i}},\tau)}{\theta'(0,\tau)}\big]^{4k+r+2}$ and $\omega=\frac{g(x)\,\mathrm{d}x}{f(x)}$. Then $\phi(B)\equiv\mathrm{Res}_{(0)}(\omega)$ mod $2\mathbf{Z}[[q]]$.

By (1), (4), the assumption that d_r is even and the transformation laws of the Jacobi theta functions (cf. [3]), one deduces that $\frac{g(x+2\pi\mathrm{i})}{f(x+2\pi\mathrm{i})}=\frac{g(x)}{f(x)}$ and $\frac{g(x+2\pi\mathrm{i}\tau)}{f(x+2\pi\mathrm{i}\tau)}=\frac{g(x)}{f(x)}$. Thus ω can be viewed as a meromorphic one form defined on a torus, which implies that $\mathrm{Res}_{(0)}(\omega)=0$ as 0 is the only pole of ω on the torus. Hence, $\phi(B)=0$. □

2. A vanishing theorem for string^c complete intersections

Let M be a closed oriented even dimensional spin^c manifold. Let L be the complex line bundle on M associated to the spin^c structure on M and $c=c_1(L)$. We use the notation $L_{\mathbf{R}}$ to denote the real plane bundle underlying L. Let $T_{\mathbf{C}}M=TM\otimes\mathbf{C}$ be the complexification of TM. Set

$$\begin{aligned}\Theta(T_{\mathbf{C}}M,L_{\mathbf{R}}\otimes\mathbf{C})=&\left(\bigotimes_{m=1}^{\infty}S_{q^{2m}}(\widetilde{T_{\mathbf{C}}M})\right)\otimes\left(\bigotimes_{n=1}^{\infty}\Lambda_{q^{2n}}(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\right)\\&\otimes\left(\bigotimes_{r=1}^{\infty}\Lambda_{-q^{2r-1}}(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\right)\otimes\left(\bigotimes_{s=1}^{\infty}\Lambda_{q^{2s-1}}(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\right),\end{aligned} \tag{5}$$

$$\Theta^*(T_{\mathbf{C}}M, L_{\mathbf{R}}\otimes\mathbf{C}) = \left(\bigotimes_{m=1}^{\infty} S_{q^{2m}}(\widetilde{T_{\mathbf{C}}M})\right)\otimes\left(\bigotimes_{n=1}^{\infty}\Lambda_{-q^{2n}}(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\right). \tag{6}$$

Define

$$W_c(M) = \left\langle \widehat{A}(TM)\exp\left(\frac{c}{2}\right)\mathrm{ch}\big(\Theta(T_{\mathbf{C}}M, L_{\mathbf{R}}\otimes\mathbf{C})\big), [M]\right\rangle \tag{7}$$

if $\dim M = 4k$ and

$$W_c(M) = \left\langle \widehat{A}(TM)\exp\left(\frac{c}{2}\right)\mathrm{ch}\big(\Theta^*(T_{\mathbf{C}}M, L_{\mathbf{R}}\otimes\mathbf{C})\big), [M]\right\rangle \tag{8}$$

if $\dim M = 4k+2$.

By a classical theorem of Atiyah and Hirzebruch [1], one sees that $W_c(M)\in\mathbf{Z}[[q]]$. Clearly, when M is actually spin with $c=0$, $W_c(M)$ is exactly the Witten genus of M.

On the other hand, for M as above, $TM\oplus L_{\mathbf{R}}$ is spin. Since $p_1(TM\oplus L_{\mathbf{R}}) = p_1(TM)+c^2$, there exists a class $\frac{p_1(TM)+c^2}{2}$ in $H^4(M,\mathbf{Z})$ determined by the spin^c structure on M, twice of which equals to $p_1(TM)+c^2$ (cf. [8, Lemma 2.2]).

A direct computation shows that (i) if $\dim M = 4k$ and $\frac{p_1(TM)-3c^2}{2} = 0$ (here it is enough to assume that this class vanishes rationally), then $W_c(M)$ is an integral modular form of weight $2k$ over $SL(2,\mathbf{Z})$; (ii) if $\dim M = 4k+2$ and $\frac{p_1(TM)-c^2}{2} = 0$ (again it is enough that this class vanishes rationally), then $W_c(M)$ is an integral modular form of weight $2k$ over $SL(2,\mathbf{Z})$.

In view of the string condition $\frac{p_1(TM)}{2} = 0$, we propose the following definition.

Definition 2.1. Let M be an even dimensional closed oriented spin^c manifold and L denote the associated complex line bundle with $c = c_1(L)$ denoting the first Chern class of L. Then the spin^c manifold M is called a string^c manifold if $\frac{1}{2}(p_1(TM) - (2+(-1)^{\frac{\dim M}{2}})c^2) = 0$ in $H^4(M,\mathbf{Z})$.

With the above terminology, $W_c(M)$ is an integral modular form over $SL(2,\mathbf{Z})$ if M is a string^c manifold.

The construction of $W_c(M)$ is inspired by the mod 2 Witten genus $\phi(B)$ considered in the previous section, as well as the considerations in [5] where the mod 2 elliptic genera are studied in the framework of Rokhlin congruences (cf. (2) and (3)). Moreover, the following vanishing theorem on complete intersections holds for W_c.

Theorem 2.2. *If V is a string^c complete intersection, of dimension greater than or equal to 4, in a complex projective space, then $W_c(V) = 0$.*

Proof. We only prove the case of $\dim V = 4k$. The case of $\dim V = 4k+2$ can be proved similarly.

A direct calculation shows that $W_c(M) = 0$ for any 4 dimensional string^c manifold M. So we assume $\dim V = 4k,\ k>1$. Let V be the intersection of r hypersurfaces of degrees $d_1, d_2, \ldots, d_r$ in the complex projective space $\mathbf{C}P^{2k+r}$. Let $i : V \hookrightarrow \mathbf{C}P^{2k+r}$ denote the embedding.

Since $(p_1(TV) - 3c^2)/2 = 0$ for some $c\in H^2(V,\mathbf{Z})$, by using the Lefschetz hyperplane theorem which implies that $i^* : H^2(\mathbf{C}P^{2k+r},\mathbf{Z})\to H^2(V,\mathbf{Z})$ is an isomorphism for $k>1$, one finds that

$$2k+r+1-\sum_{\alpha=1}^{r} d_\alpha^2 = 3l^2, \tag{9}$$

where l is an integer verifying $c = i^*(lx)$ with $x\in H^2(\mathbf{C}P^{2k+r},\mathbf{Z})$ being the generator.

Then up to a constant scalar $(2\pi\sqrt{-1})^{-2k}$, one has

$$\begin{aligned} W_c(V) &= \left\langle\left(\left(\left[\frac{i^*x}{\frac{\theta(i^*x,\tau)}{\theta'(0,\tau)}}\right]^{2k+r+1}\right)\left(\prod_{\alpha=1}^{r}\left[\frac{d_\alpha i^*x}{\frac{\theta(d_\alpha i^*x,\tau)}{\theta'(0,\tau)}}\right]^{-1}\right)\frac{\theta_1(c,\tau)}{\theta_1(0,\tau)}\frac{\theta_2(c,\tau)}{\theta_2(0,\tau)}\frac{\theta_3(c,\tau)}{\theta_3(0,\tau)}\right), [V]\right\rangle \\ &= \left\langle\left(\left(\left[\frac{x}{\frac{\theta(x,\tau)}{\theta'(0,\tau)}}\right]^{2k+r+1}\right)\left(\prod_{\alpha=1}^{r}\left[\frac{1}{\frac{\theta(d_\alpha x,\tau)}{\theta'(0,\tau)}}\right]^{-1}\right)\frac{\theta_1(lx,\tau)}{\theta_1(0,\tau)}\frac{\theta_2(lx,\tau)}{\theta_2(0,\tau)}\frac{\theta_3(lx,\tau)}{\theta_3(0,\tau)}\right), [\mathbf{C}P^{2k+r}]\right\rangle, \end{aligned} \tag{10}$$

where θ, θ_i, $1\leqslant i\leqslant 3$, are the Jacobi theta functions (cf. [3]) and we have used Poincaré duality to deduce the second equality.

Set $g(x) = (\prod_{\alpha=1}^{r}\frac{\theta(d_\alpha x,\tau)}{\theta'(0,\tau)})\frac{\theta_1(lx,\tau)}{\theta_1(0,\tau)}\frac{\theta_2(lx,\tau)}{\theta_2(0,\tau)}\frac{\theta_3(lx,\tau)}{\theta_3(0,\tau)}$, $f(x) = [\frac{\theta(x,\tau)}{\theta'(0,\tau)}]^{2k+r+1}$ and $\omega = \frac{g(x)\,dx}{f(x)}$. Then up to a constant scalar, $W_c(V) = \mathrm{Res}_{(0)}(\omega)$.

By (9), (10) and the transformation laws of the Jacobi theta functions (cf. [3]), one deduces that $\frac{g(x+1)}{f(x+1)} = \frac{g(x)}{f(x)}$ and $\frac{g(x+\tau)}{f(x+\tau)} = \frac{g(x)}{f(x)}$. Thus ω can be viewed as a meromorphic one form defined on a torus, which implies that $\mathrm{Res}_{(0)}(\omega) = 0$ as 0 is the only pole of ω on the torus. Hence, $W_c(V) = 0$. $\square$

Remark 2.3. If $c = c_1(L)$ is even, then M is spin and the genus $W_c(M)$ appears as the index of certain rigid twisted Dirac operators studied in [7]. By using the method in [7], one can show that the twisted spin^c Dirac operators associated to the genus $W_c(M)$, for general string^c M, are still rigid (in the sense of Witten [9]). We refer to [4] for more details.

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