Analytic localization and Riemann-Roch

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Riemann's legacy after 150 years

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Riemann-Roch

- Quote from the book "Riemann, Topology and Physics" by Monastyrsky :
- "Riemann succeeded in obtaining a most important result, now known as Riemann's inequality : *The number* r *of linearly independent meromorphic functions with poles of order not greater than* n_k *at* m *distinct points* P_k*, (*$k = 1.....m$*) is not less than* $\sum n_k - g + 1$*,where* g *is the genus of the surface.*"

Riemann-Roch

- "In 1864 Gustav Roch (1839-1866), a student of Riemann who also died at an early age, succeeded in strengthening this result. It turns out that

$$r = \sum n_k - g + l + i[a],$$

where $i[a]$ is the number of linearly independent differentials with zeros at the points P_k of order at least n_k and having no poles on the Riemann surface.

This is the famous Riemann-Roch theorem. At the present time numerous multidimensional generalizations of the Riemann-Roch theorem play an important role in various branches of algebraic geometry, analysis, and topology."

Hirzebruch-Riemann-Roch

- E holomorphic vector bundle on an algebraic manifold M
- Arithmetic genus : $\widetilde{\chi}(M,E) = \sum(-1)^q \dim H^{0,q}(M,E)$
- **Theorem** (Hirzebruch 1954)

$$\widetilde{\chi}(M,E) = \langle \mathrm{Td}(TM)\mathrm{ch}(E), [M] \rangle.$$

- $\dim M = 1$ case (for general E) due to Weil (1938)
- Serre (1953) first conjectured that the right hand side should be expressed by Chern classes.
- Method of proof : the cobordism theory of Thom.

Grothendieck-Riemann-Roch

- **Theorem** (Grothendieck 1958) Let X, Y be algebraic manifolds and $f : X \to Y$ is a holomorphic map. Then

$$\mathrm{ch}\,(f_! b) \cdot \mathrm{td}(TY) = f_*(\mathrm{ch}(b) \cdot \mathrm{td}(TX))$$

holds in $H^*(Y, \mathbf{Q})$ for all $b \in K_\omega(X)$.

- Here $K_\omega(\cdot)$ is the Grothendieck K-group of *coherent analytic sheafs*. This idea inspired Atiyah-Hirzebruch to develop (topological) K-theory.

- When $Y = \mathrm{pt.}$, one recovers Hirzebruch-Riemann-Roch.

Differentiable Riemann-Roch of Atiyah-Hirzebruch

- **Theorem** (Atiyah-Hirzebruch 1959) Let $i : X \hookrightarrow Y$ be an embedding between oriented closed manifolds, such that $\dim X - \dim Y$ is even, and $w_2(TX) = i^* w_2(TY)$. Then

$$\underline{\widehat{A}(TY)\mathrm{ch}\,(i_! E) = i_* \left(\widehat{A}(TX)\mathrm{ch}(E) \right)}$$

holds in $H^*(Y, \mathbf{Q})$ for any complex vector bundle E over X.

- $i_! E \in \widetilde{K}(Y)$: the direct image of E.

- Take $Y = S^{2N}$, then X is spin.
- By Bott periodicity, $\left\langle \widehat{A}(TS^{2N})\mathrm{ch}(i_! E), [S^{2N}] \right\rangle \in \mathbf{Z}$.

- **Corollary.** X spin => $\left\langle \widehat{A}(TX)\mathrm{ch}(E), [X] \right\rangle \in \mathbf{Z}$.

Geometric construction of $i_!E$

- Let $\pi : N \to X$ be the normal bundle to X in Y.
- N is spin : $w_2(N) = w_2(TY|_N) - w_2(TX) = 0$
- $S(N) = S_+(N) \oplus S_-(N)$ spinor bundle (for some g^N)
- There exists a complex vector bundle η on X such that $\underline{S_-^*(N) \otimes E \oplus \eta \text{ is a trivial bundle on } X}$
- There exists two complex vector bundles $\xi_+,\ \xi_-$ over Y and $v : \xi_+ \to \xi_-$ such that (i) $\underline{v|_{Y\setminus X} \text{ is invertible}}$; (ii)

$$\xi_\pm|_{N_r} = \pi^*\left(S_\pm^*(N) \otimes E \oplus \eta\right),$$

$$v|_{(x,Z)} = \pi^*\left(\widehat{c}(Z) \otimes \mathrm{Id}_E + \mathrm{Id}_\eta\right) : \pi^*\left(S_+^*(N) \otimes E \oplus \eta\right) \longrightarrow \pi^*\left(S_-^*(N) \otimes E \oplus \eta\right).$$

- $\underline{i_!E = \xi_+ - \xi_-} \in \widetilde{K}(Y)$

Geometric proof via Mathai-Quillen formalism

- Take any metric $g^{\xi} = g^{\xi_+} \oplus g^{\xi_-}$ on the $\mathbf{Z}_2$-graded vector bundle $\xi = \xi_+ \oplus \xi_-$
- $V = v + v^*$.
- $\nabla^{\xi} = \nabla^{\xi_+} + \nabla^{\xi_-}$ Hermitian connection on $\xi = \xi_+ \oplus \xi_-$,

$$\nabla^{\xi_\pm}\Big|_{N_r} = \pi^* \left(\nabla^{S^*_\pm(N) \otimes E + \nabla^\eta} \right)$$

- Mathai-Quillen (1986) (up to rescaling)

$$\lim_{T \to +\infty} \int_Y \widehat{A}\left(R^{TY}\right) \mathrm{tr_s}\left[\exp\left(-\left(\nabla^{\xi} + TV\right)^2\right)\right]$$

$$= \int_X \widehat{A}\left(R^{TX}\right) \exp\left(-\left(\nabla^{E}\right)^2\right).$$

A localization formula for the index of Dirac operators

- E a complex vector bundle over a closed even dim spin manifold X
- Dirac operator $D_+^E : \Gamma(S_+(TX) \otimes E) \to \Gamma(S_-(TX) \otimes E)$
- **Theorem** (Atiyah-Singer 1963)

$$\underline{\operatorname{ind}\left(D_+^E\right) = \left\langle \widehat{A}(TX)\operatorname{ch}(E), [X] \right\rangle}$$

- $i : X \hookrightarrow Y$ an embedding. With Atiyah-Hirzebruch :

$$\operatorname{ind}\left(D_+^E\right) = \operatorname{ind}\left(D_+^{i_! E}\right) \;\left(= \operatorname{ind}\left(D_+^{\xi_+}\right) - \operatorname{ind}\left(D_+^{\xi_-}\right)\right)$$

- A *direct* proof of the embedding index formula will give a new proof of the Atiyah-Singer index formula for Dirac operators. This is what we mean "**analytic localization**"

Analytic localization : from classical to modern era

- The K-theoretic proof of Atiyah-Singer involves the idea of "*localization*" already
- They use pseudodifferential operators, thus is not "*geometric*"
- The *modern era* starts with Witten's analytic proof of Morse inequalities
- The harmonic oscillator comes into the picture!
- Harmonic oscillator on $\mathbf{R}$: $-\frac{d^2}{dx^2} + x^2 - 1$

Witten's analytic proof of Morse inequalities

- $f : M \to \mathbf{R}$, Morse function on closed manifold (M, g^{TM})
- Hodge theory for $H^*(M, \mathbf{R})$: $\quad d + \delta : \Omega^*(M) \to \Omega^*(M)$
- Witten deformation (1982)

$$e^{-Tf} d\, e^{Tf} + e^{Tf} \delta\, e^{-Tf} = \underline{d + \delta + T\widehat{c}(df)}.$$

$$(d + \delta + T\widehat{c}(df))^2 = (d+\delta)^2 + T\,[d+\delta, \widehat{c}(df)] + T^2|df|^2$$

- $[d + \delta, \widehat{c}(df)]$ bounded!
- Outside $B = \{x \in M : df(x) = 0\}$, highly invertible
- Near B, harmonic oscillator
- Morse inequalities : $\#B \geq \dim H^*(M, \mathbf{R})$.

From Witten to Bismut-Lebeau

- Witten : degenerate Morse inequalities of Bott where the set of critical points consists of submanifolds instead of points.
- Harmonic oscillator analysis along the normal directions to submanifolds

- Bismut-Lebeau (1991) : far reaching generalizations to the problem on Quillen metrics for complex immersions
- Essential for Gillet-Soulé's arithmetic Riemann-Roch

- Wide applications : the systematic "analytic localization techniques" developed by Bismut-Lebeau

Simple example : index of Dirac operators

- $i : X \hookrightarrow Y$ between even dimensional spin manifolds
- E complex vector bundle over X
- $i_! E = \xi_+ - \xi_- \in \widetilde{K}(Y)$ the direct image
- $V \in \mathrm{End}(\xi)$ self-adjoint and invertible on $Y \setminus X$

- $\underline{D_T^\xi = D^\xi + TV} : \Gamma(S(TY)\widehat{\otimes}\xi) \to \Gamma(S(TY)\widehat{\otimes}\xi)$

$$\underline{(D_T^\xi)^2 = (D^\xi)^2 + T[D^\xi, V] + T^2V^2}$$

- $[D^\xi, V]$ bounded !
- When $T >> 0$, D_T^ξ highly invertible on $Y \setminus X$
- Near X : harmonic oscillator along normal directions
- Analytic Riemman-Roch for Dirac operators :

$$\mathrm{ind}\left(D_+^E\right) = \mathrm{ind}\left(D_+^{\xi_+}\right) - \mathrm{ind}\left(D_+^{\xi_-}\right)$$

The Atiyah-Singer index theorem

- Recall the Atiyah-Hirzebruch formula :

$$\left\langle \widehat{A}(TX)\mathrm{ch}(E), [X] \right\rangle = \left\langle \widehat{A}(TY)\mathrm{ch}\left(i_! E\right), [Y] \right\rangle$$

- By Bott periodicity : for any ξ on S^{2N},

$$\mathrm{ind}\left(D_+^{\xi}\right) = \left\langle \widehat{A}\left(TS^{2N}\right)\mathrm{ch}(\xi), \left[S^{2N}\right] \right\rangle$$

- Taking $Y = S^{2N}$, one gets

- **Atiyah-Singer index theorem** (1963)

$$\underline{\mathrm{ind}\left(D_+^{E}\right) = \left\langle \widehat{A}(TX)\mathrm{ch}(E), [X] \right\rangle}$$

The η-invariant of Dirac operators

- M : an odd dimensional closed spin manifold
- E : Hermtian vector bundle with a Hermitian connection
- $D^E : \Gamma(S(TM) \otimes E) \to \Gamma(S(TM) \otimes E)$ is self-adjoint
- Following Atiyah-Patodi-Singer (1973), for $\mathrm{Re}(s) >> 0$,

$$\eta(D^E, s) = \sum_{\lambda \in \mathrm{Spec}(D^E)\setminus\{0\}} \frac{\mathrm{sgn}(\lambda)}{\lambda^s}.$$

  Extend to be holomorphic at $s = 0$.

$$\overline{\eta}\left(D^E\right) = \frac{\dim\left(\ker D^E\right) + \eta\left(D^E, 0\right)}{2}$$

- $\overline{\eta}\left(D^E\right) \mod \mathbf{Z}$ depends smoothly on the defining data

A localization formula for η-invariants

- $i : X \hookrightarrow Y$ a (totally geodesic) embedding between odd dimensional closed spin Riemannian manifolds.
- $\pi : N \to X$ the normal bundle to X in Y
- E : Hermtian vector bundle with a Hermitian connection
- $i_! E = \xi_+ - \xi_-$
- **Theorem** (Bismut-Zhang 1993)

$$\overline{\eta}\left(D^E\right) \equiv \overline{\eta}\left(D^{\xi_+}\right) - \overline{\eta}\left(D^{\xi_-}\right) - \int_Y \widehat{A}\left(R^{TY}\right)\gamma^{\xi} \mod \mathbf{Z},$$

where γ^{ξ} is a Chern-Simons current verifying

$$d\gamma^{\xi} = \mathrm{ch}\left(\xi_+, \nabla^{\xi_+}\right) - \mathrm{ch}\left(\xi_-, \nabla^{\xi_-}\right) - \frac{\mathrm{ch}\left(E, \nabla^E\right)}{\widehat{A}\left(R^N\right)}\delta_X.$$

A geometric formula for η-invariants

- As a simple application, take $Y = S^{2N+1}$.
- By Bott periodicity, $\widetilde{K}(S^{2N+1}) = \{0\}$. Thus $\xi_+ - \xi_- = 0$
- $\overline{\eta}(D^{\xi_+}) - \overline{\eta}(D^{\xi_-}) \equiv \{\mathrm{Chern - Simons\ term}\} \mod \mathbf{Z}$

- **Theorem** (Zhang 2005) There exists a Chern-Simons current $\widetilde{\gamma}$ on S^{2N+1} with $\underline{d\widetilde{\gamma} = -\frac{\mathrm{ch}(E,\nabla^E)}{\widehat{A}(R^N)}\delta_X}$, such that

$$\underline{\overline{\eta}\left(D^E\right) \equiv -\int_Y \widehat{A}\left(R^{TY}\right)\widetilde{\gamma} \mod \mathbf{Z}.}$$

- This gives a <u>geometric formula</u> for $\underline{\overline{\eta}(D^E) \mod \mathbf{Z}}$.

Mod 2 index theorem for Dirac operators

- Assume $i : X \hookrightarrow S^{8N+2}$ with $\dim X = 8k+2$, X spin
- E : *real* vector bundle on X with Euclidean connection
- $\frac{1}{2}\dim(\ker D^E) \bmod 2\mathbf{Z}$ is smooth invariant.
  This is the Atiyah-Singer analytic mod 2 index $\mathrm{ind}_2(D^E)$
- Bismut-Zhang localization for η invariant takes the form

$$\mathrm{ind}_2\left(D^E\right) = \mathrm{ind}_2\left(D^{i_!E}\right) \quad \text{in} \quad \mathbf{Z}/2\mathbf{Z} = \mathbf{Z}_2.$$

- Define the Atiyah-Singer mod 2 topological index of E by

$$\mathrm{ind}_2(E) = i_!E \in \widetilde{KO}\left(S^{8N+2}\right) = \mathbf{Z}_2 \text{ (Bott periodicity)}$$

- **Theorem** (Atiyah-Singer 1970) : $\mathrm{ind}_2(D^E) = \mathrm{ind}_2(E)$.

Mod 2 index theorem for pin$^-$ manifolds

- Now assume M^{8k+2} is non-orientable, but verifying

$$w_1(M)^2 + w_2(M) = 0$$

- Call it *pin*$^-$-manifold
- E : real vector bundle on X with Euclidean connection
- Exists a "twisted" Dirac operator

$$D^E : \Gamma\left(\widetilde{S}(TM) \otimes E\right) \to \Gamma\left(\widetilde{S}(TM) \otimes E\right),$$

still self-adjoint

- $\overline{\eta}(D^E)$ mod $2\mathbf{Z}$ smooth invariant (generalized mod 2 index)

Mod 2 index theorem for pin$^-$ manifolds

- Typical example : $\mathbf{R}P^{8k+2}$
- γ the orientation line bundle of $\mathbf{R}P^{8k+2}$
- **Lemma** (Adams 1962) The group $\widetilde{KO}(\mathbf{R}P^{8k+2})$ is an abelian group of order 2^{4k+2} generated by $1-\gamma$.
- No periodicity in k
- Thus, usually one does not embedd M^{8k+2} to $\mathbf{R}P^{8N+2}$ to define a (generalized) mod 2 index

Mod 2 index theorem for pin$^-$ manifolds

- Any element α in $KO(\mathbf{R}P^{8k+2})$ can be written as

$$\alpha = m_\alpha + n_\alpha(1-\gamma), \quad m_\alpha,\ n_\alpha \in \mathbf{Z}, \ \ 0 \leq n_\alpha \leq 2^{4k+2} - 1.$$

- Let $q_{8k+2} : KO(\mathbf{R}P^{8k+2}) \to \mathbf{Z}\{\frac{1}{2^{4k+2}}\}/2\mathbf{Z}$ be the homomorphism defined by

$$q_{8k+2}(\alpha) = \frac{m_\alpha}{2^{4k+2}} + \frac{n_\alpha}{2^{4k+1}} \mod 2\mathbf{Z}.$$

Mod 2 index theorem for pin$^-$ manifolds

- Classifying map $f : M^{8k+2} \to \mathbf{R}P^{8k+2}$ such that
$$f^*(\gamma) = o(TM)$$
- $g : M^{8k+2} \hookrightarrow S^{8N}$ an embedding
- New embedding $h = (f, g) : M \hookrightarrow \mathbf{R}P^{8k+2} \times S^{8N}$
- For a real vector bundle E over a pin$^-$ M^{8k+2},
$$h_! E \in \widetilde{KO}\left(\mathbf{R}P^{8k+2} \times S^{8N}\right) = KO\left(\mathbf{R}P^{8k+2}\right).$$
- **Definition.**
$$\underline{\mathrm{ind}_t(E) = q_{8k+2}\left(h_! E\right)} \quad \in \quad \mathbf{Z}\left\{\frac{1}{2^{4k+2}}\right\}/2\mathbf{Z}.$$

Mod 2 index theorem for pin$^-$ manifolds

- **Theorem** (Zhang 1994, arXiv :1508.02619)

$$\underline{\overline{\eta}\left(D^E\right) \equiv \mathrm{ind}_t(E)} \mod 2\mathbf{Z}.$$

- Proof. By an even dim analogue of Bismut-Zhang, one has

$$\overline{\eta}\left(D^E\right) \equiv \overline{\eta}\left(D^{h_! E}\right) \mod 2\mathbf{Z}.$$

(No Chern-Simons term by dimensional reason)

- Checking on $\widetilde{KO}(\mathbf{R}P^{8k+2} \times S^{8N})$ by Adam's result. Q.E.D.

- (Generalized) mod 2 index for both spin and pin$^-$ manifolds appears recently in the physics theory of "topological insulators" .

Lichnerowicz vanishing theorem

- **Theorem** (Lichnerowicz 1963) If a closed spin manifold M carries a Riemannian metric of positive scalar curvature, then $\widehat{A}(M)=0$.
- Proof. By Lichnerowicz formula, $D^2=-\Delta+\frac{k^{TM}}{4}$. Then using the Atiyah-Singer index theorem,

$$\widehat{A}(M)=\mathrm{ind}(D_+)=0.$$

- By developing a noncommutative Riemann-Roch, Connes proved the following extension of the Lichnerowicz theorem to the case of foliations.

Connes vanishing theorem on foliations

- **Theorem** (Connes 1986) Let (M, F) be a closed oriented foliation such that the integrable subbundle F is spin. If F carries a metric such that the leafwise scalar curvature $k^F > 0$ over M, then $\widehat{A}(M) = 0$.
- Connes outlined a proof by using index theory on foliations and cyclic cohomology, as well as a noncommutative Riemann-Roch strategy.
- Question of Yau (around 1990) : a direct geometric proof?
- Kefeng Liu - Zhang (1999) : important preliminary efforts.
- Positive answer with the following new vanishing results

A new vanishing theorem on foliations

- **Theorem** (Zhang 2015, arXiv : 1508.04503). If we assume M spin instead of F spin, then $\underline{k^F > 0}$ implies $\underline{\widehat{A}(M) = 0}$.
- Proof. Analytic Riemann-Roch on Connes fibration.
- **Corollary.** On T^n, no $g^F > 0$.

  ($\underline{F = TM}$ due to Schoen-Yau and Gromov-Lawson)
- **Open question :** $k^F > 0 => k^{TM} > 0$?
- Positive answer if $\dim M \geq 5$ and M simply connected.
- General case still open.

Ray-Singer analytic torsion

- Origin : find an analytic interpretation of :
- Reidemeister torsion : first topological but not homotopic invariant
- M odd dim closed Riemannian manifold
- $\rho : \pi_1(M) \to GL(N, \mathbf{C})$ representation
- F_ρ : associated flat vector bundle on M, g^{F_ρ}
- For simplicity, assume $H^*(M, F_\rho) = \{0\}$

Ray-Singer analytic torsion

- Hodge theory : $d_\rho + \delta_\rho : \Omega^*(M, F_\rho) \to \Omega^*(M, F_\rho)$
- Hodge Laplacian (all invertible by acyclic assumption)

$$\square_{q,\rho} = d_\rho \delta_\rho + \delta_\rho d_\rho : \Omega^q(M, F_\rho) \to \Omega^q(M, F_\rho)$$

- For $\mathrm{Re}(s) >> 0$,

$$\zeta_{q,\rho}(s) = \sum_{\lambda \in \mathrm{Spec}(\square_{q,\rho})} \frac{1}{\lambda^s}$$

- $\zeta_{q,\rho}(s)$ extends to be holomorphic at $s = 0$

Ray-Singer torsion and Cheeger-Müller theorem

- **Ray-Singer** (1971) analytic torsion T_ρ

$$\log T_\rho = \frac{1}{2}\sum_{q=0}^{\dim M}(-1)^q q\,\frac{d\zeta_{q,\rho}(s)}{ds}\Big|_{s=0}.$$

- T_ρ does not depend on g^{TM}, g^{F_ρ} - smooth invariant

- **Ray-Singer conjecture :** If $\rho : \pi_1(M) \to U(N)$, then

$$\underline{T_\rho = \text{Reidemeister torsion for } \rho.}$$

(proved by **Cheeger-Müller** (1978), by surgery method)

- **Müller** (1991) : $\rho : \pi_1(M) \to SL(N, \mathbf{C})$

An analytic proof via Witten deformation

- **Bismut-Zhang** (1991) : purely analytic proof of the Cheeger-Müller theorem by using Witten deformation

$$\underline{e^{-Tf} d_\rho e^{Tf} + e^{Tf} \delta_\rho e^{-Tf}} : \Omega^*(M, F_\rho) \to \Omega^*(M, F_\rho)$$

for a Morse function f, by letting $T \to +\infty$

- Works for arbitrary $\rho : \pi_1(M) \to GL(N, \mathbf{C})$

- **Theorem** (Bismut-Zhang 1991)

$$\log\left(\frac{T_\rho}{\tau_{\rho,f}}\right) = -\frac{1}{2}\int_M \operatorname{Tr}\left[\left(g^{F_\rho}\right)^{-1} \nabla^{F_\rho} g^{F_\rho}\right] (\nabla f)^* \psi\left(TM, \nabla^{TM}\right),$$

where $\psi(TM, \nabla^{TM})$ is the Mathai-Quillen current on TM.

A simple asymptotic formula for line bundles

- Assume $\rho_p : M \to GL(1, \mathbf{C}) = \mathbf{C}^*$ and the induced flat bundle F_ρ has a flat connection $d + p\omega$ with real one form ω no where zero on M

- Then the Bismut-Zhang formula takes the form

$$\lim_{p \to +\infty} \frac{\log T_{\rho_p}}{p} = \frac{1}{2} \langle [\omega] \, e \left(TM/[\omega] \right), [M] \rangle.$$

- Bismut-Xiaonan Ma-Zhang (2011) : generalization to more general flat vector bundles

- Inspired by Müller (2010) for closed hyperbolic 3-manifolds

An asymptotic formula for analytic torsion

- $P_G \to M$ a flat principal bundle, G a Lie group
- L a positive holomorphic line bundle over a closed Kähler manifold N
- G acts holomorphically on (N, L)
- $q : \mathcal{N} = P_G \times_G N \to M$
- $\underline{F_p = P_G \times_G H^{0,0}(N, L^p)}$ flat vector bundle on M $(p >> 0)$
- (Hirzebruch-Riemann-Roch + Kodaira vanishing)

An asymptotic formula for analytic torsion

- L induces a line bundle $\mathcal{L}$ on $\mathcal{N}$ with $(g^{\mathcal{L}}, \nabla^{\mathcal{L}})$
- Flat connection on P_G induces a splitting

$$T\mathcal{N} = T^H\mathcal{N} \oplus T^V\mathcal{N}$$

- $\omega(\mathcal{L}, g^{\mathcal{L}}) \in \Gamma(q^*T^*M)$ defined by that for $U \in TM$,

$$\omega\left(\mathcal{L}, g^{\mathcal{L}}\right)(U) = \left(g^{\mathcal{L}}\right)^{-1}\left(\nabla^{\mathcal{L}}_{U^H} g^{\mathcal{L}}\right).$$

- **Nondegenerate assumption.**

$$\theta = -\frac{1}{2}\omega\left(\mathcal{L}, g^{\mathcal{L}}\right)$$

nowhere zero on $\mathcal{N}$.

An asymptotic formula for analytic torsion

- **Theorem** (Bismut-Ma-Zhang 2011). Under the N.A.
  (i) $\underline{H^*(M, F_p) = 0 \text{ for } (p >> 0)}$;

  (ii) One has the asymptotic formula

$$\underline{\lim_{p \to +\infty} p^{-\dim N - 1} \log T_{F_p}}$$

$$\underline{= \int_{\mathcal{N}} \theta \left(\widehat{\theta}^* \psi \left(q^* TM, \nabla^{q^* TM} \right) \right) \exp \left(c_1 \left(\mathcal{L}, \nabla^{\mathcal{L}} \right) \right),}$$

  where $\widehat{\theta} \in \Gamma(q^* TM)$ is the dual of θ and ψ is the Mathai-Quillen current in Bismut-Zhang theorem.

- Proof. Index theory + Toeplitz operators on $H^{0,0}(N, L^p)$

Müller's asymptotic formula for hyperbolic 3-manifolds

- M closed hyperbolic 3-manifolds
- $\Gamma \subset SL(2, \mathbf{C})$ discrete, torsion free, cocompact subgroup
- $M = \Gamma \setminus \mathbf{H}^3 = \Gamma \setminus SL(2, \mathbf{C})/SU(2)$
- $\pi_1(M) = \Gamma$
- $\rho : SL(2, \mathbf{C}) \to SL(2, \mathbf{C})$ induces $\underline{\rho_\Gamma : \Gamma \to SL(2, \mathbf{C})}$
- $\rho_{\Gamma,p} = \mathrm{Sym}^p(\rho_\Gamma)$ p-th symmetric power
- F_p on M the associated flat vector bundle

Müller's asymptotic formula for hyperbolic 3-manifolds

- **Theorem** (Müller 2010).

$$\underline{\lim_{p\to+\infty} \frac{\log T_{F_p}}{p^2} = \frac{\mathrm{vol}(M)}{4\pi}},$$

where $\mathrm{vol}(M)$ is the hyperbolic volume of M.

- Müller's proof. Use Selberg trace formula (goes back to Riemann again as Selberg first proved his trace formula for Riemann surfaces).

- Since $\mathrm{Sym}^p(\mathbf{C}^2) = H^{0,0}(\mathbf{C}P^1, \mathcal{O}(1)^p)$, Bismut-Ma-Zhang applies to give a direct geometric proof.

Bismut-Ma-Zhang for hyperbolic 3-manifolds

- Take $a > b > 0$, $\widetilde{F}_p = \mathrm{Sym}^{pa}(\mathbf{C}^2) \otimes \mathrm{Sym}^{pb}(\overline{\mathbf{C}}^2)$

- **Theorem** (Bismut-Ma-Zhang 2011)

$$\lim_{p\to+\infty} \frac{\log T_{\widetilde{F}_p}}{p^3} = \frac{3a^2b - b^3}{12\pi}\mathrm{vol}(M).$$

- Reidemeister torsion determines hyperbolic volume.

- Possible relation with the "volume conjecture" ?

  (Volume conjecture in knot theory :
  colored Jones polynomials determine hyperbolic volume)

Summary : a journey with no end

- From Riemann-Roch
- to Hirzebruch-Riemann-Roch
- to Atiyah-Singer index theorem
- to η-invariant and mod 2 index
- to analytic torsion
- possible relations to volume conjecture
- long journey with no end ...

Thanks!