Géométrie/*Geometry*

Spin^c-manifolds and Rokhlin congruences

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Abstract – We establish a general congruence formula of Rokhlin type for spin^c-manifolds. This result refines the integrality theorems of Atiyah and Hirzebruch [1]. It also extends the previous congruences due to Rokhlin [7], Atiyah-Rees [2], Esnault-Seade-Viehweg [3] and Zhang [10]-[12].

Varietés spin^c et congruences de Rokhlin

Résumé – *Nous établissons une formule de congruence générale du type Rokhlin pour des variétés spin^c. Ce résultat précise le théorème de divisibilité d'Atiyah-Hirzebruch* [1]. *Il étend également des congruences dues à Rokhlin* [7], *Atiyah-Rees* [2], *Esnault-Seade-Viehweg* [3] *et Zhang* ([10]-[12]).

Version française abrégée – Soit K une variété compacte spin de dimension $8k+4$. Soit E un fibré vectoriel orienté réel sur K. Soit $E_{\mathbf{C}}$ la complexification de E. Alors, par un théorème classique d'Atiyah-Hirzebruch [1], le nombre caractéristique

$$\langle \hat{A}(TK)\,\mathrm{ch}\,(E_{\mathbf{C}}),\ [K]\rangle$$

est un entier pair.

D'autre part, si K est une variété compacte connexe orientée non-spin de dimension 4, soit B une sous-variété compacte connexe orientable de dimension 2 de K telle que $[B]\in H_2(K, \mathbf{Z}_2)$ soit duale à la deuxième classe de Stiefel-Whitney $w_2(K)\in H^2(K, \mathbf{Z}_2)$. Rokhlin [7] a établi une formule de congruence pour la signature de K du type

$$\frac{\mathrm{Sign}\,(B.B)-\mathrm{Sign}\,(K)}{8}\equiv\Phi(B)\pmod{2\mathbf{Z}}$$

où B.B désigne l'auto-intersection de B et $\Phi(B)$ est un invariant de cobordisme spin de B associé à (K, B).

Dans cette Note, nous étendons les résultats d'Atiyah-Hirzebruch et de Rokhlin aux variétés spin^c *de dimension supérieure*.

Comme corollaire, nous donnons une formule intrinsèque pour des indices mod 2 des fibrés vectoriels orientables de dimension 2 sur une variété spin de dimension $8k+2$.

In this Note, we establish an extended Rokhlin type congruence formula for spin^c-manifold. This result generalizes the formulas proved in Zhang ([10]-[12]).

Although it turns out that both of the proofs appearing in [11] and [12] can be used to prove this formula, we here present a third proof whose idea goes back to Atiyah and Hirzebruch [1]. From the topological point of view, this proof seems more close to the heart of the problem, in comparing with the cobordism proof in [12].

1. A CONGRUENCE FORMULA FOR SPIN^c-MANIFOLDS. – Let K be a compact connected oriented spin^c-manifold of dimension $8k+4$. Let ξ be a complex line bundle on K such that the formula $c_1(\xi)\equiv w_2(TK)\pmod 2$ holds. We fix a spin structure on $\xi\oplus TK$.

Let B be an $8k+2$ dimensional compact connected orientable submanifold of K such that $[B]\in H_{8k+2}(K, \mathbf{Z})$ is Poincaré dual to $c=c_1(\xi)$. Then B carries an induced spin structure (*cf.* [5]). We call B a c-characteristic submanifold of K.

Note présentée par Jean-Michel BISMUT.

0764-4442/93/03170689 $ 2.00

If F is a real vector bundle on B, we denote by $\operatorname{ind}_2(F)$ the associated mod 2 index of Atiyah and Singer (*cf.* [4], [13]).

Let now E be a real vector bundle over K. Note $E_{\mathbf{C}}$ the complexification of E. Let $i : B \hookrightarrow K$ denote the canonical embedding of B in K.

The main result of this Note can be stated as follows.

THEOREM 1. – *The following identity holds,*

$$(1)\qquad \left\langle \hat{A}(TK)\exp\left(\frac{c}{2}\right)\operatorname{ch}(E_{\mathbf{C}}), [K]\right\rangle \equiv \operatorname{ind}_2(i^* E) \pmod{2\mathbf{Z}}.$$

Proof. – We use ideas of Atiyah-Hirzebruch [1] and Atiyah-Rees [2] to prove (1).

Let m, n be two sufficiently large positive integers. Let $f : K \to \mathbf{C}P^{4m+2}$ be a classifying map of ξ, and let $g : K \hookrightarrow S^{8n}$ be an embedding. Let $h = (f, g) : K \hookrightarrow \mathbf{C}P^{4m+2} \times S^{8n}$ be the induced embedding.

Let γ be the canonical complex line bundle over $\mathbf{C}P^{4m+2}$. Let

$$\pi :\quad \mathbf{C}P^{4m+2} \times S^{8n} \to \mathbf{C}P^{4m+2}$$

be the projection map. Then one has

$$(2)\qquad \xi = f^*(\gamma) = h^* \pi^*(\gamma).$$

Let $j : \mathbf{C}P^{4m+1} \hookrightarrow \mathbf{C}P^{4m+2}$ be the canonical embedding. Clearly $M := \mathbf{C}P^{4m+1} \times S^{8n}$ is a $c_1(\pi^*\gamma)$-characteristic submanifold of $\mathbf{C}P^{4m+2} \times S^{8n}$. Furthermore, by perturbing f and g, we get a transversal intersection $B' = K \cap M$, which is another $c_1(\xi)$-characteristic submanifold of K. Note $i' : B' \hookrightarrow K$, $j' : B' \hookrightarrow M$ the canonical embeddings.

Then by an easy modification of an argument in Ochanine ([5], Section 2.5) and by the spin cobordism invariance of the mod 2 index, one gets

$$(3)\qquad \operatorname{ind}_2(i'^* E) = \operatorname{ind}_2(i^* E).$$

Also, since $K \cap M$ is a transversal intersection, one has the following identification of KO direct images:

$$(4)\qquad (j \times \mathrm{id}_{S^{8n}})^* (h_! E) = j'_!(i'^* E).$$

On the other hand, by the Atiyah-Hirzebruch theorem the Riemann-Roch property for spin^c-manifolds as well as for the mod 2 index, one obtains that

$$(5)\qquad \left\langle \hat{A}(TB)\exp\left(\frac{c}{2}\right)\operatorname{ch}(E_{\mathbf{C}}), [K]\right\rangle$$

$$= \left\langle \hat{A}(T(\mathbf{C}P^{4m+2} \times S^{8n}))\exp\left(\frac{\pi^* c_1(\gamma)}{2}\right)\operatorname{ch}((h_! E)_{\mathbf{C}}), [\mathbf{C}P^{4m+2} \times S^{8n}]\right\rangle,$$

and

$$(6)\qquad \operatorname{ind}_2(i'^* E) = \operatorname{ind}_2(j'_!(i'^* E)).$$

By using (3)-(6), we then reduce (1) to the case where $K = \mathbf{C}P^{4m+2} \times S^{8n}$, $B = \mathbf{C}P^{4m+1} \times S^{8n}$. This in turn, via the Bott periodicity theorem, can be reduced to the case where $K = \mathbf{C}P^{4m+2}$ and $B = \mathbf{C}P^{4m+1}$ for which the validity of (1) has been proved by Atiyah and Rees [3]. □

Remark 2. – Since $KO(\mathbf{C}P^{4m+2})$ has been calculated explicitly (Sanderson [8]), the formula (1) for complex projective spaces can also be verified directly.

Remark 3. – Theorem 1 provides a partial way of calculating the mod 2 index of a real vector bundle over an $8k+2$ dimensional spin manifold, at least when this bundle can be extended through some circle bundle to a spin^c-manifold.

Remark 4. – Special cases of (1) for complex manifolds have been proved in Atiyah-Rees [2] and Esnault-Seade-Viehweg [3].

2. SOME APPLICATIONS. – We state some corollaries of theorem 1.

COROLLARY 5 (Atiyah-Hirzebruch [1]). – *Let* K *be a compact spin manifold of dimension* $8k+4$, E *a real vector bundle over* K, *then* $\langle \hat{A}(TK)\,\mathrm{ch}(E_C), [K]\rangle$ *is an even interger.*

Let (K, B) be a characteristic pair as in Section 1. Let N be the normal bundle to B in K. Note e the Euler class of N.

Let Ξ be an integral power operation on KO.

COROLLARY 6 (Zhang [11]). – *The following identity holds,*

$$(7)\quad \langle \hat{A}(TK)\,\mathrm{ch}(\Xi_C(TK)), [K]\rangle \equiv \mathrm{ind}_2(\Xi(TB\oplus\mathbf{R}^2)) + \left\langle \hat{A}(TB)\,\frac{\mathrm{ch}(\Xi_C(TB\oplus N)) - \cosh(e/2)\,\mathrm{ch}(\Xi_C(TB\oplus\mathbf{R}^2))}{2\sinh(e/2)}, [B]\right\rangle \pmod{2\mathbf{Z}}.$$

Proof. – The formula (7) follows from (1) by setting $E=\Xi(TK\oplus\mathbf{R}^2-\xi)$, where ξ is the complex line bunde over K associated to B. □

The following congruence formula for elliptic genera φ_q (*cf.* [6], [12]) and the Ochanine genus β_q [6] is a direct consequence of corollary 6.

COROLLARY 7 (Zhang [11], [12]). – *The following identity holds,*

$$(8)\quad \langle \varphi_q(TK), [K]\rangle \equiv \beta_q(B) + \left\langle \varphi_q(TB)\,\frac{\tanh(e/2)\,\varphi_q(e)-(e/2)}{e\tanh(e/2)}, [B]\right\rangle \pmod{2\mathbf{Z}[[q]]}.$$

Remark 8. – Recall that in [11], we use an analytic method of calculating the adiabatic limits of η-invariants of Dirac operators on circle bundles to prove corollary 6, while the proof of (8) in [12] is based on a cobordism theoretic method. Both two methods can be modified to prove theorem 1 immediately.

Remark 9. – In [11], we have also proved a congruence formula for the case where B is allowed to be non-orientable. This result has not received a purely topological proof.

By setting $E=\xi$ in (1), we get

COROLLARY 10. – *The following identity holds,*

$$(9)\quad \left\langle \hat{A}(TK)\exp\left(\frac{3c}{2}\right), [K]\right\rangle + \left\langle \hat{A}(TK)\exp\left(\frac{c}{2}\right), [K]\right\rangle \equiv \mathrm{ind}_2(N) \pmod{2\mathbf{Z}}.$$

Example 11. – Let $S^2=\mathbf{C}P^1 \subsetneq \mathbf{C}P^2$ be the canonical embedding, then N is the Hopf bundle H over S^2. By (9), one gets immediately that $\mathrm{ind}_2(H)=1$. Combining with the analytic approach mentioned in remark 8, this provides an explanation of [13], remark 2.4.

Remark 12. – By using theorem 1 for $E=\mathbf{R}$, one sees from (9) that $\mathrm{ind}_2(N)$ can be computed using the Atiyah invariants of some other characteristic submanifolds. This is particularly strange when B is a nonsingular spin complex hypersurface $V^d(4k+1)$ of $\mathbf{C}P^{4k+2}$ $(k>0)$, for which, in view of a result of Stolz [9], the fact that $\mathrm{ind}_2(N)$ is zero or nonzero relies on whether $V^{3d}(4k+1)$ and/or $V^d(4k+1)$ would or would not carry metrics of positive scalar curvature. We refer to [12] for a determination of whether a nonsingular spin complex hypersurface in $\mathbf{C}P^{4k+2}$ can carry a metric of positive scalar curvature.

We conclude this Note with the following generalization of example 11.

COROLLARY 13. — *Let* B *be a compact connected spin manifold of dimension* $8k+2$, *let* N *be a complex line bundle over* B. *Then the following identity holds,*

$$\operatorname{ind}_2(\mathrm{N}) \equiv \langle \hat{\mathrm{A}}(\mathrm{TB})\operatorname{ch}(\mathrm{N}), [\mathrm{B}] \rangle \pmod{2\mathbf{Z}}. \tag{10}$$

Proof. — Clearly N can be extended through the circle bundle associated to itself to a spin^c-manifold K, such that (K, B) is a characteristic pair. The formula (10) then follows easily from (9). □

Remark 14. — It might be interesting to note that according to (10), $\operatorname{ind}_2(\mathrm{N})$ does not depend on the spin structure on B.

Note remise le 13 juillet 1993, acceptée le 2 août 1993.

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