Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2009,**29B**(3):609–612
Acta Mathematica Scientia 数学物理学报
http://actams.wipm.ac.cn

COBORDISM AND ROKHLIN CONGRUENCES*

Dedicated to Professor Wu Wenjun on the occasion of his 90th birthday

Zhang Weiping (张伟平)
Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, China
E-mail: weiping@nankai.edu.cn

Abstract In this paper, we give a cobordism proof of the higher dimensional Rokhlin congruences established in [8].

Key words cobordism; Rokhlin congruence

2000 MR Subject Classification 57R; 58J

1 Introduction

In this paper, we give a cobordism proof of the higher dimensional Rokhlin congruences established in [8]. The proof in [8] is K-theoretic and the proof of some special cases given in [6] is analytic. Here we show that the cobordism method given in [7] (which is never published) for elliptic genus applies also to the general case of [8].

The original Rokhlin congruence formula is about the signature of four dimensional manifolds. Since, as was indicated in Hirzebruch's book [4], that Professor Wu Wen-Tsun was the first one who suggested the correct form of the Signature theorem for four dimensional manifolds (which was proved by Thom and Rokhlin independently), and the first proof of the Signature theorem given in [4] uses essentially the cobordism method, we dedicate this short article to Professor Wu Wen-Tsun for his 90$^{\text{th}}$ birthday.

2 Rokhlin Congruences and a Cobordism Proof

Let K be an $8k+4$ dimensional oriented manifold. Let B be an $8k+2$ dimensional orientable submanifold of K such that $[B] \subset H_{8k+2}(K, \mathbf{Z}_2)$ is dual to the Stiefel-Whitney class $w_2(K)$ (The submanifold B exists if and only if K is a Spin^c-manifold). We fix a spin structure on $K \setminus B$. Then B carries a canonically induced spin structure. Furthermore, the spin cobordism class of the induced spin structures does not depend on the spin structures chosen on $K \setminus B$ (cf. [5]).

Let $i : B \hookrightarrow K$ denote the embedding of B into K.

Let $c \in H^2(M, \mathbf{R})$ be the dual of $[B] \subset H_{8k+2}(K, \mathbf{Z})$.

*Received November 24, 2008. This work was partially supported by MOEC and NSFC

Let E be a real vector bundle over K. Then i^*E is a real vector bundle over the spin manifold B. Let $\mathrm{ind}_2(i^*E)$ be the mod 2 index in the sense of Atiyah–Singer [2] associated to i^*E. It is a spin cobordism invariant.

We can now state our higher dimensional Rokhlin congruence, proved in [8] by a K-theoretic method, as follows.

Theorem 2.1 The following identity holds,

$$\left\langle \widehat{A}(TM)\mathrm{ch}\left(E\otimes \mathbf{C}\right)\exp\left(\frac{c}{2}\right), [K]\right\rangle \equiv \mathrm{ind}_2\left(i^*E\right) \mod 2\mathbf{Z}. \tag{2.1}$$

In what follows, we will give a cobordism proof of this result. The proof uses a trick due to Ochanine [5]. Such a proof of a special case of (2.1) has been given in [7].

We first show that when the normal bundle to B in K is trivial, then Theorem 2.1 holds.

Lemma 2.2 If the normal bundle N to B in K is trivial, then the following identity holds,

$$\left\langle \widehat{A}(TM)\mathrm{ch}\left(E\otimes \mathbf{C}\right), [K]\right\rangle \equiv \mathrm{ind}_2\left(i^*E\right) \mod 2\mathbf{Z}. \tag{2.2}$$

Proof Let N_1 be a tubular manifold of B in K. Then there exists a Riemannian metric g^{TM} such that it is of product nature near ∂N_1 and when restricted to N_1 it is the direct product of the standard g^{TS^1} with a metric g^{TB}. Moreover, there is a Hermitian metric g^E as well as a Hermitian connection ∇^E on E over M such that it is of product nature near N_1 and the restriction of g^E on $N_1 = S^1\times B$ coincides with the one obtained by lifting to N_1 from g^{i^*E}, a Hermitian metric on i^*E.

Take any spin structure on $K\setminus N_1$, let $D^{E\otimes\mathbf{C}}_{K\setminus N_1}$ be the Dirac operator (twisted by the complexification $E\otimes\mathbf{C}$ of E) on $K\setminus N_1$ carrying the Atiyah-Patodi-Singer boundary condition (cf. [1]). Then by the Atiyah-Patodi-Singer index theorem ([1]), one has

$$\mathrm{ind}\left(D^{E\otimes\mathbf{C}}_{K\setminus N_1}\right) = \int_{K\setminus N_1}\widehat{A}\left(TM,\nabla^{TM}\right)\mathrm{ch}\left(E\otimes\mathbf{C}\right) - \overline{\eta}\left(D^{i^*_{N_1}E\otimes\mathbf{C}}\right), \tag{2.3}$$

where $\widehat{A}(TM,\nabla^{TM})$ is the Hirzebruch $\widehat{A}$-form associated to the Levi-Civita connection of g^{TM} (cf. [9]), $\overline{\eta}(D^{i^*_{N_1}E\otimes\mathbf{C}})$ is the reduced η-invariant in the sense of Atiyah-Patodi-Singer [1] of the induced Dirac operator on the boundary, with $i_{N_1}: N_1\hookrightarrow K$ denoting the embedding of N_1 into K.

Now, by dimensional reason, one has

$$\mathrm{ind}\left(D^{E\otimes\mathbf{C}}_{K\setminus N_1}\right)\in 2\mathbf{Z}. \tag{2.4}$$

On the other hand, one verifies easily that

$$\int_{N_1}\widehat{A}\left(TM,\nabla^{TM}\right)\mathrm{ch}\left(E\otimes\mathbf{C}\right) = 0, \tag{2.5}$$

while since $N_1 = S^1\times B$, one verifies directly that

$$\overline{\eta}\left(D^{i^*_{N_1}E\otimes\mathbf{C}}\right) = \frac{\dim\left(\ker\left(D^{i^*E}\right)\right)}{2}, \tag{2.6}$$

where D^{i^*E} is the Dirac operator on B (twisted by i^*E) with respect to the induce spin structure and the metrics and connections on B.

From (2.3)–(2.6), one gets

$$\int_K \widehat{A}\left(TM, \nabla^{TM}\right) \mathrm{ch}\left(E \otimes \mathbf{C}\right) \equiv \frac{\dim\left(\ker\left(D^{i^*E}\right)\right)}{2} \mod 2\mathbf{Z}. \tag{2.7}$$

Now by [2], $\frac{1}{2}\dim(\ker(D^{i^*E})) \bmod 2\mathbf{Z}$ is exactly the analytic definition of $\mathrm{ind}_2(i^*E)$. Combining with (2.7), one gets (2.2).

Now we come to the proof of Theorem 2.1 where we no longer assume that the normal bundle N is trivial.

We denote by the same notation c a closed two form on M which represents the corresponding cohomology class in $H^2(M, \mathbf{R})$. According to [5], there exists an $8k+4$ dimensional oriented closed manifold $P(K) \subset K \times P_1(\mathbf{C})$ satisfying the conditions of Lemma 2.2 such that the induced spin structures on $B \subset P(K)$ are spin cobordant to that on $B \subset K$. Furthermore, there are two closed two forms c_1, c_2 on $K \times P_1(\mathbf{C})$ such that 1), c_1 is the pull-back from $P_1(\mathbf{C})$ to $K \times P_1(\mathbf{C})$ of a form representing the first Chern class of the canonical line bundle over $P_1(\mathbf{C})$; 2), c_2 is the pull-back of c from K to $K \times P_1(\mathbf{C})$; 3), $c_1 + c_2$ is dual to $[P(K)] \in H_{8k+4}(K \times P_1(\mathbf{C}), \mathbf{Q})$.

Let $\pi : K \times P_1(\mathbf{C}) \to K$ denote the obvious projection. Let f be the function defined by

$$f(x) = \frac{x/2}{\sinh(x/2)}.$$

We can now apply a formula in [4, (9.3)] to get

$$\begin{aligned}
&\int_{P(K)} \widehat{A}\left(TP(K)\right) \mathrm{ch}\left(i^*_{P(K)}\pi^*\left(E \otimes \mathbf{C}\right)\right) \\
&= \int_{K \times P_1(\mathbf{C})} \left(c_1 + c_2\right) f\left(c_1 + c_2\right)^{-1} \widehat{A}\left(T\left(K \times P_1(\mathbf{C})\right)\right) \mathrm{ch}\left(\pi^*(E \otimes \mathbf{C})\right) \\
&= \int_K \widehat{A}(TK) \mathrm{ch}\left(E \otimes \mathbf{C}\right) \cosh\left(\frac{c}{2}\right).
\end{aligned} \tag{2.8}$$

Now by Lemma 2.2 and the cobordism invariance of the mod 2 index, one finds

$$\int_{P(K)} \widehat{A}\left(TP(K)\right) \mathrm{ch}\left(i^*_{P(K)}\pi^*\left(E \otimes \mathbf{C}\right)\right) \equiv \mathrm{ind}_2\left(i^*E\right). \tag{2.9}$$

From (2.8), (2.9) and the obvious dimensional reason, one gets (2.1), which completes the proof of Theorem 2.1.

We refer to [6]–[8] and [3] for further information around Theorem 2.1.

Acknowledgments Part of the paper was written while the author was visiting the School of Mathematics of Fudan University during November of 2008. He would like to thank Professor Jiaxing Hong and other members of the School for hospitality.

References

[1] Atiyah M F, Patodi V K, Singer I M. Spectral asymmetry and Riemannian geometry I. Proc Camb Philos Soc, 1975, **77**: 43–69

[2] Atiyah M F, Singer I M. The index of elliptic operators V. Ann of Math, 1971, **93**: 139–149

[3] Han F, Zhang W. Modular invariance, characteristic numbers and η-invariants. J Diff Geom, 2004, **67**: 257–288

[4] Hirzebruch F. Topological Methods in Algebraic Geometry. 3rd Edition. Springer-Verlag, 1966

[5] Ochanine S. Signature module 16, invariants de Kervaire généralisés et nombres caractéristique dans la K-théorie réelle. Mémoires de la Société Mathématique de France, Sér 2, 1981, **5**: 1–142

[6] Zhang W. Circle bundles, adiabatic limits of eta invariants and Rokhlin congruences. Ann Inst Fourier, 1994, **44**: 249–270

[7] Zhang W. Elliptic genera and Rokhlin congruences. Preprint, IHES/M/92/76

[8] Zhang W. Spin^c-manifolds and Rokhlin congruences. C R Acad Sci Paris, Serie I, 1993, **317**: 689–692

[9] Zhang W. Lectures on Chern-Weil Theory and Witten Deformations. Singapore: World Scientific, 2001