

Heat Kernels and the Index Theorems on Even and Odd Dimensional Manifolds*

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Abstract

In this talk, we review the heat kernel approach to the Atiyah-Singer index theorem for Dirac operators on closed manifolds, as well as the Atiyah-Patodi-Singer index theorem for Dirac operators on manifolds with boundary. We also discuss the odd dimensional counterparts of the above results. In particular, we describe a joint result with Xianzhe Dai on an index theorem for Toeplitz operators on odd dimensional manifolds with boundary.

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1. Introduction

As is well-known, the index theorem proved by Atiyah and Singer [AS1] in 1963, which expresses the analytically defined index of elliptic differential operators through purely topological terms, has had a wide range of implications in mathematics as well as in mathematical physics. Moreover, there have been up to now many different proofs of this celebrated result.

The existing proofs of the Atiyah-Singer index theorem can roughly be divided into three categories:

(i) The cobordism proof: this is the proof originally given in [AS1]. It uses the cobordism theory developed by Thom and modifies Hirzebruch's proof of his Signature theorem as well as his Riemann-Roch theorem;

(ii) The K -theoretic proof: this is the proof given by Atiyah and Singer in [AS2]. It modifies Grothendieck's proof of the Hirzebruch-Riemann-Roch theorem and relies on the topological K -theory developed by Atiyah and Hirzebruch. The Bott periodicity theorem plays an important role in this proof;

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(iii) The heat kernel proof: this proof originates from a simple and beautiful formula due to McKean and Singer [MS], and has closer relations with differential geometry as well as mathematical physics. It also lead directly to the important Atiyah-Patodi-Singer index theorem for Dirac operators on manifolds with boundary.

In this article, we will survey some of the developments concerning the heat kernel proofs of various index theorems, including a recent result with Dai [DZ2] on an index theorem for Toeplitz operators on odd dimensional manifolds with boundary.

2. Heat kernels and the index theorems on even dimensional manifolds

We start with a smooth closed oriented $2n$ -dimensional manifold M and two smooth complex vector bundles E, F over M , on which there is an elliptic differential operator between the spaces of smooth sections, $D_+ : \Gamma(E) \rightarrow \Gamma(F)$.

If we equip TM with a Riemannian metric and E, F with Hermitian metrics respectively, then $\Gamma(E)$ and $\Gamma(F)$ will carry canonically induced inner products.

Let $D_- : \Gamma(F) \rightarrow \Gamma(E)$ be the formal adjoint of D_+ with respect to these inner products. Then the index of D_+ is given by

$$\text{ind } D_+ = \dim(\ker D_+) - \dim(\ker D_-). \quad (2.1)$$

It is a topological invariant not depending on the metrics on TM, E and F .

The famous McKean-Singer formula [MS] says that $\text{ind } D_+$ can also be computed by using the heat operators associated to the Laplacians D_-D_+ and D_+D_- . That is, for any $t > 0$, one has

$$\text{ind } D_+ = \text{Tr}[\exp(-tD_-D_+)] - \text{Tr}[\exp(-tD_+D_-)]. \quad (2.2)$$

By introducing the \mathbf{Z}_2 -graded vector bundle $E \oplus F$ and setting $D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$, we can rewrite the difference of the two traces in the right hand side of (2.2) as a single “supertrace” as follows,

$$\text{ind } D_+ = \text{Tr}_s[\exp(-tD^2)], \quad \text{for any } t > 0. \quad (2.2)'$$

Let $P_t(x, y)$ be the smooth kernel of $\exp(-tD^2)$ with respect to the volume form on M . For any $f \in \Gamma(E \oplus F)$, one has

$$\exp(-tD^2) f(x) = \int_M P_t(x, y) f(y) dy. \quad (2.3)$$

In particular,

$$\text{Tr}_s[\exp(-tD^2)] = \int_M \text{Tr}_s[P_t(x, x)] dx. \quad (2.4)$$

Now, for simplicity, we assume that the elliptic operator D is of order one. Then by a standard result, which goes back to Minakshisundaram and Pleijel [MP], one has that when $t > 0$ tends to 0,

$$P_t(x, x) = \frac{1}{(4\pi t)^n} (a_{-n} + a_{-n+1}t + \cdots + a_0 t^n + o_x(t^n)), \quad (2.5)$$

where $a_i \in \text{End}((E \oplus F)_x)$, $i = -n, \dots, 0$.

By (2.2)', (2.4) and (2.5), and by taking $t > 0$ small enough, one deduces that

$$\int_M \text{Tr}_s[a_i] dx = 0, \quad -n \leq i < 0,$$

$$\text{ind } D_+ = \left(\frac{1}{4\pi}\right)^n \int_M \text{Tr}_s[a_0] dx. \quad (2.6)$$

Mckean and Singer conjectured in [MS] that for certain geometric operators, there should be some “fantastic cancellation” so that the following far reaching refinement of (2.6) holds,

$$\text{Tr}_s[a_i] = 0, \quad -n \leq i < 0,$$

and moreover, $\text{Tr}_s[a_0]$ can be calculated simply in the Chern-Weil geometric theory of characteristic classes.

In fact, as a typical example, let M be an even dimensional compact smooth oriented *spin* manifold carrying a Riemannian metric g^{TM} . Let R^{TM} be the curvature of the Levi-Civita connection associated to g^{TM} . Let $S(TM) = S_+(TM) \oplus S_-(TM)$ be the Hermitian vector bundle of (TM, g^{TM}) -spinors, and $D_+ : \Gamma(S_+(TM)) \rightarrow \Gamma(S_-(TM))$ the associated *Dirac* operator.

One then has the formula (cf. [BGV, Chap. 4, 5]),

$$\lim_{t \rightarrow 0} \text{Tr}_s [P_t(x, x)] dx = \left\{ \widehat{A} \left(\frac{R^{TM}}{2\pi} \right) \right\}^{\max} := \left\{ \det^{1/2} \left(\frac{\frac{\sqrt{-1}}{4\pi} R^{TM}}{\sinh \left(\frac{\sqrt{-1}}{4\pi} R^{TM} \right)} \right) \right\}^{\max}, \quad (2.7)$$

which implies the Atiyah-Singer index theorem [AS1] for D_+ :

$$\text{ind } D_+ = \widehat{A}(M) := \int_M \widehat{A} \left(\frac{R^{TM}}{2\pi} \right). \quad (2.8)$$

A result of type (2.7) is called a *local index theorem*. The first proof of such a local result was given by V. K. Patodi [P] for the de Rham-Hodge operator $d + d^*$. Other direct heat kernel proofs of (2.7) have been given by Berline-Vergne, Bismut, Getzler and Yu respectively. We refer to [BGV] and [Yu] for more details.

The heat kernel proof of the local index theorem leads to a generalization of the index theorem for Dirac operators to the case of manifolds with boundary. This was achieved by Atiyah, Patodi and Singer in [APS], and will be reviewed in the next section.

3. The index theorem for Dirac operators on even dimensional manifolds with boundary

Let M be a smooth compact oriented even dimensional *spin* manifold with (nonempty) smooth boundary ∂M . Then ∂M is again oriented and *spin*.

Let g^{TM} be a metric on TM . Let $g^{T\partial M}$ be its restriction on $T\partial M$. We assume for simplicity that g^{TM} is of *product structure* near the boundary ∂M . Let $S(TX) = S_+(TX) \oplus S_-(TX)$ be the \mathbf{Z}_2 -graded Hermitian vector bundle of (TX, g^{TX}) -spinors.

Since now M has a nonempty boundary ∂M , the associated Dirac operator $D_+ : \Gamma(S_+(TM)) \rightarrow \Gamma(S_-(TM))$ is *not* elliptic. To get an elliptic problem, one needs to introduce an elliptic boundary condition for D_+ , and this was achieved by Atiyah, Patodi and Singer in [APS]. It is remarkable that this boundary condition, to be described right now, is *global* in nature.

First of all, the Dirac operator D_+ induces canonically a formally self-adjoint first order elliptic differential operator

$$D_{\partial M} : \Gamma(S_+(TM)|_{\partial M}) \rightarrow \Gamma(S_+(TM)|_{\partial M}),$$

which is called the induced Dirac operator on the boundary ∂M .

Clearly, the L^2 -completion of $S_+(TM)|_{\partial M}$ admits an orthogonal decomposition

$$L^2(S_+(TM)|_{\partial X}) = \bigoplus_{\lambda \in \text{Spec}(D_{\partial M})} E_\lambda, \tag{3.1}$$

where E_λ is the eigenspace of λ .

Let $L^2_{\geq 0}(S_+(TM)|_{\partial M})$ denote the direct sum of the eigenspaces E_λ associated to the eigenvalues $\lambda \geq 0$. Let $P_{\geq 0}$ denote the orthogonal projection from $L^2(S_+(TM)|_{\partial M})$ to $L^2_{\geq 0}(S_+(TM)|_{\partial M})$. We call $P_{\geq 0}$ the *Atiyah-Patodi-Singer projection* associated to $D_{\partial M}$, to emphasize its role in [APS].

Then by [APS], the boundary problem

$$(D_+, P_{\geq 0}) : \{u : u \in \Gamma(S_+(TM)), P_{\geq 0}(u|_{\partial M}) = 0\} \rightarrow \Gamma(S_-(TM)), \tag{3.2}$$

is Fredholm. We call this elliptic boundary problem the Atiyah-Patodi-Singer boundary problem associated to D_+ . We denote by $\text{ind}(D_+, P_{\geq 0})$ the index of the Fredholm operator (3.2).

The Atiyah-Patodi-Singer index theorem *The following identity holds,*

$$\text{ind}(D_+, P_{\geq 0}) = \int_M \widehat{A}\left(\frac{R^{TM}}{2\pi}\right) - \overline{\eta}(D_{\partial M}). \tag{3.3}$$

The boundary correction term $\overline{\eta}(D_{\partial M})$ appearing in the right hand side of (3.3) is a spectral invariant associated to the induced Dirac operator $D_{\partial M}$ on ∂M . It is defined as follows: for any complex number $s \in \mathbf{C}$ with $\text{Re}(s) > \dim M$, define

$$\eta(D_{\partial M}, s) = \sum_{\lambda \in \text{Spec}(D_{\partial M})} \frac{\text{sgn}(\lambda)}{|\lambda|^s}. \tag{3.4}$$

By using the heat kernel method, one can show easily that $\eta(D_{\partial M}, s)$ can be extended to a meromorphic function on \mathbf{C} , which is holomorphic at $s = 0$. Following [APS], we then define

$$\bar{\eta}(D_{\partial M}) = \frac{\dim(\ker D_{\partial M}) + \eta(D_{\partial M}, 0)}{2} \quad (3.5)$$

and call it the (reduced) eta invariant of $D_{\partial M}$.

The eta invariants of Dirac operators have played important roles in many aspects of topology, geometry and mathematical physics.

In the next sections, we will discuss the role of eta invariants in the heat kernel approaches to the index theorems on odd dimensional manifolds.

4. Heat kernels and the index theorem on odd dimensional manifolds

Let M be now an *odd* dimensional smooth closed oriented *spin* manifold. Let g^{TM} be a Riemannian metric on TM and $S(TM)$ the associated Hermitian vector bundle of (TM, g^{TM}) -spinors.¹ In this case, the associated *Dirac* operator $D : \Gamma(TM) \rightarrow \Gamma(TM)$ is (formally) *self-adjoint*.² Thus, one can proceed as in Section 3 to construct the Atiyah-Patodi-Singer projection

$$P_{\geq 0} : L^2(S(TM)) \rightarrow L^2_{\geq 0}(S(TM)).$$

Now consider the trivial vector bundle \mathbf{C}^N over M . We equip \mathbf{C}^N with the canonical trivial metric and connection. Then $P_{\geq 0}$ extends naturally to an orthogonal projection from $L^2(S(TM) \otimes \mathbf{C}^N)$ to $L^2_{\geq 0}(S(TM) \otimes \mathbf{C}^N)$ by acting as identity on \mathbf{C}^N . We still denote this extension by $P_{\geq 0}$.

On the other hand, let

$$g : M \rightarrow U(N)$$

be a smooth map from M to the unitary group $U(N)$. Then g can be interpreted as automorphism of the trivial complex vector bundle \mathbf{C}^N . Moreover g extends naturally to an action on $L^2(S(TM) \otimes \mathbf{C}^N)$ by acting as identity on $L^2(S(TM))$. We still denote this extended action by g .

With the above data given, one can define a *Toeplitz* operator T_g as follows,

$$T_g = P_{\geq 0} g P_{\geq 0} : L^2_{\geq 0}(S(TM) \otimes \mathbf{C}^N) \longrightarrow L^2_{\geq 0}(S(TM) \otimes \mathbf{C}^N). \quad (4.1)$$

The first important fact is that T_g is a Fredholm operator. Moreover, it is equivalent to an elliptic pseudodifferential operator of order zero. Thus one can compute its index by using the Atiyah-Singer index theorem [AS2], as was indicated in the paper of Baum and Douglas [BD], and the result is

$$\text{ind } T_g = - \left\langle \widehat{A}(TM) \text{ch}(g), [M] \right\rangle, \quad (4.2)$$

¹Since now M is of odd dimension, the bundle of spinors does not admit a \mathbf{Z}_2 -graded structure.

²In fact, if M bounds an even dimensional spin manifold, then D can be thought of as the induced Dirac operator on boundary appearing in the previous section.

where $\text{ch}(g)$ is the odd Chern character associated to g .

There is also an analytic proof of (4.2) by using heat kernels. For this one first applies a result of Booss and Wojciechowski (cf. [BW]) to show that the computation of $\text{ind } T_g$ is equivalent to the computation of the spectral flow of the linear family of self-adjoint elliptic operators, acting on $\Gamma(S(TM) \otimes \mathbf{C}^N)$, which connects D and gDg^{-1} . The resulting spectral flow can then be computed by variations of η -invariants, where the heat kernels are naturally involved.

The above ideas have been extended in [DZ1] to give a heat kernel proof of a family extension of (4.2).

5. An index theorem for Toeplitz operators on odd dimensional manifolds with boundary

In this section, we describe an extension of (4.2) to the case of manifolds with boundary, which was proved recently in my paper with Xianzhe Dai [DZ2]. This result can be thought of as an odd dimensional analogue of the Atiyah-Patodi-Singer index theorem described in Section 3.

This section is divided into three subsections. In Subsection 4.1, we extend the definition of Toeplitz operators to the case of manifolds with boundary. In Subsection 4.2, we define an η -invariant for cylinders which will appear in the statement of the main result to be described in Subsection 4.3.

5.1. Toeplitz operators on manifolds with boundary

Let M be an odd dimensional oriented *spin* manifold with (nonempty) boundary ∂M . Then ∂M is also oriented and spin. Let g^{TM} be a Riemannian metric on TM such that it is of product structure near the boundary ∂M . Let $S(TM)$ be the Hermitian bundle of spinors associated to (M, g^{TM}) . Since $\partial M \neq \emptyset$, the Dirac operator $D : \Gamma(S(TM)) \rightarrow \Gamma(S(TM))$ is no longer elliptic. To get an elliptic operator, one needs to impose suitable boundary conditions, and it turns out that again we will adopt the boundary conditions introduced by Atiyah, Patodi and Singer [APS].

Let $D_{\partial M} : \Gamma(S(TM)|_{\partial M}) \rightarrow \Gamma(S(TM)|_{\partial M})$ be the canonically induced Dirac operator on the boundary ∂M . Then $D_{\partial M}$ is elliptic and (formally) self-adjoint. For simplicity, we assume here that $D_{\partial M}$ is *invertible*, that is, $\ker D_{\partial M} = 0$.

Let $P_{\partial M, \geq 0}$ denote the Atiyah-Patodi-Singer projection from $L^2(S(TM)|_{\partial M})$ to $L^2_{\geq 0}(S(TM)|_{\partial M})$. Then $(D, P_{\partial M, \geq 0})$ forms a *self-adjoint* elliptic boundary problem. We will also denote the corresponding elliptic self-adjoint operator by $D_{P_{\partial M, \geq 0}}$.

Let $L^2_{P_{\partial M, \geq 0}, \geq 0}(S(TM))$ be the space of the direct sum of eigenspaces of non-negative eigenvalues of $D_{P_{\partial M, \geq 0}}$. Let $P_{P_{\partial M, \geq 0}, \geq 0}$ denote the orthogonal projection from $L^2(S(TM))$ to $L^2_{P_{\partial M, \geq 0}, \geq 0}(S(TM))$.

Now let \mathbf{C}^N be the trivial complex vector bundle over M of rank N , which carries the trivial Hermitian metric and the trivial Hermitian connection. We extend $P_{P_{\partial M, \geq 0}, \geq 0}$ to act as identity on \mathbf{C}^N .

Let $g : M \rightarrow U(N)$ be a smooth unitary automorphism of \mathbf{C}^N . Then g extends to an action on $S(TM) \otimes \mathbf{C}^N$ by acting as identity on $S(TM)$.

Since g is unitary, one verifies easily that the operator $gP_{\partial M, \geq 0}g^{-1}$ is an orthogonal projection on $L^2((S(TM) \otimes \mathbf{C}^N)|_{\partial M})$, and that $gP_{\partial M, \geq 0}g^{-1} - P_{\partial M, \geq 0}$ is a pseudodifferential operator of order less than zero. Moreover, the pair $(D, gP_{\partial M, \geq 0}g^{-1})$ forms a *self-adjoint* elliptic boundary problem. We denote its associated elliptic self-adjoint operator by $D_{gP_{\partial M, \geq 0}g^{-1}}$.

Let $L^2_{gP_{\partial M, \geq 0}g^{-1}, \geq 0}(S(TM) \otimes \mathbf{C}^N)$ be the space of the direct sum of eigenspaces of nonnegative eigenvalues of $D_{gP_{\partial M, \geq 0}g^{-1}}$. Let $P_{gP_{\partial M, \geq 0}g^{-1}, \geq 0}$ denote the orthogonal projection from $L^2(S(TM) \otimes \mathbf{C}^N)$ to $L^2_{gP_{\partial M, \geq 0}g^{-1}, \geq 0}(S(TM) \otimes \mathbf{C}^N)$.

Clearly, if $s \in L^2(S(TM) \otimes \mathbf{C}^N)$ verifies $P_{\partial M, \geq 0}(s|_{\partial M}) = 0$, then gs verifies

$$gP_{\partial M, \geq 0}g^{-1}((gs)|_{\partial M}) = 0.$$

Definition 5.1 The *Toeplitz operator* T_g is defined by

$$T_g = P_{gP_{\partial M, \geq 0}g^{-1}, \geq 0}gP_{\partial M, \geq 0} : \\ L^2_{P_{\partial M, \geq 0}, \geq 0}(S(TM) \otimes \mathbf{C}^N) \rightarrow L^2_{gP_{\partial M, \geq 0}g^{-1}, \geq 0}(S(TM) \otimes \mathbf{C}^N).$$

One verifies that T_g is a Fredholm operator. The main result of this section evaluates the index of T_g by more geometric quantities.

5.2. An η -invariant associated to g

We consider the cylinder $[0, 1] \times \partial M$. Clearly, the restriction of g on ∂M extends canonically to this cylinder.

Let $D|_{[0, 1] \times \partial M}$ be the restriction of D on $[0, 1] \times \partial M$. We equip the boundary condition $P_{\partial M, \geq 0}$ at $\{0\} \times \partial M$ and the boundary condition $\text{Id} - gP_{\partial M, \geq 0}g^{-1}$ at $\{1\} \times \partial M$. Then $(D|_{[0, 1] \times \partial M}, P_{\partial M, \geq 0}, \text{Id} - gP_{\partial M, \geq 0}g^{-1})$ forms a self-adjoint elliptic boundary problem. We denote the corresponding elliptic self-adjoint operator by $D_{P_{\partial M, \geq 0}, gP_{\partial M, \geq 0}g^{-1}}$.

Let $\eta(D_{P_{\partial M, \geq 0}, gP_{\partial M, \geq 0}g^{-1}}, s)$ be the η -function of $s \in \mathbf{C}$ which, when $\text{Re}(s) \gg 0$, is defined by

$$\eta\left(D_{P_{\partial M, \geq 0}, gP_{\partial M, \geq 0}g^{-1}}, s\right) = \sum_{\lambda \neq 0} \frac{\text{sgn}(\lambda)}{|\lambda|^s},$$

where λ runs through the nonzero eigenvalues of $D_{P_{\partial M, \geq 0}, gP_{\partial M, \geq 0}g^{-1}}$.

It is proved in [DZ2] that under our situation, $\eta(D_{P_{\partial M, \geq 0}, gP_{\partial M, \geq 0}g^{-1}}, s)$ can be extended to a meromorphic function on \mathbf{C} which is holomorphic at $s = 0$.

Let $\bar{\eta}(D_{P_{\partial M, \geq 0}, gP_{\partial M, \geq 0}g^{-1}})$ be the reduced η -invariant defined by

$$\bar{\eta}\left(D_{P_{\partial M, \geq 0}, gP_{\partial M, \geq 0}g^{-1}}\right) \\ = \frac{\dim \ker\left(D_{P_{\partial M, \geq 0}, gP_{\partial M, \geq 0}g^{-1}}\right) + \eta\left(D_{P_{\partial M, \geq 0}, gP_{\partial M, \geq 0}g^{-1}}\right)}{2}.$$

5.3. An index theorem for T_g

Let ∇^{TM} be the Levi-Civita connection associated to the Riemannian metric g^{TM} . Let $R^{TM} = (\nabla^{TM})^2$ be the curvature of ∇^{TM} . Also, we use d to denote the trivial connection on the trivial vector bundle \mathbf{C}^N over M . Then $g^{-1}dg$ is a $\Gamma(\text{End}(\mathbf{C}^N))$ valued 1-form over M .

Let $\text{ch}(g, d)$ denote the odd Chern character form (cf. [Z]) of (g, d) defined by

$$\text{ch}(g, d) = \sum_{n=0}^{(\dim M - 1)/2} \frac{n!}{(2n+1)!} \left(\frac{1}{2\pi\sqrt{-1}} \right)^{n+1} \text{Tr} \left[(g^{-1}dg)^{2n+1} \right].$$

Let \mathcal{P}_M denote the Calderón projection associated to D on M (cf. [BW]). Then \mathcal{P}_M is an orthogonal projection on $L^2((S(TM) \otimes \mathbf{C}^N)|_{\partial M})$, and that $\mathcal{P}_M - P_{\partial M, \geq 0}$ is a pseudodifferential operator of order less than zero.

Let $\tau_\mu(P_{\partial M, \geq 0}, gP_{\partial M, \geq 0}g^{-1}, \mathcal{P}_M) \in \mathbf{Z}$ be the Maslov triple index in the sense of Kirk and Lesch [KL, Definition 6.8].

We can now state the main result of [DZ2], which generalizes an old result of Douglas and Wojciechowski [DoW], as follows.

Theorem 5.2 *The following identity holds,*

$$\begin{aligned} \text{ind } T_g = & - \int_M \widehat{A} \left(\frac{R^{TM}}{2\pi} \right) \text{ch}(g, d) + \overline{\eta} \left(D_{P_{\partial M, \geq 0}, gP_{\partial M, \geq 0}g^{-1}} \right) \\ & - \tau_\mu \left(P_{\partial M, \geq 0}, gP_{\partial M, \geq 0}g^{-1}, \mathcal{P}_M \right). \end{aligned}$$

The following immediate consequence is of independent interests.

Corollary 5.3 *The number*

$$\int_M \widehat{A} \left(\frac{R^{TM}}{2\pi} \right) \text{ch}(g, d) - \overline{\eta} \left(D_{P_{\partial M, \geq 0}, gP_{\partial M, \geq 0}g^{-1}} \right)$$

is an integer.

The strategy of the proof of Theorem 5.2 given in [DZ2] is the same as that of the heat kernel proof of (4.2). However, due to the appearance of the boundary ∂M , one encounters new difficulties. To overcome these difficulties, one makes use of the recent result on the splittings of η invariants (cf. [KL]) as well as some ideas involved in the Connes-Moscovici local index theorem in noncommutative geometry [CM] (see also [CH]). Moreover, the local index calculations appearing near ∂M is highly nontrivial. We refer to [DZ2] for more details.

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