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A Remark on a Residue Formula of Bott

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Abstract. We present in this note a simple proof of the Bott residue theorem in a slightly more general form.

§0. Introduction

Let M be an n-dimensional compact complex manifold, V a holomorphic vector field on M. Inspired by a paper of Witten [8], Liu Kefeng [6] introduced the operator

$$\bar{\partial}_s = \bar{\partial} + si(V), \quad s > 0$$

and considered explicitly the complex

$$0 \to A^{-n} \to A^{-n+1} \to \cdots \to A^0 \to \cdots \to A^n \to 0, \tag{0.1}$$

where

$$A^k = \bigoplus_{q-p=k} A^{p,q} \tag{0.2}$$

and the well-known fact for holomorphic vector field

$$\bar{\partial} i(V) + i(V)\bar{\partial} = 0 \tag{0.3}$$

was used.

If M is a Riemannian manifold, the analogous operator is $d + i(V)$, where V is a Killing vector field. The associated complex especially the cohomology is called equivariant cohomology. Furthermore, fairly complete localization formulas have been obtained by Duistermaat-Heckman [5] and Berline-Vergne [1].

Our main purpose here is to prove a complex analogue of these localization formulas for the $\bar{\partial}_s$-cohomology. An idea of Bismut [2] is used.

When V has only (nondegenerate) isolated zeros, this result was contained implicitly in Liu [6] and reproved in [7]. In this note we deal with the more general case where the zero set of V is allowed to be complex submanifolds also with some nondegenerate conditions. In this version our result is a generalization of the Bott residue formula [3], cf. §3.

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§ 1. The Residue Formula

Let V be a holomorphic vector field on the n-dimensional compact complex manifold M. Then V induces a bracket action on the holomorphic tangent bundle TM:

$$\theta : W \longmapsto [V, W], \ W \in \Gamma(TM). \tag{1.1}$$

Let $X = \{p \in M : V(p) = 0\}$ be the zero set of V. On any component N of X, it is easy to see that $\theta|_N$ induces an endomorphism

$$\theta|_N : TM|_N \longmapsto TM|_N, \tag{1.2}$$

where the linearity comes from the fact that $V|_N = 0$.

In this paper, we assume further that N satisfies the following two conditions:

(1.3) N is a complex submanifold;

(1.4) The homomorphism

$$\theta^{\nu}|_N : TM/TN \longmapsto TM/TN$$

on the normal bundle of N in M induced by $\theta|_N$ is an isomorphism. Under these conditions we call N a nondegenerate component of X.

Now we consider the operator

$$\bar\partial_V = \bar\partial + i(V). \tag{1.5}$$

It is well known that

$$\bar\partial_V^2 = 0, \tag{1.6}$$

so it induces the complex (0.1) and the associated cohomology is defined by

$$H_V^{(n)}(M; C) := \frac{\ker \bar\partial_V|_{A^n}}{\operatorname{Im} \bar\partial_V(A^{n-1})}. \tag{1.7}$$

It is clear that for integral formulas, the interesting part is $H_V^{(0)}(M; C)$.

Notation. Denote $A = \oplus A^{p,q}(M)$. For any $\eta \in M$, we have a decomposition $\eta = \sum \eta^{p,q}$, where $\eta^{p,q} \in A^{p,q}$.

Our main result in this note is

Theorem 1.8. *Let V be a holomorphic vector field on the n-dimensional compact complex manifold M. And the components N_i of $X = \{p \in M : V(p) = 0\}$ are nondegenerate. Then, for any $\eta \in H_V^{(0)}(M; C)$, we have*

$$\int_M \eta = \sum_i (2\pi i)^{r_i} \int_{N_i} \frac{\eta}{\det(\theta^{\nu} - k_i)},$$

where k_i is the curvature matrix associated with certain complex connection on the normal bundle TM/TN_i over N_i, and r_i is the codimension of N_i in M.

The integral on the right hand side is well defined for if k' is another curvature matrix, then by Chern-Weil theory, $\left[\dfrac{1}{\det\left(\theta^{\nu} + \dfrac{k}{2\pi i}\right)}\right]$ and $\left[\dfrac{1}{\det\left(\theta^{\nu} + \dfrac{1}{2\pi i} k'\right)}\right]$ represent the same cohomology class in $H^*(N; C)$.

§2. Proof of Theorem 1.8

First we recall an idea of Bismut[2].

Proposition 2.1. *Let $\eta\in A$ be a $\bar\partial_V$-closed form, i.e. $(\bar\partial+i(V))\eta=0$. Then for any $\omega\in A$ and $t>0$*

$$\int_M \eta=\int_M \exp\left\{-\frac{\bar\partial\omega}{t}-\frac{i(V)\omega}{t}\right\}\cdot\eta .$$

Proof. By $(\bar\partial+i(V))^2=0$ and $(\bar\partial+i(V))\eta=0$, we have

$$\frac{\partial}{\partial s}\int_M \exp\{-s(\bar\partial+i(V))\omega\}\cdot\eta$$

$$=-\int_M(\bar\partial+i(V))\omega\cdot\exp\{-s(\bar\partial+i(V))\omega\}\cdot\eta$$

$$=-\int_M(\bar\partial+i(V))(\omega\cdot\exp\{-s(\bar\partial+i(V))\omega\}\cdot\eta),$$

so (2.1) reduces to the following

Lemma 2.2. *For any $\omega\in A(M)$,*

$$\int_M\bar\partial\omega=0 .$$

Proof. It is sufficient to consider the followiing two cases:

(i) $\omega\in A^{n-1,n}$, then $\bar\partial\omega=0$,

(ii) $\omega\in A^{n,n-1}$, then $\partial\omega=0$,

so

$$\int_M\bar\partial\omega=\int_M(\bar\partial+\partial)\,\omega=\int_M d\omega=0 .$$ □

Corollary 2.3 ([6]). *If V has no zeros on M, then for any η satisfying $(\bar\partial+i(V))\eta=0$,*

$$\int_M\eta=0 .$$

Proof. Choose any Hermitian metric on TM and let ω be the 1-form dual to V via this metric. Clearly $i(V)\omega=|V|^2$, and since V has no zeros, there exists a $\delta>0$ such that $|V|^2>\delta$ on M, so by (2.1),

$$\left|\int_M\eta\right|=\left|\int_M\exp\left\{-\frac{\bar\partial\omega}{t}-\frac{|V|^2}{t}\right\}\cdot\eta\right|\leqslant C\cdot e^{-\delta/t}(t^{-n}+t^{-n+1}+\cdots+1)$$

for some constant $C>0$. Taking $t\to0$, we get the result. □

Now we suppose that $X=\{p\in M: V(p)=0\}$ has components $N_1, \cdots, N_m$, each of which is nondegenerate.

On TM we choose a Hermitian metric such that on each component N_i, we have an orthogonal decomposition

$$TM = TN \oplus \operatorname{Im}\theta|_{N_i}.$$

Now we take a sufficiently small $\varepsilon > 0$ so that $N_i(2\varepsilon)\cap N_j(2\varepsilon)=\Phi$ when $i\neq j$ where $N_i(\varepsilon)=\{x\in M: d(x, N_i) < \varepsilon\}$ is the ε-neighborhood of N_i under the chosen metric.

Now let A be the endomorphism of TM defined by

$$As=\theta(V)s-\nabla_V s, \qquad s\in\Gamma(TM), \tag{2.4}$$

where ∇ is the unique connection of bidegree (1, 0) associated with the given Hermitian metric (· , ·).

Let ω_1 be a (1, 0) form on M such that

$$\omega_1=\begin{cases} -(\nabla V, AV), & \text{on } \bigcup_i N_i(2\varepsilon); \\ 0, & \text{on } M\setminus\bigcup_i N_i(3\varepsilon), \end{cases} \tag{2.5}$$

and $\omega_1(V)\geqslant 0$ on M. Let ω_2 be a one form on M such that

$$i(V)\,\omega_2=\begin{cases} \geqslant 0, & \text{on } M; \\ 1, & \text{on } M\setminus\bigcup_i N_i(2\varepsilon); \\ 0, & \text{on } \bigcup_i N_i(\varepsilon). \end{cases} \tag{2.6}$$

The existence of such forms is clear. Now set $\omega=\omega_1+\omega_2$.

Corollary 2.7. *For the case considered,*

$$\int_M \eta=\lim_{t\to 0}\int_{N_i(\varepsilon)} \exp\left(-\frac{\bar\partial\omega_1}{t}-\frac{i(V)\omega_1}{t}\right)\cdot\eta\,. \tag{2.7}$$

Proof. We have on $\bigcup_i N_i(2\varepsilon)$,

$$i(V)\,\omega_1=-(\nabla_V V, AV)=|AV|^2. \tag{2.8}$$

Since N_i's are nondegenerate and ε is small enough we can find a $\delta>0$ such that $|AV|^2\geqslant\delta$ on $\bigcup_i N_i(2\varepsilon)\setminus\bigcup_i N_i(\varepsilon)$. So combining it with (2.6) we see that on $M\setminus\bigcup_i N_i(\varepsilon)$, $i(V)\omega\geqslant\min(\delta, 1)>0$. Now, just as in the proof of (2.3), we get

$$\lim_{t\to 0}\int_{M\setminus\bigcup_i N_i(\varepsilon)} \exp\left\{-\frac{\bar\partial\omega}{t}-\frac{i(V)\omega}{t}\right\}\cdot\eta=0\,.$$

So

$$\int_M \eta = \int_M \exp\left\{-\frac{\bar\partial\omega}{t} - \frac{i(V)\omega}{t}\right\}\cdot\eta$$

$$= \int_{M\setminus \bigcup_i N_i(\varepsilon)} \exp\left\{-\frac{\bar\partial\omega}{t} - \frac{i(V)\omega}{t}\right\}\cdot\eta$$

$$+ \int_{\bigcup_i N_i(\varepsilon)} \exp\left\{-\frac{\bar\partial\omega}{t} - \frac{i(V)\omega}{t}\right\}\cdot\eta$$

$$= \lim_{t\to 0}\sum_i \int_{N_i(\varepsilon)} \exp\left\{-\frac{\bar\partial\omega_1}{t} - \frac{i(V)\omega_1}{t}\right\}\cdot\eta\ ,$$

since $\omega_2|_{\bigcup_i N_i(\varepsilon)} = 0$. □

To complete the proof of Theorem 1.8, we need to calculate each

$$\lim_{t\to 0}\int_{N_i(\varepsilon)} \exp\left\{-\frac{\bar\partial\omega_1}{t} - \frac{i(V)\omega_1}{t}\right\}\cdot\eta .$$

For simplicity, we deal with one of the components and denote it just by N.

Since N is a holomorphic variety, we can find a coordinate patch U centered at $p\in N$, with holomorphic coordinates $\{z_1,\cdots,z_n\}$, $n=\dim_{\mathbb C}M$, such that

$$N\cap U=\{\, q\in U : z_1(q)=\cdots=z_r(q)=0\,\},$$

where r is the codimension of N in M (See Bott[3]). Because V is nondegenerate near N, one can further choose these coordinates such that on U, the Taylor expansion of V takes the form

$$V = -\sum z_a v_{ab}\frac{\partial}{\partial z_b} + O(|z|^2) \tag{2.9}$$

with a, b ranging over the integers from 1 to r and $(v_{ab})_{r\times r}$ is a nonsingular matrix. By (2.4) and (2.9), we clearly have

$$A|_N:\ \frac{\partial}{\partial z_a}\longmapsto \sum_b v_{ab}\frac{\partial}{\partial z_b}\ , \tag{2.10}$$

whence finally $\frac{\partial}{\partial z_a}$ $(a=1,\cdots,r)$ are in the image of $\theta|_N$ while the remaining $\frac{\partial}{\partial z_j}$ are restricted to the elements in the kernel of $\theta|_N$.

Now

$$\int_{U\cap N(\varepsilon)} \exp\left\{-\frac{\bar\partial\omega_1}{t}-\frac{i(V)\omega_1}{t}\right\}\cdot\eta$$

$$=\int_{U\cap N(\varepsilon)} \exp\left\{\frac{\bar\partial(\nabla V, AV)}{t}-\frac{|AV|^2}{t}\right\}\cdot\eta$$

$$=\int_{U\cap N(\varepsilon)} \exp\left\{\frac{(kV, AV)}{t}-\frac{(\nabla V, \nabla AV)}{t}-\frac{|AV|^2}{t}\right\}\cdot\eta$$

$$=\int_{U\cap N(\varepsilon)} \exp\left[-\frac{(\nabla V, \nabla AV)}{t}-\frac{((A-k)V, AV)}{t}\right\}\cdot\eta$$

$$=\int_{U\cap N}\int_{W_p(\varepsilon)} \exp\left\{-\frac{(\nabla V, \nabla AV)}{t}-\frac{((A-k)V, AV)}{t}\right\}\cdot\eta, \qquad (2.11)$$

where k is the curvature $(1, 1)$ form matrix for ∇ and $W_p(\varepsilon)$ denotes the ε-neighborhood of p in the normal space $\mathrm{Im}\theta|T_pM(=T_pM/T_pN)$. Changing the variables $z_a\to\sqrt{t}z_a$, we have by (2.11) and (2.9),

$$\int_{N(\varepsilon)\cap U} \exp\left\{-\frac{\bar\partial\omega_1}{t}-\frac{i(V)\omega_1}{t}\right\}\cdot\eta$$

$$=\int_{U\cap N}\int_{W_p(\varepsilon/\sqrt{t})} \exp\{-(\nabla V, \nabla AV)-((A-k)V, AV)\}\cdot\eta$$

$$=\int_{U\cap N}\int_{W_p(\varepsilon/\sqrt{t})} \exp\{-(A_pdz, A_p^2dz)-((A_p-k^\nu)V, A_pV)\}\cdot\eta|_N$$

$$+\int_{U\cap N}\int_{W_p(\varepsilon/\sqrt{t})} \exp\{-(A_pV, A_pV)\}\cdot\alpha(t, z)\cdot\left(\frac{i}{2\pi}\right)^r dz_1d\bar z_1\cdots dz_rd\bar z_r,$$

where k^ν is the curvature matrix of the normal bundle TM/TN over N induced by ∇ and $\lim_{t\to0}\alpha(t, z)=0$. So when we let $t\to0$, we get

$$\lim_{t\to0}\int_{N(\varepsilon)} \exp\left\{-\frac{\bar\partial\omega_1}{t}-\frac{i(V)\omega_1}{t}\right\}\cdot\eta$$

$$=\lim_{t\to0}\int_{N}\int_{W_p(\varepsilon/\sqrt{t})} \exp\{-(A_pdz, A_p^2dz)-((A_p-k^\nu)V, A_pV)\}\cdot\eta|_N$$

$$= \int_N \int_{W_p} \exp\{-(A_p dz, A_p^2 dz) - ((A_p - k^v)V, A_p V)\} \cdot \eta|_N$$

$$= \int_N \int_{W_p} \exp\{-(dz, A_p dz)\} \cdot \exp\{(-(A_p - k^v)z, A_p z)\} \cdot \eta|_N$$

$$= \int_N \int_{W_p} \exp\{(-(A_p - k^v) z, A_p z)\} \cdot \overline{\det A_p} \cdot (-1)^r \; dz_1 d\bar{z}_1 \cdots dz_r d\bar{z}_r \cdot \eta|_N$$

$$= \int_N \int_{W_p} (2i)^r \exp\{-(A_p - k^v) A_p^{-1} z, z)\} \cdot \frac{1}{\det A_p} \left(\frac{i}{2}\right)^r dz_1 d\bar{z}_1 \cdots dz_r d\bar{z}_r \cdot \eta|_N$$

$$= \int_N (2\pi i)^r \frac{1}{\det((A_p - k^v) A_p^{-1})} \cdot \frac{1}{\det A_p} \cdot \eta \Big|_N$$

$$= (2\pi i)^r \int_N \frac{\eta}{\det(A_p - k^v)} .$$

Note that here $A_p|_{TM/TN}$ is just θ^v, so our proof of Theorem 1.8 is complete. □

§3. Some Applications

In this section, we consider some applications of Theorem 1.8.

First suppose V has only nondegenerate isolated zeros. In this case let $p \in X = \{q \in M : V(q) = 0\}$; then near p, we have

$$V = v_j \frac{\partial}{\partial z_j} = z_j v_{ij} \frac{\partial}{\partial z_j}$$

and $\det(v_{ij})_p \neq 0$. The theorem becomes

Corollary 3.1 ([6], [7]). *When V has only nondegenerate zeros $\{p_i\}$ and $\eta \in H_V^{(0)}(M, C)$, then*

$$\left(\frac{i}{2\pi}\right)^n \int_M \eta = \sum_i \frac{\eta^{0,0}(p_i)}{\det(v_{ij})_{p_i}} .$$

Proof. Just note that $\theta_p^v = -(v_{ij})_p$. □

The corresponding formula for meromorphic vector fields must also be true, cf. Chern[4].

Corollary 3.2 (Bott residue formula[3]). *Let $\Lambda : \Gamma(E) \to \Gamma(E)$ be a holomorphic action of the holomorphic vector field V on the holomorphic bundle E over the n-dimensional compact holomorphic manifold M. Also, let $\varphi(z_1, \cdots, z_q)$, $q = \dim_C E$ be a homogeneous symmetric polynomial of degree $= n$, and let N_i range over the components of the zero set of V. If each N_i is nondegenerate, then*

$$\int_M \varphi(x_1,\cdots,x_q)=\sum_i \int_{N_i} \frac{\varphi(\lambda_1+x_1,\cdots,\lambda_q+x_q)}{\det(\mu_1+y_1,\ \cdots,\ \mu_r+y_r)}\ ,$$

where $x_1,\cdots,x_q$ *are the Chern roots for* E *over* M; $y_1,\cdots,y_r$ *are the Chern roots for* TM/TN *over* N; λ_i *and* μ_j *are the eigenvalues of* $\Lambda|_N$ *and* $\theta^\nu|_{TM/TN}$ *respectively.*

Proof. Choosing a Hermitian metric as in §2 and taking a Hermitian metric $\langle\cdot,\cdot\rangle_E$ for E, let R denote the curvature for E. Then the Bott theorem is, via Chern-Weil theory, equivalent to

$$\int_M \varphi\left(\frac{i}{2\pi}R\right)=\sum_i \int_{N_i} \frac{\varphi\left(\Lambda+\frac{i}{2\pi}R\right)}{\det\left(\theta^\nu+\frac{i}{2\pi}k\right)}\ . \tag{3.3}$$

Let ∇ be the $(1,0)$-connection associated with $\langle\cdot,\cdot\rangle_E$, and R be the associated curvature. Then

$$L\cdot s=\Lambda\cdot s-\nabla_V\cdot s,\qquad s\in\Gamma(E) \tag{3.4}$$

is an endomorphism of E and $\Lambda|_N=L|_N$. Furthermore, it is easy to check that

$$(\bar\partial+i(V))(\varphi(L-R))=0\,.$$

So by (1.8),

$$\int_M \varphi(-L+R)=\sum_i \int_{N_i}(2\pi i)^{r_i}\,\frac{\varphi(-\Lambda+R)}{\det(\theta^\nu-k)}\ .$$

This is just (3.3), for we have

$$\int_M \varphi\left(\frac{i}{2\pi}R\right)=\left(\frac{i}{2\pi}\right)^n\int_M \varphi(-L+R)$$

$$=\sum_i \int_{N_i}(2\pi i)^{r_i}\cdot\left(\frac{i}{2\pi}\right)^n\frac{\varphi(-\Lambda+R)}{\det(\theta^\nu-k)}$$

$$=\sum_i \int_{N_i}\left(\frac{i}{2\pi}\right)^{n-r_i}\frac{\varphi(-\Lambda+R)}{\det(-\theta^\nu+k)}$$

$$=\sum_i \int_{N_i} \frac{\varphi\left(-\Lambda+\frac{i}{2\pi}R\right)}{\det\left(-\theta^\nu+\frac{i}{2\pi}k\right)}$$

$$= \sum_i \int_{N_i} (-1)^{n-r_i} \frac{\varphi\left(\Lambda - \frac{i}{2\pi} R\right)}{\det\left(\theta^v - \frac{i}{2\pi} k\right)} = \sum_i \int_{N_i} \frac{\varphi\left(\Lambda + \frac{i}{2\pi} R\right)}{\det\left(\theta^v + \frac{i}{2\pi} k\right)} . \qquad \square$$

Next we prove the complex analogue of a formula of Duistermaat-Heckman[5]. For the case where V has only nondegenerate isolated zeros, this has been proved in [6].

Corollary 3.5. *Let V have nondegenerate zero components $\{N_i\}$. If ω is a $\bar\partial_V$-closed (1,1) form and there is an $f \in C^\infty(M)$ such that $i(V)\omega = \bar\partial f$, then for any $s > 0$,*

$$\int_M e^{-sf} \frac{\omega^n}{n!} = \sum_i (2\pi i)^{r_i} \int_{N_i} \frac{e^{\omega - sf}}{\det(s\theta^v - k_i)} ,$$

where k_i is a (1, 1) curvature form of TM/TN_i.

Proof. Just note that under the given conditions,

$$(\bar\partial + s\,i(V))\, e^{\omega - sf} = 0 ,$$

so by (1.8),

$$\int_M e^{-sf} \frac{\omega^n}{n!} = \int_M e^{\omega - sf} = \sum_i (2\pi i)^{r_i} \int_{N_i} \frac{e^{\omega - sf}}{\det(s\theta^v - k_i)} . \qquad \square$$

Remark 1. The original method of Bott seems to work here also. But our proof is technically simpler.

Remark 2. If we take $\deg\varphi < n$ in (3.2), we may get some interesting vanishing formulas.

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