Advances in Mathematics 254 (2014) 41–48

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

η-Invariant and a problem of Bérard-Bergery on the existence of closed geodesics [☆]

Zizhou Tang [a], Weiping Zhang [b]

[a] School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, PR China
[b] Chern Institute of Mathematics and LPMC, Nankai University, Tianjin 300071, PR China

ARTICLE INFO

Article history:
Received 5 August 2013
Accepted 20 December 2013
Available online xxxx
Communicated by Gang Tian

MSC:
51H25
53C22
58J20

Keywords:
Eells–Kuiper projective plane
SC^p structure
μ invariant
η invariant

ABSTRACT

We use the η-invariant of Atiyah–Patodi–Singer to compute the Eells–Kuiper invariant for the Eells–Kuiper quaternionic projective plane. By combining with a known result of Bérard-Bergery, it shows that every Eells–Kuiper quaternionic projective plane carries a Riemannian metric such that all geodesics passing through a certain point are simply closed and of the same length.

1. Introduction

The η-invariant introduced by Atiyah, Patodi and Singer [2], as well as its various ramifications, has played important roles in many problems in geometry and topology. In this short paper, we use the η-invariant to compute the Eells–Kuiper invariant for the Eells–Kuiper quaternionic projective plane. By combining with a known result of

☆ The project is partially supported by MOEC and NNSFC.
E-mail addresses: zztang@bnu.edu.cn (Z. Tang), weiping@nankai.edu.cn (W. Zhang).

0001-8708/$ – see front matter
http://dx.doi.org/10.1016/j.aim.2013.12.019

Bérard-Bergery, it shows that every Eells–Kuiper quaternionic projective plane carries a Riemannian metric such that all geodesics passing through a certain point are simply closed and of the same length.

To be more precise, let p be a point in a closed manifold M. Let g be a Riemannian metric on M. The Riemannian structure (M, g) is called an SC^p Riemannian structure if all geodesics issued from p are simply closed (periodic) geodesics with the same length. We refer to the classic book [4] for a systematic account of the SC^p structures.

It is clear that there are SC^p Riemannian structures on the compact symmetric spaces of rank one (briefed in [4] as CROSS), namely the unit spheres, the real projective spaces, the complex projective spaces, the quaternionic projective spaces and the Cayley projective plane, endowed with the corresponding canonical metrics. Moreover, a fundamental result of Bott [5] states that any smooth manifold carrying an SC^p structure should have the same integral cohomology ring as that of a CROSS. On the other hand, there are manifolds verifying the above cohomological condition but not diffeomorphic to any CROSS. For typical examples, we mention the (exotic) homotopy spheres and the Eells–Kuiper (exotic) quaternionic projective planes.

In 1975, Bérard-Bergery [3] discovered an SC^p structure on an exotic sphere of dimension 10. He then raised the natural question: *is there any (exotic) Eells–Kuiper quaternionic projective plane carrying an SC^p structure*? The same question was also posed explicitly by Besse in the classic book [4, 0.15 on p. 4]. Moreover, it is pointed out in [4, p. 143] that a positive answer to the above question would also give a positive nontrivial example to the following open question: *whether a Blaschke manifold at a point*[1] *would carry an SC^p Riemannian structure*?

The purpose of this article is to provide a positive answer to the above two questions concerning the Eells–Kuiper quaternionic projective planes.

Before going on, we describe the Eells–Kuiper quaternionic projective planes as follows, starting with the standard construction of Milnor [15].

For any pair of integers (h, j), let $\xi_{h,j}$ be the S^3-bundle over S^4 determined by the characteristic map $f_{h,j} : S^3 \to SO(4)$ with $f_{h,j}(u)v = u^h v u^j$ for $u \in S^3$, $v \in \mathbf{R}^4$, where we identify $\mathbf{R}^4$ with the space of quaternions. It is shown in [15] that when $h + j = 1$, the total space of the above sphere bundle is homeomorphic to the unit sphere S^7. From now on, we denote by M_h this total space corresponding to $(h, j) = (h, 1 - h)$, and denote by N_h the associated disk bundle.

Remark 1.1. When $h = 0$ or 1, M_h is just the unit 7-sphere and the sphere bundle is just the Hopf fibration (corresponding to the left or right multiplications of the quaternions, respectively). On the other hand, M_2 is the exotic sphere generating the group $\Theta(7)$ (the set of the orientation preserving diffeomorphism classes of 7-dimensional oriented homotopy spheres), which is isomorphic to the cyclic group $\mathbf{Z}_{28}$.

---

[1] Cf. [4, 5.37 on p. 135] for a definition.

It is shown by Eells and Kuiper [9] that the homotopy sphere M_h is diffeomorphic to S^7 if and only if the following congruence holds for h,

$$\frac{h(h-1)}{56} \equiv 0 \mod \mathbf{Z}. \tag{1.1}$$

From now on, we assume that h satisfies (1.1). Then there is a diffeomorphism $\sigma : M_h \to S^7$. Let $X_{h,\sigma}$ denote the 8-dimensional closed smooth manifold constructed from N_h by attaching the unit disk D^8 by the diffeomorphism $\sigma : \partial(N_h) = M_h \to \partial(D^8) = S^7$. This is what we call an Eells–Kuiper quaternionic projective plane, first constructed in [8].[2] We remark that when $h = 0$ or 1, and $\sigma = \mathrm{id}$, $X_{h,\sigma}$ is just the standard quaternionic projective plane $\mathbf{H}P^2$. We also mention a deep result due to Kramer and Stolz [11] which asserts that the diffeomorphism type of the resulting manifold $X_{h,\sigma}$ does not depend on the choice of the diffeomorphism $\sigma : M_h \to S^7$.

Let τ_h be the canonical involution on M_h obtained by the fiberwise antipodal involution on S^3. By [3, Theorem 1] and the above result of Kramer–Stolz, to prove that $X_{h,\sigma}$ carries an SC^p Riemannian structure, one only needs to show that there is a diffeomorphism $\sigma' : M_h \to S^7$ such that $\tau\sigma' = \sigma'\tau_h$, where τ is the standard antipodal involution of S^7. Equivalently, one needs only to show that the quotient manifold M_h/τ_h is diffeomorphic to $\mathbf{R}P^7$. This is the content of the following main result of this paper.

Theorem 1.1. *The involution τ_h on $M_h \cong S^7$ is equivalent to the standard antipodal involution on S^7. In other words, M_h/τ_h is diffeomorphic to $\mathbf{R}P^7$.*

Corollary 1.1. *Every Eells–Kuiper quaternionic projective plane admits an SC^p Riemannian structure.*

Remark 1.2. Since there are infinitely many Eells–Kuiper quaternionic projective planes not diffeomorphic to each other, the above corollary actually shows that there is an infinite family of pairwise non-diffeomorphic manifolds M with the cohomology ring of $\mathbf{H}P^2$ such that each M admits an SC^p Riemannian structure.

The rest of this article is organized as follows. In Section 2, we reduce the proof of Theorem 1.1 to a problem of computing the Eells–Kuiper μ invariant introduced in [9]. In Section 3, we recall the results of Donnelly [6] and Kreck and Stolz [12] (cf. [10]) which use η-invariants to express the μ-invariant, and then carry out the required computation of the involved η invariant.

[2] Indeed, Eells and Kuiper showed in [8] that the $X_{h,\sigma}$'s are the only 8-dimensional closed smooth manifolds admitting a Morse function with 3 critical points.

2. Theorem 1.1 and the Eells–Kuiper μ invariant

As was indicated in [3, p. 240], by results of Mayer [14], there could only be two possibilities for M_h/τ_h. That is, it is diffeomorphic either to $\mathbf{R}P^7$ or to the connected sum $\mathbf{R}P^7\#14M_2$, where $14M_2$ is the connected sum $M_2\#\cdots\#M_2$ of 14 copies of M_2.

On the other hand, Milnor [16] showed that the Eells–Kuiper μ invariant of $\mathbf{R}P^7$ and $\mathbf{R}P^7\#14M_2$ takes different values. Thus, in order to prove Theorem 1.1, one needs only to show that the μ invariant of M_h/τ_h is different from that of $\mathbf{R}P^7\#14M_2$.

For completeness, we recall the definition of the Eells–Kuiper μ invariant in our situation. Let M be a 7-dimensional closed oriented spin manifold such that the 4-th cohomology group $H^4(M;\mathbf{R})$ vanishes.[3] If M bounds a compact oriented spin manifold N, then the first Pontrjagin class $p_1(N) \in H^4(N, M;\mathbf{Q})$ is well-defined.

Following [9, (11)], we define $\mu(M) \in \mathbf{R}/\mathbf{Z}$ by

$$\mu(M) \equiv \frac{p_1^2(N)}{2^7 \times 7} - \frac{\operatorname{Sign}(N)}{2^5 \times 7} \mod \mathbf{Z}, \tag{2.1}$$

where $p_1^2(N)$ denotes the corresponding Pontrjagin number and $\operatorname{Sign}(N)$ is the Signature of N.

Now set $M = M_h$, $N = N_h$. Let $x \in H^4(S^4;\mathbf{Z})$ be the generator. By [15], one has

$$e(\xi_{h,1-h}) = x, \qquad p_1(\xi_{h,1-h}) = \pm 2(2h-1)x, \tag{2.2}$$

where $e(\xi_{h,1-h})$ and $p_1(\xi_{h,1-h})$ are the Euler class and the first Pontrjagin class of the sphere bundle $\xi_{h,1-h}$ respectively. Also by [15], one has

$$\operatorname{Sign}(N_h) = 1. \tag{2.3}$$

From (2.2) and (2.3), one deduces as in [15] and [9] that

$$\frac{p_1^2(N_h)}{2^7 \times 7} - \frac{\operatorname{Sign}(N_h)}{2^5 \times 7} = \frac{h(h-1)}{56}, \tag{2.4}$$

which is an integer in view of the assumption (1.1).

Recall that by [16], one has $\mu(\mathbf{R}P^7) = \pm\frac{1}{32}$ while $\mu(\mathbf{R}P^7\#14M_2) = \pm\frac{1}{32} + \frac{1}{2}$. Thus, in order to prove Theorem 1.1, one needs only to prove the following result.

Theorem 2.1. *The following identity holds for any integer h verifying (1.1),*

$$\mu(M_h/\tau_h) \equiv \pm\frac{1}{32} \mod \mathbf{Z}. \tag{2.5}$$

Theorem 2.1 will be proved in Section 3.

[3] By the above diffeomorphism type result, it is clear that M_h/τ_h verifies this condition.

3. A proof of Theorem 2.1

In this section, we compute $\mu(M_h/\tau_h)$. The obvious difficulty is that one does not find easily an 8-dimensional spin manifold with boundary M_h/τ_h. Instead, we will make use of an intrinsic formula for the μ invariant, which is given by Donnelly [6] and Kreck and Stolz [12] (cf. the survey paper of Goette [10]).

Indeed, for any 7-dimensional closed oriented spin manifold M with $H^4(M;\mathbf{R}) = 0$, let g^{TM} be a Riemannian metric on TM. Let ∇^{TM} be the associated Levi-Civita connection. Let $p_1(TM, \nabla^{TM})$ be the corresponding first Pontrjagin form (cf. [17, Section 1.6.2]). Then there is a 3-form $\widehat{p}_1(TM, \nabla^{TM})$ on M such that

$$d\,\widehat{p}_1\big(TM, \nabla^{TM}\big) = p_1\big(TM, \nabla^{TM}\big). \tag{3.1}$$

Let D_M (resp. B_M) be the Dirac (resp. Signature) operator associated to g^{TM}. Let $\eta(D_M)$, $\eta(B_M)$ be the Atiyah–Patodi–Singer η invariant of D_M, B_M (cf. [2]). Let

$$\bar{\eta}(D_M) = \frac{1}{2}\big(\dim(\ker D_M) + \eta(D_M)\big)$$

be the corresponding reduced η-invariant.

By [6] and [12] (cf. [10, p. 424]), the μ invariant defined in (2.1) can be represented by

$$\mu(M) \equiv \bar{\eta}(D_M) + \frac{\eta(B_M)}{2^5 \times 7} - \frac{1}{2^7 \times 7}\int_M p_1\big(TM, \nabla^{TM}\big) \wedge \widehat{p}_1\big(TM, \nabla^{TM}\big) \mod \mathbf{Z}. \tag{3.2}$$

Now consider the double covering $M_h \to M_h/\tau_h$. We fix a spin structure on M_h/τ_h and lift everything from M_h/τ_h to M_h. We get that

$$\begin{aligned}\mu(M_h/\tau_h) \equiv{}& \bar{\eta}(P_h D_{M_h}) + \frac{\eta(P_h B_{M_h})}{2^5 \times 7}\\ &- \frac{1}{2^8 \times 7}\int_{M_h} p_1\big(TM_h, \nabla^{TM_h}\big) \wedge \widehat{p}_1\big(TM_h, \nabla^{TM_h}\big) \mod \mathbf{Z},\end{aligned} \tag{3.3}$$

where $P_h = \frac{1}{2}(1 + \tau_h)$ is the canonical projection. Here τ_h denotes the lifted actions on the corresponding vector bundles.

Indeed, recall that M_h is a fiber bundle over S^4 with fiber S^3. It is the boundary of the unit disk bundle N_h over S^4, while τ_h is the canonical involution which maps on each fiber by mapping a point to its antipodal. This involution extends canonically to an involution on N_h which we still denote by τ_h. Clearly, the fixed point set of τ_h on N_h is S^4, the image of the zero section of the disk bundle.

Let g^{TN_h} be a τ_h invariant Riemannian metric on TN_h such that it restricts to g^{TM_h} on $\partial N_h = M_h$ and is of product structure near M_h (the existence of such a metric is clear). Let ∇^{TN_h} be the associated Levi-Civita connection.

By dimensional reason we see that we are in the situation of even type in the sense of [1, Proposition 8.46]. Thus there exists a τ_h-equivariant spin structure on N_h, such that it induces a τ_h-equivariant spin structure on M_h, which equals the one lifted from the spin structure given on M_h/τ_h. In particular, τ_h lifts to an action on the associated spinor bundle $S(TN_h) = S_+(TN_h) \oplus S_-(TN_h)$ associated to (TN_h, g^{TN_h}), preserving the corresponding $\mathbf{Z}_2$-grading. It induces an action on $S(TM_h) = S_+(TN_h)|_{M_h}$. Moreover, the lifted τ_h action commutes with the Dirac operator $D_{N_h} : \Gamma(S(TN_h)) \to \Gamma(S(TN_h))$, and thus also commutes with the induced Dirac operator $D_{M_h} : \Gamma(S(TM_h)) \to \Gamma(S(TM_h))$, which in turn determines a Dirac operator on M_h/τ_h on which one can apply (3.2) and (3.3).

Let $D_{N_h,+} : \Gamma(S_+(TN_h)) \to \Gamma(S_-(TN_h))$ be the natural restriction of D_{N_h}. By the Atiyah–Patodi–Singer index theorem [2] and its equivariant extension by Donnelly [7], one finds

$$\bar{\eta}(P_h D_{M_h}) \equiv \frac{1}{2} \int_{N_h} \widehat{A}(TN_h, \nabla^{TN_h}) + \frac{1}{2} \int_{S^4} A_1 \mod \mathbf{Z}, \tag{3.4}$$

where the mod $\mathbf{Z}$ term comes from the Atiyah–Patodi–Singer type index $\operatorname{ind}_{\mathrm{APS}}(P_h D_{N_h,+})$, $\widehat{A}(TN_h, \nabla^{TN_h})$ is the Hirzebruch $\widehat{A}$-form associated to ∇^{TN_h} (cf. [17, Section 1.6.3]) and A_1 is the canonical contribution on the fixed point set (which by the local index theory is the same as the usual fixed point set contribution appearing in the equivariant Atiyah–Singer index theorem for compact group actions on closed manifolds).

Similarly,

$$\eta(P_h B_{M_h}) = \frac{1}{2} \int_{N_h} L(TN_h, \nabla^{TN_h}) + \frac{1}{2} \int_{S^4} A_2 - \frac{1}{2}\mathrm{Sign}(N_h) - \frac{1}{2}\mathrm{Sign}(N_h, \tau_h), \tag{3.5}$$

where $L(TN_h, \nabla^{TN_h})$ is the Hirzebruch L-form associated to ∇^{TN_h} (cf. [17, Section 1.6.3]), A_2 is the canonical contribution on the fixed point set and $\mathrm{Sign}(N_h, \tau_h)$ is the notation for the equivariant Signature with respect to τ_h.

By a direct computation, one has

$$\begin{aligned} &\frac{1}{2} \int_{N_h} \widehat{A}(TN_h, \nabla^{TN_h}) + \frac{1}{2^6 \times 7} \left(\int_{N_h} L(TN_h, \nabla^{TN_h}) - \mathrm{Sign}(N_h) \right) \\ &\quad - \frac{1}{2^8 \times 7} \int_{M_h} p_1(TM_h, \nabla^{TM_h}) \wedge \widehat{p}_1(TM_h, \nabla^{TM_h}) = \frac{p_1^2(N_h)}{2^8 \times 7} - \frac{\mathrm{Sign}(N_h)}{2^6 \times 7}. \end{aligned} \tag{3.6}$$

From (2.4) and (3.3)–(3.6), we find that

$$\mu(M_h/\tau_h) \equiv \frac{h(h-1)}{112} + \frac{1}{2} \int_{S^4} A_1 + \frac{1}{2^6 \times 7} \int_{S^4} A_2 - \frac{\mathrm{Sign}(N_h, \tau_h)}{2^6 \times 7} \mod \mathbf{Z}. \tag{3.7}$$

Now let W_h denote the normal bundle in N_h to the submanifold S^4, the fixed point set of τ_h. It is clear that τ_h acts on W_h by multiplication by -1.

By (2.2) and [13, p. 267], one finds

$$\int_{S^4} A_1 = \pm\frac{1}{32}\int_{S^4} p_1(W_h) = \pm\frac{(2h-1)}{16}. \tag{3.8}$$

Similarly, by [13, p. 265] and (2.2), one has

$$\int_{S^4} A_2 = \int_{S^4} e(W_h) = 1. \tag{3.9}$$

On the other hand, since S^4 is the fixed point set of τ_h, τ_h preserves $x \in H^4(S^4;\mathbf{Z})$. Thus one has

$$\operatorname{Sign}(N_h, \tau_h) = 1. \tag{3.10}$$

From (3.7)–(3.10), one gets

$$\mu(M_h/\tau_h) \equiv \frac{h(h-1)}{112} \pm \frac{2h-1}{32} \mod \mathbf{Z}. \tag{3.11}$$

We now claim that under the condition (1.1), (2.5) follows from (3.11).

Indeed, under the assumption (1.1), one has $h \equiv 0, 1, 8, 49 \mod 56\mathbf{Z}$. Thus we only need to do the case by case checking as follows, where by "≡" we mean that the congruence is mod **Z**.

(i) For $h = 56k$, then $\frac{h(h-1)}{112} \equiv \frac{k}{2}$, while $\frac{2h-1}{32} \equiv -\frac{1}{32} + \frac{k}{2}$;
(ii) For $h = 56k + 1$, then $\frac{h(h-1)}{112} \equiv \frac{k}{2}$, while $\frac{2h-1}{32} \equiv \frac{1}{32} + \frac{k}{2}$;
(iii) For $h = 56k + 8$, one has $\frac{h(h-1)}{112} \equiv \frac{1}{2} + \frac{k}{2}$, while $\frac{2h-1}{32} \equiv -\frac{1}{32} + \frac{1}{2} + \frac{k}{2}$;
(iv) For $h = 56k + 49$, one has $\frac{h(h-1)}{112} \equiv \frac{k}{2}$, while $\frac{2h-1}{32} \equiv \frac{1}{32} + \frac{k}{2}$.

Combining (i)–(iv) with (3.11), we always have (2.5).

The proof of Theorem 2.1, as well as of Theorem 1.1 and Corollary 1.1 is complete.

Acknowledgments

The first author thanks Dr. Chao Qian for useful discussion. We also thank the referee of this paper for very helpful suggestions.

References

[1] M.F. Atiyah, R. Bott, A Lefschetz fixed point formula for elliptic complexes: II. Applications, Ann. of Math. 88 (1968) 451–491.

[2] M.F. Atiyah, V.K. Patodi, I.M. Singer, Spectral asymmetry and Riemannian geometry I, Proc. Cambridge Philos. Soc. 77 (1975) 43–69.
[3] L. Bérard-Bergery, Quelques exemples de variétés riemanniennes où toutes les géodésiques issues dun point sont fermées et de même longueur suivis de quelques résultats sur leur topologie, Ann. Inst. Fourier 27 (1977) 231–249.
[4] A.L. Besse, Manifolds All of Whose Geodesics are Closed, With Appendices by D.B.A. Epstein, J.-P. Bourguignon, L. Bérard-Bergery, M. Berger and J.L. Kazdan, Ergeb. Math. Grenzgeb., vol. 93, Springer, Berlin, 1978.
[5] R. Bott, On manifolds all of whose geodesics are closed, Ann. of Math. 60 (1954) 375–382.
[6] H. Donnelly, Spectral geometry and invariants from differential topology, Bull. Lond. Math. Soc. 7 (1975) 147–150.
[7] H. Donnelly, Eta invariants for G-spaces, Indiana Univ. Math. J. 27 (1978) 889–918.
[8] J. Eells, N.H. Kuiper, Manifolds which are like projective planes, Publ. Math. Inst. Hautes Etudes Sci. 14 (1962) 5–46.
[9] J. Eells, N.H. Kuiper, An invariant for certain smooth manifolds, Ann. Mat. Pura Appl. 60 (1962) 93–110.
[10] S. Goette, Computations and applications of η invariants, in: C. Bär, J. Lohkamp, M. Schwarz (Eds.), Global Differential Geometry, in: Springer Proc. Math., Springer-Verlag, 2012, pp. 401–433.
[11] L. Kramer, S. Stolz, A diffeomorphism classification of manifolds which are like projective planes, J. Differential Geom. 77 (2007) 177–188.
[12] M. Kreck, S. Stolz, A diffeomorphism classification of 7-dimensional homogeneous Einstein manifolds with $SU(3) \times SU(2) \times U(1)$-symmetry, Ann. of Math. 127 (1988) 373–388.
[13] H.B. Lawson, M.-L. Michelsohn, Spin Geometry, Princeton Univ. Press, Princeton, 1989.
[14] K.H. Mayer, Fixpunktfreie Involutionen von 7-sphären, Math. Ann. 185 (1970) 250–258.
[15] J. Milnor, On manifolds homeomorphic to the 7-sphere, Ann. of Math. 64 (1956) 399–405.
[16] J. Milnor, Remarks concerning spin manifolds, in: Differential and Combinatorial Topology, a Symposium in Honor of Marston Morse, Princeton Univ. Press, 1965, pp. 55–62.
[17] W. Zhang, Lectures on Chern–Weil Theory and Witten Deformations, Nankai Tracts in Mathematics, vol. 4, World Scientific, Singapore, 2001.