Geometry and Topology of Submanifolds X
eds. W. H. Chen *et al.* (pp. 336–345)

η-invariants and the Poincaré-Hopf index formula

Weiping Zhang*

Abstract

We present an analytic proof of the Poincaré-Hopf index theorem. Our proof makes use of an old idea of Atiyah and works for the case where the isolated zeros of the vector field can be degenerate.

Key words: Euler characteristic, elliptic boundary problem, spectral flow, η-invariant.
Math. Subject Classification: 58G10.

0 Introduction

Let M be an even dimensional oriented smooth closed manifold. Let $V \in \Gamma(TM)$ be a smooth tangent vector field on M. We assume that the singularity set

$$B(V) := \{x \in M : V(x) = 0\}$$

consists of isolated points. Then for any $x \in B(V)$ one can define an integer $\deg_V(x)$, which we call the degree of V at x, as follows: let U_x be a sufficiently small open neighborhood of x such that V is nowhere zero on $U_x \setminus \{x\}$ and that the closure of U_x is diffeomorphic to the standard closed ball in the $\dim M$ dimensional Euclidean space, then V induces a map v from ∂U_x, which is diffeomorphic to the standard sphere $S^{\dim M-1}(1)$, to $S^{\dim M-1}(1)$ in the following manner: for any $y \in \partial U_x$, $V(y)$ may be viewed as a unit vector in the Euclidean space containing U_x, which determines the point $v(y)$ on $S^{\dim M-1}(1)$. We define $\deg_V(x)$ to be the Brouwer degree of this induced map.

Let $\chi(M)$ denote the Euler characteristic of M.

The Poincaré-Hopf index formula (cf. [6] Theorem 11.25) can be stated as follows.

Theorem 1 *The following identity holds:* $\chi(M) = \sum_{x\in B(V)} \deg_V(x)$.

In this paper, we present an analytic proof of this classical result by developing an old idea of Atiyah [1] on manifolds with boundary. In doing so, we reduce the problem to a calculation of variations of η-invariants on the spheres around the zeros of V. The main point here is that we do not deform the vector field V to make its zeros nondegenerate. Thus in particular, our proof works for the case where V has degenerate zeros. This is different from the analytic proof proposed by Witten [10]. We hope that the ideas involved

This paper is in final form and no version of it will be submitted for publication elsewhere.

*Partially supported by the NSFC, MOEC and the Qiu Shi Foundation.

in this proof may be useful in other situations; in particular we hope this will yield a deeper analytic understanding of some of the results in the paper of Atiyah-Dupont [2], where further generalizations of the Poincaré-Hopf index formula have been studied extensively.

Here is a brief outline of the paper. In Sections 1 and 2, we reduce our proof to computations of spectral flows of Dirac type operators on spheres around zeros of V. In Section 3, we compute these spectral flows via variations of η-invariants.

1 Splitting of the index

1.1 An analytic interpretation of $\chi(M)$

Let g be a Riemannian metric on M. Let $\Lambda^*(T^*M)$ denote the exterior algebra bundle of the cotangent bundle T^*M. Let d^* be the formal adjoint of the exterior differential operator d with respect to the standard L^2 inner product on the space of smooth differential forms:

$$\Omega^*(M) := \Gamma(\Lambda^*(T^*M)).$$

Let D denote the de Rham-Hodge operator defined by

$$D = d + d^* : \Omega^*(M) \to \Omega^*(M).$$

Let $D_{\mathrm{e/o}}$ be the restriction of D to $\Omega^{\mathrm{even/odd}}(M)$ respectively. Then $D_{\mathrm{e}} : \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{odd}}(M)$ is a first order elliptic differential operator whose formal adjoint is $D_{\mathrm{o}} : \Omega^{\mathrm{odd}}(M) \to \Omega^{\mathrm{even}}(M)$. Furthermore, by the Hodge decomposition theorem (cf. [9] Corollary III.5.6), one has

$$\chi(M) = \mathrm{ind}\, D_{\mathrm{e}}. \tag{1}$$

1.2 de Rham-Hodge operator on manifolds with boundary

We adopt the following notational conventions. We use the Riemannian metric g to identify the tangent bundle TM with the cotangent bundle T^*M; if $e \in TM$ is a tangent vector, let $e^* \in T^*M$ be the corresponding dual cotangent vector. Let ext and int denote exterior and interior multiplications respectively. Let $c(e)$ and $\widehat{c}(e)$ be the Clifford operators acting on $\Lambda^*(T^*M)$ given by

$$c(e) := \mathrm{ext}(e^*) - \mathrm{int}(e^*) \text{ and } \widehat{c}(e) := \mathrm{ext}(e^*) + \mathrm{int}(e^*).$$

Choose $\delta > 0$ be small enough so that the balls

$$B_\delta(x) := \{y \in M : d(x,y) \leq \delta\} \text{ for } x \in B(V)$$

are disjoint. We choose the Riemannian metric to be flat on these balls; these balls are then isometric to the ball of radius δ in Euclidean space. To simplify subsequent notation, we let

$$\mathcal{B}(x) := B_{\delta/2}(x).$$

Let $\mathcal{M}$ be the closure of the complement of $\cup_{x\in B(V)}\mathcal{B}(x)$ in M:

$$\mathcal{M} := \mathrm{Closure}\{M \setminus \cup_{x\in B(V)}\mathcal{B}(x)\}.$$

We now study the elliptic boundary problems of the type of Atiyah-Patodi-Singer [3] for the de Rham-Hodge operator on $\mathcal{M}$. The point here is that since we have assumed that the Riemannian metric g is flat on $\cup_{x\in B(V)}\mathcal{B}(x)$, one has to deal with the situation where the metric near the boundary $\partial\mathcal{M}$ is not of product nature. For this, we will make use of the more refined analysis developed in the paper of Gilkey [8].

Let $D_{\mathcal{M}}$ (resp. $D_{\mathcal{M},\mathrm{e/o}}$) be the restriction of D (resp. $D_{\mathrm{e/o}}$) to $\mathcal{M}$.

Let $e_1,\dots,e_{\dim M}$ be an oriented orthonormal basis for TM and let $\nabla^{\Lambda^*(T^*M)}$ be the canonical Euclidean connection on $\Lambda^*(T^*M)$ lifted from the Levi-Civita connection ∇^{TM} of g. Then one has that (cf. [8] (5.2))

$$D_{\mathcal{M}} = \sum_{i=1}^{\dim M} c(e_i)\nabla_{e_i}^{\Lambda^*(T^*M)} : \Omega^*(M)|_{\mathcal{M}} \to \Omega^*(M)|_{\mathcal{M}}. \tag{2}$$

Let $\vec{\mathrm{n}}$ be the inward unit normal vector field on $\partial\mathcal{M}$ and let $f_1,\dots,f_{\dim M-1}$ be an oriented orthonormal basis of $T\partial\mathcal{M}$. Let $L_{jk} = \langle\nabla_{e_j}^{TM}e_k,\vec{\mathrm{n}}\rangle$, $1\le j,\ k\le \dim M-1$, be the second fundamental form of the isometric embedding $\partial\mathcal{M}\hookrightarrow\mathcal{M}$.

Following [8] Lemma 2.2, we define the tangential operators acting on $\Omega^{\mathrm{even/odd}}(M)|_{\partial\mathcal{M}}$ by

$$D_{\partial\mathcal{M},\mathrm{e/o}} = -c(\vec{\mathrm{n}})\sum_{j=1}^{\dim M-1} c(f_j)\nabla_{f_j}^{\Lambda^*(T^*M)} + \frac{1}{2}\sum_{j=1}^{\dim M-1} L_{jj} \tag{3}$$

respectively. To be more precise, for any $1\le j\le\dim M-1$, set

$$\widetilde{c}(f_j) = -c(\vec{\mathrm{n}})c(f_j) : \Omega^{\mathrm{even/odd}}(M)|_{\partial\mathcal{M}} \to \Omega^{\mathrm{even/odd}}(M)|_{\partial\mathcal{M}} \tag{4}$$

and

$$\widetilde{\nabla}_{f_j} = \nabla_{f_j}^{\Lambda^*(T^*M)} - \frac{1}{2}\sum_{k=1}^{\dim M-1} L_{jk}\widetilde{c}(f_k) : \Omega^{\mathrm{even/odd}}(M)|_{\partial\mathcal{M}} \to \Omega^{\mathrm{even/odd}}(M)|_{\partial\mathcal{M}}. \tag{5}$$

Then one verifies the following analogue of [8] Lemma 2.2(d):

$$D_{\partial\mathcal{M},\mathrm{e/o}} = \sum_{j=1}^{\dim M-1} \widetilde{c}(f_j)\widetilde{\nabla}_{f_j} : \Omega^{\mathrm{even/odd}}(M)|_{\partial\mathcal{M}} \to \Omega^{\mathrm{even/odd}}(M)|_{\partial\mathcal{M}}. \tag{6}$$

Let $P_{\partial\mathcal{M},\ge 0,\mathrm{e}}$ (resp. $P_{\partial\mathcal{M},\ge 0,\mathrm{o}}$, resp. $P_{\partial\mathcal{M},>0,\mathrm{o}}$) be the orthogonal projection from the L^2-completion space of $\Omega^{\mathrm{even}}(M)|_{\partial\mathcal{M}}$ (resp. $\Omega^{\mathrm{odd}}(M)|_{\partial\mathcal{M}}$, resp. $\Omega^{\mathrm{odd}}(M)|_{\partial\mathcal{M}}$) to its sub-Hilbert space obtained from the orthogonal direct sum of nonnegative (resp. nonnegative, resp. positive) eigenspaces of $D_{\partial\mathcal{M},\mathrm{e}}$ (resp. $D_{\partial\mathcal{M},\mathrm{o}}$, resp. $D_{\partial\mathcal{M},\mathrm{o}}$).

Let $(D_{\partial\mathcal{M},\mathrm{e/o}}, P_{\partial\mathcal{M},\ge 0,\mathrm{e/o}})$ (resp. $(D_{\partial\mathcal{M},\mathrm{o}}, P_{\partial\mathcal{M},>0,\mathrm{o}})$) be the realizations of the operators $D_{\mathcal{M},\mathrm{e/o}}$ (resp. $D_{\partial\mathcal{M},\mathrm{o}}$) with respect to the the Atiyah-Patodi-Singer type boundary conditions given by $P_{\partial\mathcal{M},\ge 0,\mathrm{e/o}}$ (resp. $P_{\partial\mathcal{M},>0,\mathrm{o}}$) respectively (cf. [3] and in particular [8]). These boundary value problems are all elliptic. Moreover, $(D_{\partial\mathcal{M},\mathrm{o}}, P_{\partial\mathcal{M},>0,\mathrm{o}})$ is adjoint to $(D_{\partial\mathcal{M},\mathrm{e}}, P_{\partial\mathcal{M},\ge 0,\mathrm{e}})$ ([8] Theorem 2.3(a)).

The above strategy can also be developed for each $\mathcal{B}(x)$, $x\in B(V)$, with similar notation.

1.3 A splitting formula for $\chi(M)$

We state the main result of this section as follows.

Proposition 2 *We have that* $\chi(M) = \operatorname{ind}(D_{\mathcal{M},\mathrm{e}}, P_{\partial\mathcal{M},\geq 0,\mathrm{e}})$.

Proof. By (3), one sees that on each $\partial\mathcal{B}(x)$, $x \in B(V)$, one has

$$D_{\partial\mathcal{M},\mathrm{e}} = -D_{\partial\mathcal{B}(x),\mathrm{e}}.$$

We can then apply [8] Theorem 6.4(g), which generalizes the Atiyah-Patodi-Singer index theorem [3] to the case where the metric near the boundary is not of product nature, to M, $\mathcal{M}$ and $\mathcal{B}(x)$, $x \in B(V)$, to get

$$\operatorname{ind} D_{\mathrm{e}} = \operatorname{ind}(D_{\mathcal{M},\mathrm{e}}, P_{\partial\mathcal{M},\geq 0,\mathrm{e}}) + \sum_{x\in B(V)} \operatorname{ind}(D_{\mathcal{B}(x),\mathrm{e}}, P_{\partial\mathcal{B}(x),\geq 0,\mathrm{e}}) + \sum_{x\in B(V)} \dim(\ker D_{\partial\mathcal{B}(x),\mathrm{e}}). \quad (7)$$

Now since each ball $\mathcal{B}(x)$, $x \in B(V)$, is a standard ball in an Euclidean space, one sees that the operators $D_{\mathcal{B}(x),\mathrm{e/o}}$ are the standard Dirac operators twisted by a trivial vector bundle on $\mathcal{B}(x)$. In view of (6) and [8] Lemma 4.1, one then sees that the induced operators $D_{\partial\mathcal{B}(x),\mathrm{e/o}}$ on the boundary are the standard Dirac operators twisted by trivial vector bundles with trivial twisted connections.[1] If $\dim M \geq 4$, then the scalar curvature on each $\partial\mathcal{B}(x)$ is positive and one uses the Lichnerowicz formula to see that there are no harmonic spinors on $\partial\mathcal{B}(x)$. If $\dim M = 2$, one computes directly that the 'bounding' spin structure is the Möbius spin structure and hence there are no harmonic spinors on the boundary. Thus the kernels of the boundary operators are trivial, i.e., for any $x \in B(V)$,

$$\dim(\ker D_{\partial\mathcal{B}(x),\mathrm{e/o}}) = 0. \quad (8)$$

On the other hand, by using Green's formula (cf. [8] (2.28)) as well as the fact that the metric on $\mathcal{B}(x)$ is flat, one deduces easily that for any $s \in \Omega^{\mathrm{even/odd}}(M)|_{\mathcal{B}(x)}$, $x \in B(V)$, one has

$$\int_{\mathcal{B}(x)} \langle D_{\mathrm{e/o}}s, D_{\mathrm{e/o}}s\rangle dv_{\mathcal{B}(x)} = -\int_{\partial\mathcal{B}(x)} \langle s, D_{\partial\mathcal{B}(x),\mathrm{e/o}}s\rangle dv_{\partial\mathcal{B}(x)} - \frac{1}{2}\int_{\partial\mathcal{B}(x)} \langle s, \sum_{j=1}^{\dim M-1} L_{jj}s\rangle dv_{\partial\mathcal{B}(x)}$$

$$+ \sum_{i=1}^{\dim M} \int_{\mathcal{B}(x)} \langle \nabla_{e_i}^{\Lambda^*(T^*M)}s, \nabla_{e_i}^{\Lambda^*(T^*M)}s\rangle dv_{\mathcal{B}(x)}, \quad (9)$$

where $e_1, \ldots, e_{\dim M}$ is an orthonormal basis of $T\mathcal{B}(x)$, and $dv_{\mathcal{B}(x)}$ (resp. $dv_{\partial\mathcal{B}(x)}$) is the volume form on $\mathcal{B}(x)$ (resp. $\partial\mathcal{B}(x)$) induced by the Riemannian metric g.

Now since the mean curvature $-\sum_{j=1}^{\dim M-1} L_{jj}$ of the isometric embedding $\partial\mathcal{B}(x) \hookrightarrow \mathcal{B}(x)$ is positive, one verifies directly from (9) that

$$\operatorname{ind}(D_{\mathcal{B}(x),\mathrm{e}}, P_{\partial\mathcal{B}(x),\geq 0,\mathrm{e}}) = 0, \quad \text{for any } x \in B(V). \quad (10)$$

The proposition now follows from equations (1), (7), (8) and (10). □

[1] See Section 3.2 for a more detailed explanation.

Remark 3 *It is important to note that the reason that $D_{\partial\mathcal{B}(x),\mathrm{e/o}}$ are not equivalent to the de Rham-Hodge operators on $\partial\mathcal{B}(x)$ is due to the fact that here we are dealing with the Atiyah-Patodi-Singer type boundary problems in situations where one does not assume that the metric near the boundary is of product structure.*

Remark 4 *One can also prove the splitting formula (7) alternatively without using the index theorem ([8] Theorem 6.4(g)) for manifolds with boundary. To this order, one deforms the Riemannian metric g near the hypersurface $\partial\mathcal{M}$ in M so that the metric near $\partial\mathcal{M}$ is of product nature. One also deforms the de Rham-Hodge operator on M to a Dirac type operator such that near $\partial\mathcal{M}$, it is of product nature with the induced tangential operators on $\partial\mathcal{M}$ given by $D_{\partial\mathcal{M},\mathrm{e/o}}$. One then applies the splitting formula for this deformed Dirac type operator, which can be proved by using the Bojarski theorem (cf. [5] Theorem 24.1), to get (7). We leave the details to the interested reader.*

2 Euler characteristic and the spectral flow

2.1 Review of the definition of the spectral flow

The concept of the spectral flow was introduced by Atiyah-Patodi-Singer in [4] for a curve of self-adjoint elliptic differential operators.

Let $D(u)$, $0 \le u \le 1$, be a smooth curve of self-adjoint first order elliptic differential operators on a compact smooth manifold, then the spectral flow of the family $\{D(u)\}_{0\le u\le 1}$, denoted by $\mathrm{sf}\{D(u), 0 \le u \le 1\}$, counts the *net number* of eigenvalues of $D(u)$ which change sign when u varies from 0 to 1.

We now assume that all $D(u)$, $0 \le u \le 1$, have the same principal symbol.

Let $P_{0,\ge 0}$ (resp. $P_{1,\ge 0}$) be the orthogonal projection from the L^2-completion space of the domain of $D(0)$ (resp. $D(1)$) to its sub-Hilbert space obtained from the orthogonal direct sum of nonnegative eigenspaces of $D(0)$ (resp. $D(1)$). Then the operator

$$T(P_{0,\ge 0}, P_{1,\ge 0}) = P_{0,\ge 0}P_{1,\ge 0} : \mathrm{Im}(P_{1,\ge 0}) \to \mathrm{Im}(P_{0,\ge 0}) \tag{11}$$

is a Fredholm operator. Furthermore, by a result of Dai and Zhang [7] Theorem 1.4, one has

$$\mathrm{sf}\{D(u), 0 \le u \le 1\} = \mathrm{ind}\, T(P_{0,\ge 0}, P_{1,\ge 0}). \tag{12}$$

2.2 Euler characteristic and the spectral flow

We normalize V to assume that

$$|V|_g = 1 \quad \text{on} \quad \mathcal{M}.$$

Set

$$\widehat{D}_{\partial\mathcal{M},\mathrm{e}} = \widehat{c}(V) D_{\partial\mathcal{M},\mathrm{o}} \widehat{c}(V) : \Omega^{\mathrm{even}}(M)|_{\partial\mathcal{M}} \to \Omega^{\mathrm{even}}(M)|_{\partial\mathcal{M}}. \tag{13}$$

From equations (3)-(6) and (13), one deduces that

$$\widehat{D}_{\partial\mathcal{M},\mathrm{e}} = D_{\partial\mathcal{M},\mathrm{e}} + \widehat{c}(V) \sum_{j=1}^{\dim M-1} \widetilde{c}(f_j)\widehat{c}(\nabla^{TM}_{f_j} V). \tag{14}$$

We use this equation to see that the boundary problem $(D_{\mathcal{M},\mathrm{e}}, \widehat{c}(V)P_{\partial\mathcal{M},\geq 0,\mathrm{o}}\widehat{c}(V))$ is elliptic. Moreover, as was already observed in [1] (2.2), the following operator is a 0^{th} order operator:

$$A(V) = D_{\mathcal{M},\mathrm{o}} + \widehat{c}(V)D_{\mathcal{M},\mathrm{e}}\widehat{c}(V) : \Omega^{\mathrm{odd}}(\mathcal{M}) \to \Omega^{\mathrm{odd}}(\mathcal{M}).$$

One uses equation (8) and [8] Theorem 2.3(a) to see that:

$$\begin{aligned} &\mathrm{ind}\,(D_{\mathcal{M},\mathrm{e}}, \widehat{c}(V)P_{\partial\mathcal{M},\geq 0,\mathrm{o}}\widehat{c}(V)) = \mathrm{ind}\,(\widehat{c}(V)D_{\mathcal{M},\mathrm{e}}\widehat{c}(V), P_{\partial\mathcal{M},\geq 0,\mathrm{o}}) \\ &= \mathrm{ind}\,(-D_{\mathcal{M},\mathrm{o}} + A(V), P_{\partial\mathcal{M},\geq 0,\mathrm{o}}) = \mathrm{ind}\,(D_{\mathcal{M},\mathrm{o}} - A(V), P_{\partial\mathcal{M},\geq 0,\mathrm{o}}) \\ &= \mathrm{ind}\,(D_{\mathcal{M},\mathrm{o}}, P_{\partial\mathcal{M},\geq 0,\mathrm{o}}) = -\mathrm{ind}\,(D_{\mathcal{M},\mathrm{e}}, P_{\partial\mathcal{M},\geq 0,\mathrm{e}}). \end{aligned} \tag{15}$$

Clearly, $\widehat{P}_{\partial\mathcal{M},\geq 0,\mathrm{e}} := \widehat{c}(V)P_{\partial\mathcal{M},\geq 0,\mathrm{o}}\widehat{c}(V)$ is the orthogonal projection mapping from the L^2-completion space of the domain of $\widehat{D}_{\partial\mathcal{M},\mathrm{e}}$ to its sub-Hilbert space obtained from the orthogonal direct sum of nonnegative eigenspaces of $\widehat{D}_{\partial\mathcal{M},\mathrm{e}}$. Thus, from (15) and (11) one has

$$\begin{aligned} \mathrm{ind}\,(D_{\mathcal{M},\mathrm{e}}, P_{\partial\mathcal{M},\geq 0,\mathrm{e}}) &= \frac{1}{2}(\mathrm{ind}\,(D_{\mathcal{M},\mathrm{e}}, P_{\partial\mathcal{M},\geq 0,\mathrm{e}}) - \mathrm{ind}\,(D_{\mathcal{M},\mathrm{e}}, \widehat{P}_{\partial\mathcal{M},\geq 0,\mathrm{e}})) \\ &= \frac{1}{2}\mathrm{ind}\,T(P_{\partial\mathcal{M},\geq 0,\mathrm{e}}, \widehat{P}_{\partial\mathcal{M},\geq 0,\mathrm{e}}), \end{aligned} \tag{16}$$

where the last equality follows from the variation formula for Dirac type operators with global elliptic boundary condition (cf. [5] Proposition 2.14).[2]

Now for any $u \in [0,1]$, set

$$D_{\partial\mathcal{M},\mathrm{e}}(u) = (1-u)D_{\partial\mathcal{M},\mathrm{e}} + u\widehat{D}_{\partial\mathcal{M},\mathrm{e}}. \tag{17}$$

Then $\{D_{\partial\mathcal{M},\mathrm{e}}(u)\}_{0\leq u\leq 1}$ is a smooth curve of first order self-adjoint elliptic differential operators all having the same principal symbol.

From Proposition 2, equations (12) and (16), one finds

$$\chi(M) = \frac{1}{2}\mathrm{sf}\{D_{\partial\mathcal{M},\mathrm{e}}(u) : 0 \leq u \leq 1\}.$$

For $0 \leq u \leq 1$, let $D_{\partial\mathcal{B}(x),\mathrm{e}}(u)$ be the restrictions of $-D_{\partial\mathcal{M},\mathrm{e}}(u)$ to each boundary sphere around $x \in B(V)$. As $\partial\mathcal{M} = -\cup_{x\in B}\partial\mathcal{B}(x)$, where one also takes account the orientations, one gets the following formula, which is the main result of this section,

$$\chi(M) = -\frac{1}{2}\sum_{x\in B(V)} \mathrm{sf}\{D_{\partial\mathcal{B}(x),\mathrm{e}}(u) : 0 \leq u \leq 1\}. \tag{18}$$

Remark 5 *The deduction (15) has been inspired by an idea of Atiyah [1] which was used to show that the Euler characteristic of a closed manifold admitting a nowhere zero vector field is zero.*

[2]The book [5] only deals with the situation of product nature near the boundary. However, one can use deformations as was indicated in Remark 4 to reduce our problem to the case of product nature near the boundary.

3 A computation of the spectral flow

In this section, we compute the spectral flows appearing in the right hand side of (18) through variations of η-invariants.

3.1 η-invariants and spectral flow

For any $u \in [0,1]$, following [3], let $\eta(D_{\partial\mathcal{B}(x),\mathrm{e}}(u), s)$ be the η-function of $D_{\partial\mathcal{B}(x),\mathrm{e}}(u)$ defined for $s \in \mathbf{C}$ with $\mathrm{Re}\,(s) > \dim M + 1$,

$$\eta(D_{\partial\mathcal{B}(x),\mathrm{e}}(u), s) = \sum_{\lambda \in \mathrm{Spec}(D_{\partial\mathcal{B}(x),\mathrm{e}}(u))\setminus\{0\}} \frac{\mathrm{sgn}\,\lambda}{|\lambda|^s}.$$

It can be extended to a meromorphic function on $\mathbf{C}$ which is holomorphic at $s=0$ (cf. [4]). The value of $\eta(D_{\partial\mathcal{B}(x),\mathrm{e}}(u), s)$ at $s=0$, denoted by $\eta(D_{\partial\mathcal{B}(x),\mathrm{e}}(u))$, is the η-invariant of $D_{\partial\mathcal{B}(x),\mathrm{e}}(u)$ in the sense of Atiyah, Patodi and Singer [3]. Let $\overline{\eta}(D_{\partial\mathcal{B}(x),\mathrm{e}}(u))$ be the reduced η-invariant of $D_{\partial\mathcal{B}(x),\mathrm{e}}(u)$, which was also defined in [3]:

$$\overline{\eta}(D_{\partial\mathcal{B}(x),\mathrm{e}}(u)) = \frac{\dim(\ker D_{\partial\mathcal{B}(x),\mathrm{e}}(u)) + \eta(D_{\partial\mathcal{B}(x),\mathrm{e}}(u))}{2}.$$

By (14) and (17), one sees that for any $u \in [0,1]$, $\frac{\partial}{\partial u}D_{\partial\mathcal{B}(x),\mathrm{e}}(u)$ is a bounded operator. By standard results for heat kernel asymptotics, one has the following asymptotic expansion as $t \to 0^+$,

$$\mathrm{Tr}[\frac{\partial}{\partial u}D_{\partial\mathcal{B}(x),\mathrm{e}}(u)\exp(-t(D_{\partial\mathcal{B}(x),\mathrm{e}}(u))^2)] = \frac{c_{-k/2}}{t^{k/2}} + \cdots + \frac{c_{-1/2}}{t^{1/2}} + O(t^{1/2}),$$

where $k = \dim M - 1$ and $c_{-k/2}, \ldots, c_{-1/2}$ are smooth functions of $u \in [0,1]$.

The following well-known result (cf. [11] Proposition 3.6) illustrates the relations between spectral flow and variations of reduced η-invariants.

Proposition 6 *For any $s \in [0,1]$, one has*

$$\mathrm{sf}\{D_{\partial\mathcal{B}(x),\mathrm{e}}(u), 0 \le u \le s\} = \int_0^s \frac{c_{-1/2}}{\sqrt{\pi}}du + \overline{\eta}(D_{\partial\mathcal{B}(x),\mathrm{e}}(s)) - \overline{\eta}(D_{\partial\mathcal{B}(x),\mathrm{e}}(0)).$$

Now, from (3), (13) and (17), one verifies that

$$D_{\partial\mathcal{B}(x),\mathrm{e}}(1) = \widehat{c}(V)D_{\partial\mathcal{M},\mathrm{o}}\widehat{c}(V) = -c(\vec{\mathrm{n}})\widehat{c}(V)D_{\partial\mathcal{M},\mathrm{e}}(0)\widehat{c}(V)c(\vec{\mathrm{n}}).$$

Thus, one finds that

$$\overline{\eta}(D_{\partial\mathcal{B}(x),\mathrm{e}}(1)) = \overline{\eta}(D_{\partial\mathcal{B}(x),\mathrm{e}}(0)). \tag{19}$$

From Proposition 6 and equation (19), one gets

$$\mathrm{sf}\{D_{\partial\mathcal{B}(x),\mathrm{e}}(u), 0 \le u \le 1\} = \int_0^1 \frac{c_{-1/2}}{\sqrt{\pi}}du. \tag{20}$$

3.2 Evaluation of the spectral flow

We compute in this subsection the right hand side of (20).

As we have noted in the proof of Proposition 2, the operator $D_{\partial\mathcal{B}(x),\mathrm{e}}$ is the standard Dirac operator twisted by a trivial vector bundle. We first make this more precise. Let

$$S(T\mathcal{B}(x)) := S_+(T\mathcal{B}(x)) \oplus S_-(T\mathcal{B}(x))$$

be the $\mathbf{Z}_2$-graded bundle of spinors over $\mathcal{B}(x)$ associated to $g|_{\mathcal{B}(x)}$ (which is the standard Euclidean metric). We have

$$\Lambda^{\mathrm{even}}(T^*\mathcal{B}(x)) = S_+(T\mathcal{B}(x)) \otimes S_+(T\mathcal{B}(x)) \oplus S_-(T\mathcal{B}(x)) \otimes S_-(T\mathcal{B}(x)),$$
$$\Lambda^{\mathrm{odd}}(T^*\mathcal{B}(x)) = S_+(T\mathcal{B}(x)) \otimes S_-(T\mathcal{B}(x)) \oplus S_-(T\mathcal{B}(x)) \otimes S_+(T\mathcal{B}(x)).$$

In order to avoid confusion, we will use the symbol $'$ to indicate the twisted spinor bundles (That is, the second factors in the tensor products in the right hand sides of the above equations). Then one sees directly that $S'_\pm(T\mathcal{B}(x))$ are trivial vector bundles on which ∇^{TM} lifts to the trivial flat connections.

Let τ denote the $\mathbf{Z}_2$-grading operator of the splitting

$$S'(T\mathcal{B}(x)) = S'_+(T\mathcal{B}(x)) \oplus S'_-(T\mathcal{B}(x)),$$

that is, $\tau|_{S'_\pm(T\mathcal{B}(x))} = \pm\mathrm{Id}$. Then one has

$$\tau = \left(\frac{1}{\sqrt{-1}}\right)^{\frac{\dim M}{2}} \widehat{c}(e_1)\cdots\widehat{c}(e_{\dim M}). \tag{21}$$

From equation (6) and [8] Lemma 2.2, one gets the identification of differential operators

$$\begin{aligned}&-c(\vec{\mathrm{n}})D_{\partial\mathcal{B}(x),\mathrm{e}}c(\vec{\mathrm{n}}) = \tau D_{\partial\mathcal{B}(x),\mathrm{o}} \text{ as maps from}\\ &\Gamma(S_+(T\mathcal{B}(x))\otimes S'_-(T\mathcal{B}(x)))|_{\partial\mathcal{B}(x)} \text{ to } \Gamma(S_+(T\mathcal{B}(x))\otimes S'_-(T\mathcal{B}(x)))|_{\partial\mathcal{B}(x)}.\end{aligned} \tag{22}$$

Now denote the canonical Dirac operator on $\partial\mathcal{B}(x)$ with twisted coefficient the trivial vector bundle $S'(T\mathcal{B}(x))|_{\partial\mathcal{B}(x)}$ by

$$\widetilde{D}_{\partial\mathcal{B}(x)} : \Gamma(S_+(T\mathcal{B}(x))\otimes S'(T\mathcal{B}(x)))|_{\partial\mathcal{B}(x)} \to \Gamma(S_+(T\mathcal{B}(x))\otimes S'(T\mathcal{B}(x)))|_{\partial\mathcal{B}(x)}.$$

Set for any $0 \le u \le 1$ that

$$\widetilde{D}_{\partial\mathcal{B}(x)}(u) = (1-u)\widetilde{D}_{\partial\mathcal{B}(x)} + u\widehat{c}(V)\widetilde{D}_{\partial\mathcal{B}(x)}\widehat{c}(V). \tag{23}$$

From equations (6), (13), (17), (22), (23) and by proceeding as in [8] Sect. 3, one sees that the two families $\{D_{\partial\mathcal{B}(x),\mathrm{e}}(u)\}_{0\le u\le 1}$ and $\{\tau\widetilde{D}_{\partial\mathcal{B}(x)}(u)\}_{0\le u\le 1}$ are unitary equivalent. Thus, one has

$$\mathrm{sf}\{D_{\partial\mathcal{B}(x),\mathrm{e}}(u) : 0 \le u \le 1\} = \mathrm{sf}\{\tau\widetilde{D}_{\partial\mathcal{B}(x)}(u) : 0 \le u \le 1\}. \tag{24}$$

Furthermore, one can apply [7] (4.41) to our special case, with $g = \widehat{c}(V)$ acting on the trivial vector bundle $S'(T\mathcal{B}(x))$, to get that (with $n = \dim M$)

$$\int_0^1 \frac{c_{-1/2}}{\sqrt{\pi}}du = -\frac{1}{(2\pi\sqrt{-1})^{n/2}}\int_{\partial\mathcal{B}(x)} \widehat{A}(T\partial\mathcal{B}(x))\sum_{k=0}^{+\infty}\frac{k!}{(2k+1)!}\mathrm{Tr}_s[(\sum_{j=1}^{n-1}\widehat{c}(V)\widehat{c}(\nabla^{TM}_{f_j}V)f_j^*)^{2k+1}], \tag{25}$$

where Tr_s is the notation of the supertrace of operators on $S'(T\mathcal{B}(x))$ with respect to the $\mathbf{Z}_2$-grading operator τ.

Now by (21), one verifies that

$$\mathrm{Tr}_s[(\sum_{j=1}^{n-1}\widehat{c}(V)\widehat{c}(\nabla_{f_j}^{TM}V)f_j^*)^{2k+1}]=0 \ \text{ if } \ k\neq\frac{n}{2}-1, \tag{26}$$

while

$$\mathrm{Tr}_s[(\sum_{j=1}^{n-1}\widehat{c}(V)\widehat{c}(\nabla_{f_j}^{TM}V)f_j^*)^{n-1}]=\frac{2^{n/2}(n-1)!}{(-\sqrt{-1})^{n/2}}f_1^*\wedge\cdots\wedge f_{n-1}^*$$
$$\int^B V^*\wedge(\nabla_{f_1}^{TM}V)^*\wedge\cdots\wedge(\nabla_{f_{n-1}}^{TM}V)^*, \tag{27}$$

where $\int^B V^*\wedge(\nabla_{f_1}^{TM}V)^*\wedge\cdots\wedge(\nabla_{f_{n-1}}^{TM}V)^*$ is the function on $\partial\mathcal{B}(x)$ such that

$$V^*\wedge(\nabla_{f_1}^{TM}V)^*\wedge\cdots\wedge(\nabla_{f_{n-1}}^{TM}V)^*=f_1^*\wedge\cdots\wedge f_{n-1}^*\wedge(-\vec{\mathrm{n}})^*\int^B V^*\wedge(\nabla_{f_1}^{TM}V)^*\wedge\cdots\wedge(\nabla_{f_{n-1}}^{TM}V)^*. \tag{28}$$

Let $v:\partial\mathcal{B}(x)\to S^{n-1}(1)$ denote the canonical map induced by $V|_{\partial\mathcal{B}(x)}$, let ω be the volume form on $S^{n-1}(1)$. Then by (28) one verifies directly that

$$f_1^*\wedge\cdots\wedge f_{n-1}^*\int^B V^*\wedge(\nabla_{f_1}^{TM}V)^*\wedge\cdots\wedge(\nabla_{f_{n-1}}^{TM}V)^*=v^*\omega. \tag{29}$$

From (20), (24)-(27) and (29), one deduces that

$$\mathrm{sf}\{D_{\partial\mathcal{B}(x),e}(u),0\leq u\leq 1\}=-\frac{1}{\pi^{n/2}}\int_{\partial\mathcal{B}(x)}(\frac{n}{2}-1)!v^*\omega. \tag{30}$$

On the other hand, since the volume of $S^{n-1}(1)$ equals to $2\pi^{n/2}/(\frac{n}{2}-1)!$, from the standard differential geometric interpretation of the Brouwer degree, one has

$$\frac{(\frac{n}{2}-1)!}{2\pi^{n/2}}\int_{\partial\mathcal{B}(x)}v^*\omega=\deg_V(x),$$

which, together with (30), gives that

$$\mathrm{sf}\{D_{\partial\mathcal{B}(x),e}(u),0\leq u\leq 1\}=-2\deg_V(x). \tag{31}$$

The Poincaré-Hopf index formula then follows from (18) and (31). □

Acknowledgements The author is grateful to the referee for his critical reading and very helpful suggestions.

References

[1] M. F. Atiyah, Vector fields on manifolds. *Arbeitsgemeinschaft für Forschung des Landes Nordrhein-Westfalen, Düsseldorf 1969*, 200 (1970), 7–24.

[2] M. F. Atiyah and J. L. Dupont, Vector fields with finite singularities. *Acta Math.* 128 (1972), 1–40.

[3] M. F. Atiyah, V. K. Patodi and I. M. Singer, Spectral asymmetry and Riemannian geometry I. *Math. Proc. Cambridge Philos. Soc.* 77 (1975), 43–69.

[4] M. F. Atiyah, V. K. Patodi and I. M. Singer, Spectral asymmetry and Riemannian geometry III. *Math. Proc. Cambridge Philos. Soc.* 79 (1976), 71–99.

[5] B. Booss and K. P. Wojciechowski, *Elliptic Boundary Problems for Dirac Operators.* Birkhäuser, 1993.

[6] R. Bott and L. Tu, *Differential Forms in Algebraic Topology.* Graduate Text in Math. **82**, Springer-Verlag, 1982.

[7] X. Dai and W. Zhang, Higher spectral flow. *J. Funct. Anal.* 157 (1998), 432–469.

[8] P. B. Gilkey, On the index of geometrical operators for Riemannian manifolds with boundary. *Adv. in Math.* 102 (1993), 129-183.

[9] H. B. Lawson and M.-L. Michelsohn, *Spin Geometry.* Princeton Univ. Press, 1989.

[10] E. Witten, Supersymmetry and Morse theory. *J. Diff. Geom.* 17 (1982), 661–692.

[11] W. Zhang, Analytic and topological invariants associated to nowhere zero vector fields. *Pacific J. Math.* 187 (1999), 379-398.

Nankai Institute of Mathematics, Nankai University, Tianjin 300071, P. R. China
E-mail: weiping@nankai.edu.cn

Received January 29, 2000.