PERGAMON

Topology 39 (2000) 643–655

TOPOLOGY

www.elsevier.com/locate/top

A counting formula for the Kervaire semi-characteristic

Weiping Zhang*,[1]

Nankai Institute of Mathematics, Nankai University, Tianjin 300071, People's Republic of China

Received 4 September 1998; received in revised form 16 February 1999; accepted 17 March 1999

Dedicated to Professor Yanlin Yu on his sixtieth birthday

Abstract

We establish a generic counting formula for the Kervaire semi-characteristic of $4q+1$ dimensional manifolds.

Keywords: Kervaire semi-characteristic; Mod 2 index; Analytic localization

0. Introduction

Let M be a $4q+1$ dimensional smooth oriented closed manifold. The Kervaire semi-characteristic $k(M)$ of M is a mod 2 invariant defined by

$$k(M)=\sum_{i=0}^{2q}\dim H^{2i}(M;\mathbf{R})\qquad \mathrm{mod}\,2. \tag{0.1}$$

It admits an analytic interpretation in terms of the mod 2 index of Atiyah and Singer [3] (see also [1]).

In this paper, we will prove a topological counting formula for $k(M)$. Our main result may be viewed as a mod 2 analogue of the classical Poincaré–Hopf index formula which counts the Euler characteristic of a manifold through singularities of vector fields on that manifold.

To be more precise, let E be a codimension one orientable sub-bundle of the tangent vector bundle TM, the existence of which is a consequence of the Hopf Theorem (cf. [8]) saying that there always exist nowhere zero vector fields on closed orientable manifolds with vanishing Euler characteristic. Let X be a transversal section of E. Then the zero set of X, denoted by F, consists of

* Tel.: + 86-22-23500760; fax: + 86-22-23501532.

[1] Partially supported by CNNSF, EMC and the Qiu Shi Foundation.

E-mail address: weiping@sun.nankai.edu.cn (W. Zhang)

PII: S0040-9383(99)00036-1

a finite number of circles $F_1, \dots, F_p$ on M. Over each of these circles, one can associate canonically a line bundle through the behavior of X around the circle (See the main text for more details). Let F_o denote the subset of F consisting of those circles over each of which the associated line bundle is orientable.

The counting formula mentioned above can be stated as follows:

$$k(M) \equiv \sharp\{i \mid F_i \subset F_o\} \qquad \bmod 2. \tag{0.2}$$

Identity (0.2) formally looks very much like the Poincaré–Hopf formula. However, one notable difference is that while the Poincaré–Hopf formula counts the number of isolated points, here one counts the number of circles.[2]

While the formulation of (0.2) is purely topological, our proof of it is analytic. We first construct as in [11] a real skew-adjoint first-order elliptic differential operator whose mod 2 index provides an alternative analytic interpretation of $k(M)$. We then use the transversal section X to deform this skew-adjoint operator in a way similar to what Witten [10] used in his analytic approach to the Poincaré–Hopf formula. By applying the localization techniques developed in the paper of Bismut and Lebeau [4] to these deformed operators, one gets (0.2).

In fact, the above strategy applies to any orientable closed manifold with vanishing Euler characteristic (see Section 3 for more details). The remarkable fact is that, as was proved in [11], these mod 2 invariants actually equal to the Kervaire semi-characteristic in dimensions of form $4q+1$. This gives rise to the intrinsic formula (0.2).

This paper is organized as follows. In Section 1, we present an analytic interpretation of the Kervaire semi-characteristic in dimension $4q+1$ and state the main result of this paper. In Section 2, we introduce the deformation mentioned above and prove the main result stated in Section 1. The final Section 3 contains a brief discussion of extensions of the main result in arbitrary dimensions. There is also an appendix in which we present, for the sake of self-completeness of this paper, somewhat more details of the analysis described in Section 2 when one needs to apply the techniques of Bismut and Lebeau [4] to prove the main result stated in Section 1.

1. The Kervaire semi-characteristic in dimension $4q+1$: a counting formula

This section is organized as follows. In Section 1.1 we recall from Zhang [11] an analytic interpretation of the Kervaire semi-characteristic in dimension $4q+1$. In Section 1.2 we state the main result of this paper, which gives a counting formula for the Kervaire semi-characteristic in dimension $4q+1$.

1.1. An analytic interpretation of the Kervaire semi-characteristic in dimension $4q+1$

Let M be a $4q+1$ dimensional smooth oriented closed manifold. Let g^{TM} be a Riemannian metric on M whose associated Levi–Civita connection will be denoted by ∇^{TM}. For any $e \in TM$, let

[2] In fact, formulas counting the Kervaire semi-characteristic through isolated singularities of vector fields have been studied extensively by Atiyah and Dupont [2].

$e^* \in T^*M$ corresponds to e via g^{TM}. Let $c(e)$, $\hat{c}(e)$ be the Clifford operators acting on the exterior algebra bundle $\wedge^*(T^*M)$ given by

$$c(e) = e^* \wedge - i_e, \quad \hat{c}(e) = e^* \wedge + i_e, \tag{1.1}$$

where $e^* \wedge$ and i_e are the standard notation for exterior and inner multiplications. If $e, e' \in TM$, one has

$$\begin{aligned} &c(e)c(e') + c(e')c(e) = -2\langle e, e' \rangle, \\ &\hat{c}(e)\hat{c}(e') + \hat{c}(e')\hat{c}(e) = 2\langle e, e' \rangle, \\ &c(e)\hat{c}(e') + \hat{c}(e')c(e) = 0. \end{aligned} \tag{1.2}$$

Also, g^{TM} defines canonically an Euclidean inner product on $\Gamma(\wedge^*(T^*M))$. Let $\delta = d^*$ be the formal adjoint of the exterior differential operator d with respect to this inner product.

Now let V be a smooth nowhere zero vector field on M. The existence of V follows from a theorem of Hopf (cf. [8]) saying that there always exist nowhere zero vector fields on closed orientable manifolds with vanishing Euler characteristic. Without loss of generality, we can and we will assume that

$$|V|^2_{g^{TM}} = 1. \tag{1.3}$$

Definition 1.1 (Zhang [11, Definition 2.1]). The operator D_V is the operator acting on $\Gamma(\wedge^{\mathrm{even}}(T^*M))$ defined by

$$D_V = \tfrac{1}{2}(\hat{c}(V)(d + \delta) - (d + \delta)\hat{c}(V)). \tag{1.4}$$

By (1.2), one verifies that D_V is a real skew-adjoint elliptic first-order differential operator. Furthermore, if $e_0, e_1, \ldots, e_{4q}$ is an oriented orthonormal base of TM, then one has the following formula proved in [11, (2.4)]:

$$D_V = \hat{c}(V)(d + \delta) - \frac{1}{2}\sum_{i=0}^{4q} c(e_i)\hat{c}(\nabla^{TM}_{e_i} V). \tag{1.5}$$

Let

$$\mathrm{ind}_2\, D_V = \dim(\ker D_V) \mod 2 \tag{1.6}$$

be the mod 2 index of D_V in the sense of Atiyah and Singer [3]. The following result, which was proved in [11], gives an analytic interpretation of the Kervaire semi-characteristic $k(M)$ of M.

Theorem 1.2 (Zhang [11, Theorem 2.5]). *The following identity holds*:

$$\mathrm{ind}_2\, D_V = k(M). \tag{1.7}$$

Proof. We outline the proof of (1.7) for the completeness of this paper. Let D_R be the elliptic differential operator defined by

$$D_R = \hat{c}(e_0) \cdots \hat{c}(e_{4q})(d + \delta): \Gamma(\wedge^{\mathrm{even}}(T^*M)) \to \Gamma(\wedge^{\mathrm{even}}(T^*M)). \tag{1.8}$$

Then one verifies easily (cf. [3]) that D_R is real skew-adjoint with

$$\dim(\ker D_R) \equiv k(M) \qquad \bmod 2. \tag{1.9}$$

On the other hand, one verifies directly that the elliptic operator

$$D'_R = D_R - \frac{1}{2}\hat{c}(V)\hat{c}(e_0)\cdots\hat{c}(e_{4q})\sum_{i=0}^{4q} c(e_i)\hat{c}(\nabla^{TM}_{e_i}V) \tag{1.10}$$

is also skew-adjoint.

Formula (1.7) then follows from (1.5), (1.6) and (1.8)–(1.10), as well as the homotopy invariance property of the mod 2 index [3]. □

Theorem 1.2 will play an essential role in our proof of the main result of this paper to be stated in the next subsection.

1.2. *A counting formula for the Kervaire semi-characteristic in dimension* $4q+1$

Let M be as in Section 1.1 and γ_V denote the oriented line bundle generated and oriented by V. Let E be an orientable codimension one sub-bundle of TM. Without loss of generality, we may take E to be the orthogonal complement to γ_V in TM. Then E carries an induced orientation from those of TM and γ_V. Let g^E be the metric on E induced from g^{TM}.

Let X be a transversal section of E. Let F be the zero set of X. Then F consists of a union of disjoint circles $F_1, \ldots, F_p$. Let $i: F \hookrightarrow M$ denote the obvious embedding. Without loss of generality, one may well assume that $V|_F$ is tangent to F and that i^*E is the normal bundle to F in M.[3]

For any $x \in F$, let $e_0 = V, e_1, \ldots, e_{4q}$ be an oriented orthonormal base near x, and let $y_0, \ldots, y_{4q}$ be the normal coordinate system near x associated to $e_0(x), \ldots, e_{4q}(x)$. Then near x, X can be written as

$$X = \sum_{i=1}^{4q} f_i(y)e_i. \tag{1.11}$$

By the transversality of X, one sees that the following endomorphism of E_x is invertible:

$$C(x) = \{c_{ij}(x)\}_{1\leqslant i,j\leqslant 4q} \quad \text{with } c_{ij}(x) = \frac{\partial f_i}{\partial y_j}(0), \tag{1.12}$$

where the matrix is with respect to the base $e_1(x), \ldots, e_{4q}(x)$.

Let $C^*(x)$ be the adjoint of $C(x)$ with respect to $g^E|_{E_x}$ and $|C(x)| = \sqrt{C^*(x)C(x)}$. Let $K(x)$ be the endomorphism of $\wedge^*(E_x^*)$ defined by

$$K(x) = \mathrm{Tr}[|C(x)|] + \sum_{i,j=1}^{4q} c_{ij}(x)c(e_j(x))\hat{c}(e_i(x)). \tag{1.13}$$

[3] In fact, let f denote a unit tangent vector field of F. Then since $\dim E$ is of codimension one, one verifies easily from the transversality assumption that $\langle V|_F, f\rangle$ is nowhere zero on F. One can then deform V easily through nowhere zero vector fields to a nowhere zero vector field V', which is still transversal to E, such that $V'|_F = \mathrm{sign}(\langle V|_F, f\rangle)f$. One can then start with V' and, by the homotopy invariance of the mod 2 index [3], this does not affect the final result.

One verifies easily that $K(x)$ does not depend on the choice of the base $e_1(x), \dots, e_{4q}(x)$. Thus it defines an endomorphism K of the exterior algebra bundle $\wedge^*(E^*)|_F$ over F.

Now by [7, Proposition 2.21], one deduces easily that $\ker K$ forms a real line bundle $o_F(X)$ over F. Clearly, the orientability of $o_F(X)$ does not depend on the metric g^{TM}.

For any connected component F_i of F, denote by $o_{F_i}(X)$ the restriction of $o_F(X)$ on F_i. We can now state the main result of this paper as follows.

Theorem 1.3. *The following identity holds*:

$$k(M) \equiv \sharp\{i \,|\, o_{F_i}(X) \text{ is orientable over } F_i\} \qquad \text{mod } 2. \tag{1.14}$$

As an immediate consequence, if X has no zero, then one gets the following special case of a theorem of Atiyah [1, Theorem 1.2].

Corollary 1.4. *If there exist two linearly independent vector fields on M, then $k(M) = 0$.*

Remark 1.5. The formulation of (1.14) has been inspired by a result of Gompf described in the ICM-98 talk of Taubes [9, Section 5].

Theorem 1.3 will be proved in the next section by an analytic method.

2. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 by using the analytic interpretation of $k(M)$ given in Theorem 1.2. To do so, we introduce a deformation of the operator D_V and then apply the techniques of Bismut and Lebeau [4, Sections 8, 9] to the deformed operators. The algebraic result proved in [7, Proposition 2.21] also plays an important role in the proof.

This section is organized as follows. In Section 2.1, we introduce the deformation mentioned above. In Section 2.2 we apply the techniques in [4] to complete the proof of Theorem 1.3.

2.1. A deformation of the skew-adjoint operator D_V

We continue the discussions in Section 1. Recall that X is a transversal section of E and the skew-adjoint operator D_V is defined by (1.4). We first introduce the following deformation of D_V.

Definition 2.1. For any $T \in \mathbf{R}$, let $D_{V,T}$ be the operator defined by

$$D_{V,T} = D_V + T\hat{c}(V)\hat{c}(X) : \Gamma(\wedge^{\text{even}}(T^*M)) \to \Gamma(\wedge^{\text{even}}(T^*M)). \tag{2.1}$$

As X is perpendicular to V, by (1.2) one verifies that $D_{V,T}$ is also skew-adjoint. Thus, by the homotopy invariance of the mod 2 index [3], one has that for any $T \in \mathbf{R}$,

$$\dim(\ker D_{V,T}) \equiv \dim(\ker D_V) \qquad \text{mod } 2. \tag{2.2}$$

We now prove a Bochner-type formula for $-D_{V,T}^2$.

Let $e_0, e_1, \dots, e_{4q}$ be an oriented orthonormal base of TM. From (1.2), (1.5) and (2.1), one finds

$$D_{V,T} = \hat{c}(V)\left(d + \delta - \frac{1}{2}\hat{c}(V)\sum_{i=0}^{4q} c(e_i)\hat{c}(\nabla^{TM}_{e_i} V) + T\hat{c}(X)\right). \tag{2.3}$$

From (1.2) and (2.3), one deduces that

$$\begin{aligned} - D^2_{V,T} &= \left(d + \delta - \frac{1}{2}\hat{c}(V)\sum_{i=0}^{4q} c(e_i)\hat{c}(\nabla^{TM}_{e_i} V) + T\hat{c}(X)\right)^2 \\ &= - D^2_V + T[d + \delta, \hat{c}(X)] - T\sum_{i=0}^{4q} \langle \nabla^{TM}_{e_i} X, V\rangle c(e_i)\hat{c}(V) + T^2|X|^2. \end{aligned} \tag{2.4}$$

On the other hand, one has standardly that

$$d + \delta = \sum_{i=0}^{4q} c(e_i)\nabla^{\wedge^*(T^*M)}_{e_i}, \tag{2.5}$$

where $\nabla^{\wedge^*(T^*M)}$ is the canonical lift of ∇^{TM} to $\wedge^*(T^*M)$.

From (1.2), (2.4) and (2.5), one gets the following Bochner-type formula which will play an important role in the next subsection:

$$- D^2_{V,T} = - D^2_V + T\sum_{i=0}^{4q} \left(c(e_i)\hat{c}(\nabla^{TM}_{e_i} X) - \langle \nabla^{TM}_{e_i} X, V\rangle c(e_i)\hat{c}(V)\right) + T^2|X|^2. \tag{2.6}$$

2.2. *Proof of Theorem 1.3*

We first prove a simple estimate which enables us to localize the problem to arbitrarily small neighborhoods of the zero set F.

Proposition 2.2. *For any open neighborhood U of F, there exist constants $C' > 0$, $b > 0$ such that for any $T \geqslant 1$ and any $s \in \Gamma(\wedge^{\mathrm{even}}(T^*M))$ with* $\mathrm{Supp}\, s \subset M\backslash U$, *one has the following estimate of Sobolev norms*:

$$\|D_{V,T}s\|_0^2 \geqslant C'(\|s\|_1^2 + (T - b)\|s\|_0^2). \tag{2.7}$$

Proof. Since $M\backslash U$ is compact and X is nowhere zero on $M\backslash U$, Proposition 2.2 follows trivially from the Bochner-type formula (2.6). $\square$

By Proposition 2.2, we need only to concentrate on the analysis near F. By this we follow closely the arguments in [4, Sections 8, 9]. In particular, we will take the advantage of the topological nature of the problem to simplify the analysis greatly.

The main observation is that since the normal bundle i^*E to F is oriented and F consists of a union of circles, i^*E is actually a trivial bundle over F. Thus a sufficiently small neighborhood of

F should be of the form $F \times \mathbf{R}^{4q}$. Furthermore, one can choose the metric g^{TM} such that near F it is of the product form

$$g^{TM} = g^{TF} \oplus g^{T\mathbf{R}^{4q}}, \tag{2.8}$$

with each restriction $g^{TF_i} = g^{TF}|_{F_i}$ comes from the standard metric on the circle S^1 while $g^{T\mathbf{R}^{4q}}$ comes from the standard Euclidean metric.

Thus one can fix a covariantly constant oriented Euclidean base $e_1, \dots, e_{4q}$ of i^*E such that $e_0 = V, e_1, \dots, e_{4q}$ forms an oriented orthonormal base of $i^*(TM)$. Let $y_1, \dots, y_{4q}$ be the standard Euclidean coordinates associated to $e_1, \dots, e_{4q}$. Clearly, any point sufficiently closed to F can be represented uniquely by $(x, y) = (x, y_1, \dots, y_{4q})$ with $x \in F$. In particular, the vector field X can be written, near F, as

$$X(x,y) = \sum_{i=1}^{4q} f_i(x, y) e_i \quad \text{with each } f_i(x, 0) \equiv 0. \tag{2.9}$$

The transversal condition of X takes the same form as in (1.12), with the matrices

$$C(x) = \{c_{ij}(x)\}_{1 \leqslant i,j \leqslant 4q} \quad \text{with } c_{ij}(x) = \frac{\partial f_i}{\partial y_j}(x,0) \tag{2.10}$$

being invertible for all $x \in F$. When there is no confusion we denote by $C = \{c_{ij}\}_{1 \leqslant i,j \leqslant 4q}$ the endomorphism over F obtained from $C(x) = \{c_{ij}(x)\}_{1 \leqslant i,j \leqslant 4q}$, $x \in F$. Let $|C| = \sqrt{C^* C}$ be defined as in Section 1.1.

With all of the above simplifications, one deduces easily that near F, one has the following very simple form of the Bochner-type formula (2.6):

$$-D_{V,T}^2 = -\sum_{i=0}^{4q} \nabla_{e_i}^2 + T\left(\sum_{i,j=1}^{4q} c_{ij} c(e_j) \hat{c}(e_i) + O(|y|)\right) + T^2(\langle |C|y, |C|y\rangle + O(|y|^3)). \tag{2.11}$$

Now for any $x \in F$, by [7, Proposition 2.21] one knows that the operator

$$K(x) = \mathrm{Tr}[|C(x)|] + \sum_{i,j=1}^{4q} c_{ij}(x) c(e_j) \hat{c}(e_i) \colon \wedge^*(E_x^*) \to \wedge^*(E_x^*) \tag{2.12}$$

is nonnegative with $\dim(\ker K(x)) = 1$. Furthermore, one has

$$\ker K(x) \subset \wedge^{\mathrm{even}}(E_x^*) \text{ if } \det C(x) > 0 \tag{2.13}$$

and

$$\ker K(x) \subset \wedge^{\mathrm{odd}}(E_x^*) \text{ if } \det C(x) < 0. \tag{2.13$'$}$$

For any $x \in F$, we fix an element $\rho(x)$ of unit length in $\ker K(x)$.

On the other hand, for any $T > 0$, one verifies easily that on each fiber E_x, the operator $-\sum_{i=1}^{4q} \nabla_{e_i}^2 - T\,\mathrm{Tr}[|C|] + T^2 \langle |C|y, |C|y\rangle$ acting on $C^\infty(E_x)$ is a (rescaled) harmonic oscillator whose kernel is one dimensional and is generated by $\exp(-T\langle |C|y, |C|y\rangle/2)$.

To summarize, one has the following result (compare with [7, Corollary 2.22]) which will play the same role as [4, Theorem 7.4] played in [4, Section 9].

Lemma 2.3. *Take $T > 0$. Then for any $x \in F$, as an operator acting on $\Gamma(\wedge^*(E_x^*))$ over E_x, $-\sum_{i=1}^{4q}\nabla_{e_i}^2 + T\sum_{i,j=1}^{4q} c_{ij}c(e_j)\hat{c}(e_i) + T^2\langle |C|y, |C|y\rangle$ is nonnegative with the kernel being of one dimension and generated by $\exp(-T\langle |C|y, |C|y\rangle/2)\rho(x)$ with $\rho(x) \subset \wedge^{\mathrm{even}}(E_x^*)$ if $\det C(x) > 0$ and $\rho(x) \subset \wedge^{\mathrm{odd}}(E_x^*)$ if $\det C(x) < 0$. Furthermore, the nonzero eigenvalues of it are all $\geqslant TA$ for some positive constant A which can be chosen not depending on x.*

Now let $o_F(X) \subset \wedge^*(E^*|_F)$ denote the line bundle formed by $\ker K(x)$, $x \in F$. Then $\wedge^*(T^*F)\otimes o_F(X)$ is a sub-bundle of $\wedge^*(T^*M)|_F$. Let p denote the canonical orthogonal projection mapping from $\Gamma(\wedge^*(T^*M)|_F)$ onto $\Gamma(\wedge^*(T^*F)\otimes o_F(X))$. Let $D^F = d^F + \delta^F$ be the de Rham–Hodge operator acting on $\Gamma(\wedge^*(T^*F)\otimes o_F(X))$.

Set

$$D^H = \sum_{i=0}^{4q} c(e_i)(i^*\nabla^{\wedge^*(T^*M)})(e_i). \tag{2.14}$$

From (2.14) and our simplified assumptions near F, one gets easily the following result, which is the analogue of [4, (8.93)].

Proposition 2.4. *The following identity for differential operators acting on $\Gamma(\wedge^{(1-\mathrm{sgn}\det(C))/2}(T^*F)\otimes o_F(X))$, where we use the standard notation that* $\mathrm{sgn}\det(C) = 1$ *if* $\det(C) > 0$ *and* $\mathrm{sgn}\det(C) = -1$ *if* $\det(C) < 0$, *holds*,

$$p\hat{c}(V)D^H p = \hat{c}(V)D^F. \tag{2.15}$$

Proof of Theorem 1.3. One first verifies that

$$-(\hat{c}(V)D^F)^2 = D^{F,2}. \tag{2.16}$$

Let $c_0 > 0$ be such that the operator $D^{F,2}$ acting on $\Gamma(\wedge^{(1-\mathrm{sgn}\det(C))/2}(T^*F)\otimes o_F(X))$ contains no eigenvalues in $(0, 2c_0)$.

By Propositions 2.2, 2.4, Lemma 2.3 and (2.16), one can proceed as in [4, Section 9] to prove the following analogue of [4, (9.156)].[4] That is, there exists $T_0 > 0$ such that for any $T \geqslant T_0$,

$$\#\{\lambda : \lambda \in \mathrm{Sp}(-D_{V,T}^2), \lambda \leqslant c_0\} = \dim(\ker D^{F,2}). \tag{2.17}$$

From (2.2), (2.17), Theorem 1.2, the skew-adjointness of $D_{V,T}$ as well as the Hodge theorem for D^F, one gets

$$k(M) \equiv \dim H^{(1-\mathrm{sgn}\det(C))/2}(F; o_F(X)) \mod 2. \tag{2.18}$$

[4] See Appendix A for a more detailed discussion.

Now as each connected component F_i of F is a circle, it is clear that if $o_{F_i}(X)$ is orientable over F_i, then $\dim H^{(1-\mathrm{sgn}\det(C))/2}(F_i;o_{F_i}(X))=1$; while if $o_{F_i}(X)$ is nonorientable over F_i, then $\dim H^{(1-\mathrm{sgn}\det(C))/2}(F_i;o_{F_i}(X))=0$.

Theorem 1.3 follows from (2.18) and the above discussion. □

3. Applications and extensions

Let M be a $4q+1$-dimensional smooth closed oriented manifold. Let $k_2(M)$ be the $\mathbf{Z}_2$-Kervaire semi-characteristic defined by

$$k_2(M)=\sum_{i=0}^{2q}\dim H^{2i}(M;\mathbf{Z}_2) \qquad \mathrm{mod}\,2. \tag{3.1}$$

By a result of Lusztig et al. [6], one knows that

$$k(M)-k_2(M)=\langle w_2(TM)w_{4q-1}(TM),[M]\rangle, \tag{3.2}$$

where w_i is the ith Stiefel–Whitney class of TM.

Since $w_2(TM)=0$ if M is spin, one gets from (3.2) and Theorem 1.3 the following consequence (compare also with the Remark in [1, pp. 16]).

Theorem 3.1. *Under the same condition as in Theorem 1.3, if M is also a spin manifold, then*

$$k_2(M)\equiv \sharp\{i\,|\,o_{F_i}(X) \text{ is orientable over } F_i\} \qquad \mathrm{mod}\,2. \tag{3.3}$$

We now drop the condition that $\dim M=4q+1$ and let M be a smooth closed oriented manifold with vanishing Euler characteristic. Let V be a nowhere zero vector field on M whose existence is given by the theorem of Hopf mentioned in Section 1.1. Then one can define as in Section 1.1 the operator D_V as well as the associated mod 2 index

$$\alpha(V)=\mathrm{ind}_2\,D_V\equiv\dim(\ker D_V) \qquad \mathrm{mod}\,2. \tag{3.4}$$

We now state the following result which extends Theorem 1.2 to all dimensions.

Theorem 3.2. (i) *If* $\dim M=4q+2$ *or* $4q+3$, *then* $\alpha(V)=0$; (ii) *if* $\dim M=4q$, *then*

$$\alpha(V)=\frac{\mathrm{sign}(M)}{2} \qquad \mathrm{mod}\,2, \tag{3.5}$$

where $\mathrm{sign}(M)$ *is the signature of* M.

Proof. (i) Let $e_1,\dots,e_{\dim M-1}$ be an oriented orthonormal base of E, the orthogonal complement to the line bundle γ_V generated by V in TM. Set

$$\hat{c}(E)=\prod_{i=1}^{\dim M-1}\hat{c}(e_i). \tag{3.6}$$

From (1.2)–(1.4), one verifies that when $\dim M = 4q+3$, then

$$D_V \hat{c}(E) = \hat{c}(E) D_V, \quad \hat{c}(E)^2 = -1 \tag{3.7}$$

and that when $\dim M = 4q+2$, then

$$D_V \hat{c}(V)\hat{c}(E) = -\hat{c}(V)\hat{c}(E) D_V, \quad (\hat{c}(V)\hat{c}(E))^2 = -1. \tag{3.8}$$

From (3.7) and (3.8) one sees that when $\dim M = 4q+2$ or $4q+3$, $\ker(D_V)$ admits a complex structure and is hence of even dimension.

(ii) Now we assume $\dim M = 4q$. Set

$$c(TM) = c(V) \prod_{i=1}^{\dim M - 1} c(e_i). \tag{3.9}$$

From (1.2) and (1.3) one verifies that

$$c(TM)^2 = 1. \tag{3.10}$$

Set

$$\wedge_{\pm}(T^*M) = \{ s \in \wedge^*(T^*M) \colon (-1)^q c(TM)s = \pm s \}. \tag{3.11}$$

One verifies easily that $\hat{c}(V)$ induces an isomorphism

$$\hat{c}(V) \colon \Gamma(\wedge^{\mathrm{odd}}(T^*M)) \cap \Gamma(\wedge_-(T^*M)) \to \Gamma(\wedge^{\mathrm{even}}(T^*M)) \cap \Gamma(\wedge_-(T^*M)). \tag{3.12}$$

On the other hand, from (1.2)–(1.5) one deduces the following identity of operators acting on $\Gamma(\wedge^*(T^*M))$:

$$\begin{aligned} \hat{c}(V)(\hat{c}(V)(d+\delta) - (d+\delta)\hat{c}(V)) &= -(\hat{c}(V)(d+\delta) - (d+\delta)\hat{c}(V))\hat{c}(V) \\ &= 2(d+\delta) - \hat{c}(V) \sum_{i=0}^{4q} c(e_i)\hat{c}(\nabla^{TM}_{e_i} V). \end{aligned} \tag{3.13}$$

From (3.11)–(3.13) and the definition of the signature operator (cf. [5, Example 6.2]), one deduces that

$$\begin{aligned} \dim(\ker D_V) &= \dim(\ker D_V|_{\Gamma(\wedge^{\mathrm{even}}(T^*M)) \cap \Gamma(\wedge_+(T^*M))}) \\ &\quad + \dim(\ker D_V|_{\Gamma(\wedge^{\mathrm{even}}(T^*M)) \cap \Gamma(\wedge_-(T^*M))}) \\ &\equiv \dim(\ker D_V|_{\Gamma(\wedge^{\mathrm{even}}(T^*M)) \cap \Gamma(\wedge_+(T^*M))}) \\ &\quad - \dim(\ker (D_V \hat{c}(V))|_{\Gamma(\wedge^{\mathrm{odd}}(T^*M)) \cap \Gamma(\wedge_-(T^*M))}) \qquad \mathrm{mod}\, 2 \\ &= \mathrm{ind}\, [d+\delta - \hat{c}(V)(d+\delta)\hat{c}(V) \colon \Gamma(\wedge^{\mathrm{even}}(T^*M)) \cap \Gamma(\wedge_+(T^*M)) \\ &\quad \to \Gamma(\wedge^{\mathrm{odd}}(T^*M)) \cap \Gamma(\wedge_-(T^*M))] \\ &= \mathrm{ind}\, [d+\delta \colon \Gamma(\wedge^{\mathrm{even}}(T^*M)) \cap \Gamma(\wedge_+(T^*M)) \\ &\quad \to \Gamma(\wedge^{\mathrm{odd}}(T^*M)) \cap \Gamma(\wedge_-(T^*M))] \\ &= \tfrac{1}{2}(e(M) + \mathrm{sign}(M)). \end{aligned} \tag{3.14}$$

From (3.14) and the assumption that $e(M) = 0$, one gets (3.5).
The proof of Theorem 3.2 is completed. $\square$

Now let X be a transversal section of E. Let F be the zero set of X which consists of a union of disjoint circles F_i's. Let $o_F(X)$ be the line bundle over F defined in a similar way as the one defined in Section 1.2 for the $4q+1$ dimensional case. Let $o_{F_i}(X)$ be the restriction of $o_F(X)$ on the component F_i. From Theorem 3.2 and by proceeding similarly as in Section 2.2, one gets easily the following result which is the analogue of Theorem 1.3 in other dimensions (compare with [2, Theorem 1.1]).

Theorem 3.3. (i) *If* $\dim M = 4q+2$ *or* $4q+3$, *then*

$$\sharp\{i \mid o_{F_i}(X) \text{ is orientable over } F_i\} \equiv 0 \mod 2, \tag{3.15}$$

(ii) *if* $\dim M = 4q$, *then*

$$\sharp\{i \mid o_{F_i}(X) \text{ is orientable over } F_i\} \equiv \frac{\operatorname{sign}(M)}{2} \mod 2. \tag{3.16}$$

Appendix A. Some estimates needed for the proof of Theorem 1.3

The purpose of this appendix is to provide a more detailed version of the proof of (2.17). We will follow closely [4, Sections 8, 9].

For any $\mu \geqslant 0$, let $\mathbf{H}^\mu(M)$, $\mathbf{H}^\mu(F)$ be the μth Sobolev spaces of sections of the bundles $\wedge^{\mathrm{even}}(T^*M)$, $\wedge^{(1-\operatorname{sgn}\det(C))/2}(T^*F)\otimes o_F(X)$, respectively. We use the standard L^2-norm for 0th Sobolev norm.

Let $\varepsilon_0 > 0$ be sufficiently small so that over the tubular neighborhood $B_{2\varepsilon_0}(F) = \{(x,y) \mid x \in F, |y| \leqslant 2\varepsilon_0\}$ of F, one has the product metric of form (2.8) and that the restricted tubular neighborhoods around the connected components of F do not intersect with each other. Without confusion, we identify $B_{2\varepsilon_0}(F)$ with the corresponding disc bundle in the normal bundle i^*E.

Let $\gamma : \mathbf{R} \to [0,1]$ be a smooth function such that $\gamma(a) = 1$ if $a \leqslant \frac{1}{2}$ and that $\gamma(a) = 0$ if $a \geqslant 1$. Let $0 < \varepsilon < 2\varepsilon_0$ which will be further fixed later. If $(x,y) \in E|_F$, set $\gamma_\varepsilon(x,y) = \gamma(|y|/\varepsilon)$.

For $T > 0$, $x \in F$, set

$$\alpha_T(x) = \int_{E_x} \exp(-T\langle |C|y, |C|y\rangle)\gamma_\varepsilon^2(x,y)|\det(C)|\,dv_{E_x}(y). \tag{A.1}$$

For any $\mu \geqslant 0$, $T > 0$, let $J_T : \mathbf{H}^\mu(F) \to \mathbf{H}^\mu(M)$ be the linear map defined by

$$u \in \mathbf{H}^\mu(F) \mapsto (\alpha_T)^{-1/2}\gamma_\varepsilon(x,y)\sqrt{\det|C|}\exp\left(-\frac{T\langle |C|y, |C|y\rangle}{2}\right)u(x) \in \mathbf{H}^\mu(M). \tag{A.2}$$

By the definition of γ_ε, the map J_T is well defined.

For $\mu \geqslant 0$, $T > 0$, let $\mathbf{H}^\mu_T(M)$ be the image of $\mathbf{H}^\mu(F)$ in $\mathbf{H}^\mu(M)$ by J_T. Let $\mathbf{H}^{0,\perp}_T(M)$ be the orthogonal space to $\mathbf{H}^0_T(M)$ in $\mathbf{H}^0(M)$, let p_T, $p_T^\perp$ be the orthogonal projection from $\mathbf{H}^0(M)$ to $\mathbf{H}^0_T(M)$, $\mathbf{H}^{0,\perp}_T(M)$, respectively. Set $\mathbf{H}^{\mu,\perp}_T(M) = \mathbf{H}^\mu(M) \cap \mathbf{H}^{0,\perp}_T(M)$. Clearly, J_T maps $\mathbf{H}^0(F)$ onto $\mathbf{H}^0_T(M)$ isometrically.

Following [4, Section 9b], we now write $D_{V,T}$ as a $(2,2)$ matrix and prove the corresponding estimates for them.

For any $T \in \mathbf{R}$, set

$$\begin{aligned} D_{V,T,1} &= p_T D_{V,T} p_T, \qquad & D_{V,T,2} &= p_T D_{V,T} p_T^{\perp}, \\ D_{V,T,3} &= p_T^{\perp} D_{V,T} p_T, \qquad & D_{V,T,4} &= p_T^{\perp} D_{V,T} p_T^{\perp}. \end{aligned} \tag{A.3}$$

We now state the following result which consists of the analogues of [4, Theorems 9.8, 9.10 and 9.14] in our situation.

Proposition A.1. 1. *As $T \to +\infty$, the following formula for operators acting on $\Gamma(\wedge^{(1-\operatorname{sgn}\det(C))/2}(T^*F) \otimes o_F(X))$ holds*:

$$J_T^{-1} D_{V,T,1} J_T = \hat{c}(V) D^F + O\left(\frac{1}{\sqrt{T}}\right). \tag{A.4}$$

2. *There exists $C_1 > 0$ such that for any $T \geqslant 1$, $s \in \mathbf{H}_T^{1,\perp}(M)$, $s' \in \mathbf{H}_T^1(M)$, we have*

$$\|D_{V,T,2} s\|_0 \leqslant C_1 \left(\frac{\|s\|_1}{\sqrt{T}} + \|s\|_0 \right), \tag{A.5}$$

$$\|D_{V,T,3} s'\|_0 \leqslant C_1 \left(\frac{\|s'\|_1}{\sqrt{T}} + \|s'\|_0 \right). \tag{A.6}$$

3. *There exists $\varepsilon \in (0, \varepsilon_0]$, $T_0 > 0$, $C_2 > 0$ such that for any $T \geqslant T_0$, $s \in \mathbf{H}_T^{1,\perp}(M)$, we have*

$$\|D_{V,T,4} s\|_0 \geqslant C_2 (\|s\|_1 + \sqrt{T} \|s\|_0). \tag{A.7}$$

Proof. Proposition (A.1) can be proved in the same way as [4, Theorems 9.8, 9.10 and 9.14] were proved in [4, Section 9b]. As the situation here is much simpler, we outline the main steps.

First of all, by (2.3), (2.5) and the simplified geometric assumptions made in this subsection, one has the following formula for $D_{V,T}$ near F,

$$D_{V,T} = \hat{c}(V) c(V) \nabla_V + \hat{c}(V) \sum_{i=1}^{4q} c(e_i) \nabla_{e_i} + T \sum_{i,j=1}^{4q} c_{ij} y_j \hat{c}(V) \hat{c}(e_i) + O(|y|) + T O(|y|^2), \tag{A.8}$$

which is the analogue of [4, (8.58)].

Now one verifies directly that

$$\begin{aligned} & - \left(\hat{c}(V) \sum_{i=1}^{4q} c(e_i) \nabla_{e_i} + T \hat{c}(V) \sum_{i,j=1}^{4q} c_{ij} y_j \hat{c}(e_i) \right)^2 \\ & = - \sum_{i=1}^{4q} \nabla_{e_i}^2 + T \sum_{i,j=1}^{4q} c_{ij} c(e_j) \hat{c}(e_i) + T^2 \langle |C| y, |C| y \rangle \end{aligned} \tag{A.9}$$

which is exactly the operator dealt with in Lemma 2.3.

On the other hand, by (2.15) one verifies easily that

$$J_T^{-1} p_T \hat{c}(V) c(V) \nabla_V p_T J_T = \hat{c}(V) D^F. \tag{A.10}$$

From (A.8)–(A.10) and Lemma 2.3, one can proceed easily as in [4, pp. 104–108] to get Parts 1, 2 of Proposition A.1.

Similarly, by using (A.8)–(A.10) and Lemma 2.3, one can proceed as in [4, pp. 109–114], in a much simpler form, to find $\varepsilon \in (0, 3\varepsilon_0/2)$, $C_3 > 0$, $b > 0$ such that for any $T \geqslant 1$, any $s \in \mathbf{H}_T^{1,\perp}(M)$ with Supp $s \subset B_\varepsilon(F)$, one has

$$\|D_{V,T}s\|_0^2 \geqslant C_3(\|s\|_1^2 + (T - b)\|s\|_0^2) \tag{A.11}$$

which together with Proposition 2.2 and the gluing arguments in [4, pp. 115–116] give the following

Proposition A.2. *There exist* $\varepsilon \in (0, \varepsilon_0]$, $C_4 > 0$, $b' > 0$ *such that for any* $T \geqslant 1$, *any* $s \in \mathbf{H}_T^{1,\perp}(M)$, *then*

$$\|D_{V,T}s\|_0^2 \geqslant C_4(\|s\|_1^2 + (T - b')\|s\|_0^2). \tag{A.12}$$

Part 3 of Proposition A.1 follows easily from Proposition A.2 as well as the proved Part 2 of Proposition A.1. □

With Proposition A.1 in hand, one can then proceed as in [4, pp. 117–125] to complete the proof of (2.17) easily. □

Acknowledgements

The author would like to thank Professor Johan Dupont for helpful discussions related to Theorem 3.2. He is also grateful to the referee for his very useful suggestions.

References

[1] M.F. Atiyah, Vector fields on manifolds, Arbeitsgemeinschaft für Forschung des Landes Nordrhein-Westfalen, Dusseldorf 1969 200 (1970) 7–24.
[2] M.F. Atiyah, J.L. Dupont, Vector fields with finite singularities, Acta Mathematica 128 (1972) 1–40.
[3] M.F. Atiyah, I.M. Singer, The index of elliptic operators V, Annals of Mathematics 93 (1971) 139–149.
[4] J.-M. Bismut, G. Lebeau, Complex immersions and Quillen metrics, Publicacions Mathematiques IHES 74 (1991) 1–297.
[5] H.B. Lawson, M.-L. Michelsohn, Spin Geometry, Princeton Univ. Press, Princeton, NJ, 1989.
[6] G. Lusztig, J. Milnor, F. Peterson, Semi-characteristics and cobordism, Topology 8 (1969) 357–359.
[7] M. Shubin, Novikov inequalities for vector fields, The Gelfand Mathematical Seminar, 1993–1995, Birkhäuser, Boston, 1996, pp. 243–274.
[8] N. Steenrod, The Topology of Fiber Bundles, Princeton Univ. Press, Princeton, NJ, 1951.
[9] C.H. Taubes, The geometry of the Seiberg–Witten invariants, Documenta Math. Extra Vol. ICM 1998, II, 493–504.
[10] E. Witten, Supersymmetry and Morse theory, Journal of Differential Geometry 17 (1982) 661–692.
[11] W. Zhang, Analytic and topological invariants associated to nowhere zero vector fields, Pacific Journal of Mathematics 187 (1999) 379–398.