Asian J. Math.
Vol. 4, No. 4, pp. 933–960, December 2000

012

SPIN^C MANIFOLDS AND RIGIDITY THEOREMS IN K-THEORY*

KEFENG LIU[†], XIAONAN MA[‡], and WEIPING ZHANG[§]

Abstract. We extend our family rigidity and vanishing theorems in [LiuMaZ] to the Spin^c case. In particular, we prove a K-theory version of the main results of [H], [Liu1, Theorem B] for a family of almost complex manifolds.

0. Introduction. Let M, B be two compact smooth manifolds, and $\pi : M \to B$ be a smooth fibration with compact fibre X. Let TX be the relative tangent bundle. Assume that a compact Lie group G acts fiberwise on M, that is, the action preserves each fiber of π. Let P be a family of G-equivariant elliptic operators along the fiber X. Then the family index of P, $\mathrm{Ind}(P)$, is a well-defined element in $K(B)$ (cf. [AS]) and is a virtual G-representation (cf. [LiuMa1]). We denote by $(\mathrm{Ind}(P))^G \in K(B)$ the G-invariant part of $\mathrm{Ind}(P)$.

A family of elliptic operator P is said to be *rigid on the equivariant Chern character level* with respect to this G-action, if the equivariant Chern character $\mathrm{ch}_g(\mathrm{Ind}(P)) \in H^*(B)$ is independent of $g \in G$. If $\mathrm{ch}_g(\mathrm{Ind}(P))$ is identically zero for any g, then we say P has *vanishing property on the equivariant Chern character level*. More generally, we say that P is *rigid on the equivariant K-theory level*, if $\mathrm{Ind}(P) = (\mathrm{Ind}(P))^G$. If this index is identically zero in $K_G(B)$, then we say that P has *vanishing property on the equivariant K-theory level*. To study rigidity and vanishing, we only need to restrict to the case where $G = S^1$. From now on we assume $G = S^1$.

As was remarked in [LiuMaZ], the rigidity and vanishing properties on the K-theory level are more subtle than that on the Chern character level. The reason is that the Chern character can kill the torsion elements involved in the index bundle.

In [LiuMaZ], we proved several rigidity and vanishing theorems on the equivariant K-theory level for elliptic genera. In this paper, we apply the method in [LiuMaZ] to prove rigidity and vanishing theorems on the equivariant K-theory level for Spin^c manifolds, as well as for almost complex manifolds. To prove the main results of this paper, to be stated in Section 2.1, we will introduce some shift operators on certain vector bundles over the fixed point set of the circle action, and compare the index bundles after the shift operation. Then we get a recursive relation of these index bundles which will in turn lead us to the final result (cf. [LiuMaZ]).

Let us state some of our main results in this paper more explicitly. As was remarked in [LiuMaZ], our method is inspired by the ideas of Taubes [T] and Bott-Taubes [BT].

For a complex (resp. real) vector bundle E over M, let

$$\begin{aligned} &\mathrm{Sym}_t(E) = 1 + tE + t^2\mathrm{Sym}^2 E + \cdots, \\ &\Lambda_t(E) = 1 + tE + t^2\Lambda^2 E + \cdots \end{aligned} \tag{0.1}$$

---

*Received June 13, 2000; accepted for publication September 14, 2000.

[†]Department of Mathematics, Stanford University, Stanford, CA 94305, USA (kefeng@math.stanford.edu). Partially supported by the Sloan Fellowship and an NSF grant.

[‡]Humboldt-Universitat zu Berlin, Institut für Mathematik, Rudower Chausse 25, 12489 Berlin, Germany (xiaonan@mathematik.hu-berlin.de). Partially supported by SFB 288.

[§]Nankai Institute of Mathematics, Nankai university, Tianjin 300071, P. R. China (weiping@nankai.edu.cn). Partially supported by NSFC, MOEC and the Qiu Shi Foundation.

be the symmetric and exterior power operations of E (resp. $E \otimes_{\mathbf{R}} C$) in $K(M)[[t]]$ respectively.

We assume that TX has an S^1-invariant almost complex structure J. Then we can construct canonically the Spin^c Dirac operator D^X on $\Lambda^*(T^{(0,1)*}X)$ along the fiber X. Let W be an S^1-equivariant complex vector bundle over M. We denote by $K_W = \det W$ and $K_X = \det(T^{(1,0)}X)$ the determinant line bundles of W and $T^{(1,0)}X$ respectively. Let

$$Q_1(W) = \otimes_{n=0}^{\infty} \Lambda_{-q^n}(\overline{W}) \otimes \bigotimes\nolimits_{n=1}^{\infty} \Lambda_{-q^n}(W). \tag{0.2}$$

For $N \in \mathbf{N}, N \geq 2$, let $y = e^{2\pi i/N} \in \mathbf{C}$. Let G_y be the multiplicative group generated by y. Following Witten [W], we consider the fiberwise action G_y on W and $\overline{W}$ by sending $y \in G_y$ to y on W and y^{-1} on $\overline{W}$. Then G_y acts naturally on $Q_1(W)$. We define $Q_1(T^{(1,0)}X)$ and the action G_y on it in the above way.

The following theorem generalizes the result in [H] to the family case.

Theorem 0.1. *Assume* $c_1(T^{(1,0)}X) = 0 \bmod(N)$, *the family of* $G_y \times S^1$ *equivariant Spin*^{c} *Dirac operators* $D^X \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX \otimes_{\mathbf{R}} \mathbf{C}) \otimes Q_1(T^{(1,0)}X)$ *is rigid on the equivariant K-theory level, for the* S^1 *action.*

The following family rigidity and vanishing theorem generalizes [Liu1, Theorem B] to the family case.

Theorem 0.2. *Assume* $\omega_2(TX - W)_{S^1} = 0$, $\frac{1}{2}p_1(TX - W)_{S^1} = e\cdot\overline{\pi}^* u^2$ $(e \in \mathbf{Z})$ *in* $H^*_{S^1}(M, \mathbf{Z})$, *and* $c_1(W) = 0 \bmod(N)$. *Consider the family of* $G_y \times S^1$ *equivariant Spin*^{c} *Dirac operators*

$$D^X \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX \otimes_{\mathbf{R}} \mathbf{C}) \otimes Q_1(W).$$

i) If $e = 0$, *then these operators are rigid on the equivariant K-theory level for the* S^1 *action.*

ii) If $e < 0$, *then the index bundles of these operators are zero in* $K_{G_y \times S^1}(B)$. *In particular, these index bundles are zero in* $K_{G_y}(B)$.

We refer to Section 2 for more details on the notation in Theorem 0.2. Actually, our main result, Theorem 2.2, holds on a family of Spin^c-manifolds with Theorem 0.2 being one of its special cases.

This paper is organized as follows. In Section 1, we recall a K-Theory version of the equivariant family index theorem for the circle action case [LiuMaZ, Theorem 1.2]. As an immediate corollary, we get a K-theory version of the vanishing theorem of Hattori for a family of almost complex manifolds. In Section 2, we prove the rigidity and vanishing theorem for elliptic genera in the Spin^c case, on the equivariant K-theory level. The proof of the main results in Section 2 is based on two intermediate results which will be proved in Sections 3 and 4 respectively.

Acknowledgements. Part of this work was done while the authors were visiting the Morningside Center of Mathematics in Beijing during the summer of 1999. The authors would like to thank the Morningside Center for hospitality. The second author would also like to thank the Nankai Institute of Mathematics for hospitality.

1. A K-theory version of the equivariant family index theorem. In this section, we recall a K-theory version of the equivariant family index theorem [LiuMaZ, Theorem 1.2] for S^1-actions, which will play a crucial role in the following sections.

This section is organized as follows: In Section 1.1, we recall the K-theory version of the equivariant family index theorem for S^1-actions on a family of Spin^c manifolds. In Section 1.2, as a simple application of Theorem 1.1, we obtain a K-theory version of the vanishing theorem of Hattori [Ha] for the case of almost complex manifolds.

1.1. A K-theory version of the equivariant family index theorem. Let $M,\ B$ be two compact manifolds, let $\pi : M \to B$ be a fibration with compact fibre X such that $\dim X = 2l$ and that S^1 acts fiberwise on M. Let h^{TX} be a metric on TX. We assume that TX is oriented. Let (W, h^W) be a Hermitian complex vector bundle over M.

Let V be a $2p$ dimensional oriented real vector bundle over M. Let L be a complex line bundle over M with the property that the vector bundle $U = TX \oplus V$ obeys $\omega_2(U) = c_1(L) \bmod (2)$. Then the vector bundle U has a Spin^c-structure. Let $h^V,\ h^L$ be the corresponding metrics on $V,\ L$. Let $S(U,L)$ be the fundamental complex spinor bundle for (U,L) [LaM, Appendix D.9] which locally may be written as

$$S(U,L) = S_0(U) \otimes L^{1/2}, \tag{1.1}$$

where $S_0(U)$ is the fundamental spinor bundle for the (possibly non-existent) spin structure on U, and where $L^{1/2}$ is the (possibly non-existent) square root of L.

Assume that the S^1-action on M lifts to $V,\ L$ and W, and assume the metrics $h^{TX},\ h^V,\ h^L,\ h^W$ are S^1-invariant. Also assume that the S^1-actions on $TX,\ V,\ L$ lift to $S(U,L)$.

Let ∇^{TX} be the Levi-Civita connection on (TX, h^{TX}) along the fibre X. Let ∇^V, ∇^L and ∇^W be the S^1-invariant and metric-compatible connections on (V, h^V), (L, h^L) and (W, h^W) respectively. Let $\nabla^{S(U,L)}$ be the Hermitian connection on $S(U,L)$ induced by $\nabla^{TX} \oplus \nabla^V$ and ∇^L (cf. [LaM, Appendix D], [LiuMaZ, §1.1]). Let $\nabla^{S(U,L)\otimes W}$ be the tensor product connection on $S(U,L) \otimes W$ induced by $\nabla^{S(U,L)}$ and ∇^W,

$$\nabla^{S(U,L)\otimes W} = \nabla^{S(U,L)} \otimes 1 + 1 \otimes \nabla^W. \tag{1.2}$$

Let $\{e_i\}_{i=1}^{2l}$ (resp. $\{f_j\}_{j=1}^{2p}$) be an oriented orthonormal basis of (TX, h^{TX}) (resp. (V, h^V)). We denote by $c(\cdot)$ the Clifford action of $TX \oplus V$ on $S(U,L)$. Let $D^X \otimes W$ be the family Spin^c-Dirac operator on the fiber X defined by

$$D^X \otimes W = \sum_{i=1}^{2l} c(e_i) \nabla_{e_i}^{S(U,L)\otimes W}. \tag{1.3}$$

There are two canonical ways to consider $S(U,L)$ as a $\mathbf{Z}_2$-graded vector bundle. Let

$$\begin{aligned} \tau_s &= i^l c(e_1) \cdots c(e_{2l}), \\ \tau_e &= i^{l+p} c(e_1) \cdots c(e_{2l}) c(f_1) \cdots c(f_{2p}) \end{aligned} \tag{1.4}$$

be two involutions of $S(U,L)$. Then $\tau_s^2 = \tau_e^2 = 1$. We decompose $S(U,L) = S^+(U,L) \oplus S^-(U,L)$ corresponding to τ_s (resp. τ_e) such that $\tau_s|_{S^\pm(U,L)} = \pm 1$ (resp. $\tau_e|_{S^\pm(U,L)} = \pm 1$).

For $\tau = \tau_s$ or τ_e, by [LiuMa1, Proposition 1.1], the index bundle $\mathrm{Ind}_\tau(D^X)$ over B is well-defined in the equivariant K-group $K_{S^1}(B)$.

Let $F=\{F_\alpha\}$ be the fixed point set of the circle action on M. Then $\pi : F_\alpha \to B$ (resp. $\pi : F \to B$) is a smooth fibration with fibre Y_α (resp. Y). Let $\widetilde{\pi} : N \to F$ denote the normal bundle to F in M. Then $N = TX/TY$. We identify N as the orthogonal complement of TY in $TX_{|F}$. Let h^{TY}, h^N be the corresponding metrics on TY and N induced by h^{TX}. Then, we have the following S^1-equivariant decomposition of TX over F,

$$TX_{|F} = N_{m_1} \oplus \cdots \oplus N_{m_l} \oplus TY,$$

where each N_γ is a complex vector bundle such that $g \in S^1$ acts on it by g^γ. To simplify the notation, we will write simply that

$$TX_{|F} = \oplus_{v\neq 0} N_v \oplus TY, \tag{1.5}$$

where N_v is a complex vector bundle such that $g \in S^1$ acts on it by g^v with $v \in \mathbf{Z}^*$. Clearly, $N = \oplus_{v\neq 0} N_v$. We will denote by N a complex vector bundle, and $N_{\mathbf{R}}$ the underlying real vector bundle of N.

Similarly let

$$W_{|F} = \oplus_v W_v \tag{1.6}$$

be the S^1-equivariant decomposition of the restriction of W over F. Here W_v ($v \in \mathbf{Z}$) is a complex vector bundle over F on which $g \in S^1$ acts by g^v.

We also have the following S^1-equivariant decomposition of V restricted to F,

$$V_{|F} = \oplus_{v\neq 0} V_v \oplus V_0^{\mathbf{R}}, \tag{1.7}$$

where V_v is a complex vector bundle such that g acts on it by g^v, and $V_0^{\mathbf{R}}$ is the real subbundle of V such that S^1 acts as identity. For $v \neq 0$, let $V_{v,\mathbf{R}}$ denote the underlying real vector bundle of V_v. Denote by $2p' = \dim V_0^{\mathbf{R}}$ and $2l' = \dim Y$.

Let us write

$$L_F = L \otimes \Big(\bigotimes_{v\neq 0} \det N_v \bigotimes_{v\neq 0} \det V_v \Big)^{-1}. \tag{1.8}$$

Then $TY \oplus V_0^{\mathbf{R}}$ has a Spin^c structure as $\omega_2(TY \oplus V_0^{\mathbf{R}}) = c_1(L_F) \bmod (2)$. Let $S(TY \oplus V_0^{\mathbf{R}}, L_F)$ be the fundamental spinor bundle for $(TY \oplus V_0^{\mathbf{R}}, L_F)$ [LaM, Appendix D, pp. 397].

Let D^Y, D^{Y_α} be the families of Spin^c Dirac operators acting on $S(TY \oplus V_0^{\mathbf{R}}, L_F)$ over F, F_α as (1.3). If R is an Hermitian complex vector bundle equipped with an Hermitian connection over F, let $D^Y \otimes R$, $D^{Y_\alpha} \otimes R$ denote the twisted Spin^c Dirac operators on $S(TY \oplus V_0^{\mathbf{R}}, L_F) \otimes R$ and on $S(TY_\alpha \oplus V_0^{\mathbf{R}}, L_F) \otimes R$ respectively.

Recall that $N_{v,\mathbf{R}}$ and $V_{v,\mathbf{R}}$ are canonically oriented by their complex structures. The decompositions (1.5), (1.7) induce the orientations on TY and $V_0^{\mathbf{R}}$ respectively. Let $\{e_i\}_{i=1}^{2l'}$, $\{f_j\}_{j=1}^{2p'}$ be the corresponding oriented orthonormal basis of (TY, h^{TY}) and $(V_0^{\mathbf{R}}, h^{V_0^{\mathbf{R}}})$. There are two canonical ways to consider $S(TY \oplus V_0^{\mathbf{R}}, L_F)$ as a $\mathbf{Z}_2$-graded vector bundle. Let

$$\begin{aligned} \tau_s &= i^{l'} c(e_1) \cdots c(e_{2l'}), \\ \tau_e &= i^{l'+p'} c(e_1) \cdots c(e_{2l'}) c(f_1) \cdots c(f_{2p'}) \end{aligned} \tag{1.9}$$

be two involutions of $S(TY \oplus V_0^{\mathbf{R}}, L_F)$. Then $\tau_s^2 = \tau_e^2 = 1$. We decompose $S(TY \oplus V_0^{\mathbf{R}}, L_F) = S^+(TY \oplus V_0^{\mathbf{R}}, L_F) \oplus S^-(TY \oplus V_0^{\mathbf{R}}, L_F)$ corresponding to τ_s (resp. τ_e) such that $\tau_s|_{S^\pm(TY \oplus V_0^{\mathbf{R}}, L_F)} = \pm 1$ (resp. $\tau_e|_{S^\pm(TY \oplus V_0^{\mathbf{R}}, L_F)} = \pm 1$).

Upon restriction to F, one has the following isomorphism of $\mathbf{Z}_2$-graded Clifford modules over F,

$$S(U, L) \simeq S(TY \oplus V_0^{\mathbf{R}}, L_F) \widehat{\bigotimes_{v \neq 0}} \Lambda N_v \widehat{\bigotimes_{v \neq 0}} \Lambda V_v. \tag{1.10}$$

We denote by Ind_{τ_s}, Ind_{τ_e} the index bundles corresponding to the involutions τ_s, τ_e respectively.

Let S^1 act on L by sending $g \in S^1$ to g^{l_c} ($l_c \in \mathbf{Z}$) on F. Then l_c is locally constant on F. We define the following elements in $K(F)[[q^{1/2}]]$,

$$\begin{aligned} R_\pm(q) = q^{\frac{1}{2}\Sigma_v |v| \dim N_v - \frac{1}{2}\Sigma_v v \dim V_v + \frac{1}{2} l_c} \otimes_{0<v} \Big(\mathrm{Sym}_{q^v}(N_v) \otimes \det N_v\Big) \\ \otimes_{v<0} \mathrm{Sym}_{q^{-v}}(\overline{N}_v) \otimes_{v \neq 0} \Lambda_{\pm q^v}(V_v) \otimes_v q^v W_v = \textstyle\sum_n R_{\pm,n} q^n, \\ R'_\pm(q) = q^{-\frac{1}{2}\Sigma_v |v| \dim N_v - \frac{1}{2}\Sigma_v v \dim V_v + \frac{1}{2} l_c} \otimes_{0<v} \mathrm{Sym}_{q^{-v}}(\overline{N}_v) \\ \otimes_{v<0} \Big(\mathrm{Sym}_{q^v}(N_v) \otimes \det N_v\Big) \otimes_{v \neq 0} \Lambda_{\pm q^v}(V_v) \otimes_v q^v W_v = \textstyle\sum_n R'_{\pm,n} q^n. \end{aligned} \tag{1.11}$$

The following result was proved in [LiuMaZ, Theorem 1.2]:

THEOREM 1.1. *For $n \in \mathbf{Z}$, we have the following identity in $K(B)$,*

$$\begin{aligned} \mathrm{Ind}_{\tau_s}(D^X \otimes W, n) &= \textstyle\sum_\alpha (-1)^{\Sigma_{0<v} \dim N_v} \mathrm{Ind}_{\tau_s}(D^{Y_\alpha} \otimes R_{+,n}) \\ &= \textstyle\sum_\alpha (-1)^{\Sigma_{v<0} \dim N_v} \mathrm{Ind}_{\tau_s}(D^{Y_\alpha} \otimes R'_{+,n}), \\ \mathrm{Ind}_{\tau_e}(D^X \otimes W, n) &= \textstyle\sum_\alpha (-1)^{\Sigma_{0<v} \dim N_v} \mathrm{Ind}_{\tau_e}(D^{Y_\alpha} \otimes R_{-,n}) \\ &= \textstyle\sum_\alpha (-1)^{\Sigma_{v<0} \dim N_v} \mathrm{Ind}_{\tau_e}(D^{Y_\alpha} \otimes R'_{-,n}). \end{aligned} \tag{1.12}$$

REMARK 1.1. If TX has an S^1-equivariant Spin structure, by setting $V = 0, L = \mathbf{C}$, we get [LiuMaZ, Theorem 1.1].

1.2. K-theory version of the vanishing theorem of Hattori. In this subsection, we assume that TX has an S^1-equivariant almost complex structure J. Then one has the canonical splitting

$$TX \otimes_{\mathbf{R}} \mathbf{C} = T^{(1,0)}X \oplus T^{(0,1)}X, \tag{1.13}$$

where

$$\begin{aligned} T^{(1,0)}X = \{z \in TX \otimes_{\mathbf{R}} \mathbf{C}, Jz = \sqrt{-1}z\}, \\ T^{(0,1)}X = \{z \in TX \otimes_{\mathbf{R}} \mathbf{C}, Jz = -\sqrt{-1}z\}. \end{aligned}$$

Let $K_X = \det(T^{(1,0)}X)$ be the determinant line bundle of $T^{(1,0)}X$ over M. Then the complex spinor bundle $S(TX, K_X)$ for (TX, K_X) is $\Lambda(T^{(0,1)*}X)$. In this case, the almost complex structure J on TX induces an almost complex structure on TY. Then we can rewrite (1.5) as,

$$T^{(1,0)}X = \oplus_{v \neq 0} N_v \oplus T^{(1,0)}Y, \tag{1.14}$$

where N_v are complex vector subbundles of $T^{(1,0)}X$ on which $g \in S^1$ acts by multiplication by g^v.

We suppose that $c_1(T^{(1,0)}X) = 0 \bmod (N)$ $(N \in \mathbf{Z}, N \geq 2)$. Then the complex line bundle $K_X^{1/N}$ is well defined over M. After replacing the S^1 action by its N-fold action, we can always assume that S^1 acts on $K_X^{1/N}$. For $s \in \mathbf{Z}$, let $D^X \otimes K_X^{s/N}$ be the twisted Dirac operator on $\Lambda(T^{(0,1)*}X) \otimes K_X^{s/N}$ defined as in (1.3).

The following result generalizes the main result of [Ha] to the family case.

Theorem 1.2. *We assume that M is connected and that the S^1 action is nontrivial. If $c_1(T^{(1,0)}X) = 0 \bmod (N)$ $(N \in \mathbf{Z}, N \geq 2)$, then for $s \in \mathbf{Z}$, $-N < s < 0$,*

$$\operatorname{Ind}(D^X \otimes K_X^{s/N}) = 0 \text{ in } K_{S^1}(B). \tag{1.15}$$

Proof. Consider $R_+(q)$, $R'_+(q)$ of (1.11) with $V = 0, W = K_X^{s/N}$. We know

$$\begin{aligned} R_{+,n} &= 0 \text{ if } n < a_1 = \inf_\alpha\big(\tfrac{1}{2}\textstyle\sum_v |v| \dim N_v + (\tfrac{1}{2} + \tfrac{s}{N}) \sum_v v \dim N_v\big), \\ R'_{+,n} &= 0 \text{ if } n > a_2 = \sup_\alpha\big(-\tfrac{1}{2}\textstyle\sum_v |v| \dim N_v + (\tfrac{1}{2} + \tfrac{s}{N}) \sum_v v \dim N_v\big). \end{aligned} \tag{1.16}$$

As $-N < s < 0$, by (1.16), we know that $a_1 \geq 0$, $a_2 \leq 0$, with a_1 or a_2 equal to zero iff $\sum_v |v| \dim N_v = 0$ for all α, which means that the S^1 action does not have fixed points.

From Theorem 1.1 (cf. [Z, Theorem A.1]) and the above discussion, we get Theorem 1.2. □

Remark 1.2. From the proof of Theorem 1.2, one also deduces that $D^X \otimes K_X^{-1}$, D^X are rigid on the equivariant K-theory level (cf. [Z, (2.17)]).

2. Rigidity and vanishing theorems in K-Theory.

The purpose of this section is to establish the main results of this paper: the rigidity and vanishing theorems on the equivariant K-theory level for a family of Spin^c manifolds. The results in this section refine some of the results in [LiuMa2] to the K-theory level.

This section is organized as follows: In Section 2.1, we state our main results, the rigidity and vanishing theorems on the equivariant K-theory level for a family of Spin^c manifolds. In Section 2.2, we state two intermediate results which will be used to prove our main results stated in Section 2.1. In Section 2.3, we prove the family rigidity and vanishing theorems.

Throughout this section, we keep the notations of Section 1.1.

2.1. Family rigidity and vanishing Theorem.

Let $\pi : M \to B$ be a fibration of compact manifolds with fiber X and $\dim X = 2l$. We assume that S^1 acts fiberwise on M, and TX has an S^1-invariant Spin^c structure. Let V be an even dimensional real vector bundle over M. We assume that V has an S^1-invariant spin structure. Let W be an S^1-equivariant complex vector bundle of rank r over M. Let $K_W = \det(W)$ be the determinant line bundle of W.

Let K_X be the S^1-equivariant complex line bundle over M which is induced by the S^1-invariant Spin^c structure of TX. Its equivariant first Chern class $c_1(K_X)_{S^1}$ may also be written as $c_1(TX)_{S^1}$.

Let $S(TX, K_X)$ be the complex spinor bundle of (TX, K_X) as in Section 1.1. Let $S(V) = S^+(V) \oplus S^-(V)$ be the spinor bundle of V.

We define the following elements in $K(M)[[q^{1/2}]]$:

$$\begin{aligned}
&Q_1(W) = \bigotimes_{n=0}^{\infty} \Lambda_{-q^n}(\overline{W}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^n}(W),\\
&R_1(V) = (S^+(V) + S^-(V)) \otimes_{n=1}^{\infty} \Lambda_{q^n}(V),\\
&R_2(V) = (S^+(V) - S^-(V)) \otimes_{n=1}^{\infty} \Lambda_{-q^n}(V),\\
&R_3(V) = \otimes_{n=1}^{\infty} \Lambda_{-q^{n-1/2}}(V),\\
&R_4(V) = \otimes_{n=1}^{\infty} \Lambda_{q^{n-1/2}}(V).
\end{aligned} \tag{2.1}$$

For $N \in \mathbf{N}, N \geq 2$, let $y = e^{2\pi i/N} \in \mathbf{C}$. Let G_y be the multiplicative group generated by y. Following Witten [W], we consider the fiberwise action G_y on W and $\overline{W}$ by sending $y \in G_y$ to y on W and y^{-1} on $\overline{W}$. Then G_y acts naturally on $Q_1(W)$.

Recall that the equivariant cohomology group $H^*_{S^1}(M, \mathbf{Z})$ of M is defined by

$$H^*_{S^1}(M, \mathbf{Z}) = H^*(M \times_{S^1} ES^1, \mathbf{Z}), \tag{2.2}$$

where ES^1 is the usual universal S^1-principal bundle over the classifying space BS^1 of S^1. So $H^*_{S^1}(M, \mathbf{Z})$ is a module over $H^*(BS^1, \mathbf{Z})$ induced by the projection $\overline{\pi} : M \times_{S^1} ES^1 \to BS^1$. Let $p_1(V)_{S^1},\ p_1(TX)_{S^1} \in H^*_{S^1}(M, \mathbf{Z})$ be the S^1-equivariant first Pontrjagin classes of V and TX respectively. As $V \times_{S^1} ES^1$ is spin over $M \times_{S^1} ES^1$, one knows that $\frac{1}{2} p_1(V)_{S^1}$ is well-defined in $H^*_{S^1}(M, \mathbf{Z})$ (cf. [T, pp. 456-457]). Also recall that

$$H^*(BS^1, \mathbf{Z}) = \mathbf{Z}[[u]] \tag{2.3}$$

with u a generator of degree 2.

In the following, we denote by $D^X \otimes R$ the family of Dirac operators acting fiberwise on $S(TX, K_X) \otimes R$ as was defined in Section 1.1.

We can now state the main results of this paper as follows.

THEOREM 2.1. *If* $\omega_2(W)_{S^1} = \omega_2(TX)_{S^1}$, $\frac{1}{2} p_1(V + W - TX)_{S^1} = e \cdot \overline{\pi}^* u^2$ $(n \in \mathbf{Z})$ *in* $H^*_{S^1}(M, \mathbf{Z})$, *and* $c_1(W) = 0 \bmod(N)$. *For* $i = 1, 2, 3, 4$, *consider the family of* $G_y \times S^1$*-equivariant elliptic operators*

$$D^X \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX) \otimes Q_1(W) \otimes R_i(V).$$

i) If $e = 0$, *then these operators are rigid on the equivariant* K*-theory level for the* S^1 *action.*

ii) If $e < 0$, *then the index bundles of these operators are zero in* $K_{G_y \times S^1}(B)$. *In particular, these index bundles are zero in* $K_{G_y}(B)$.

REMARK 2.1. As $\omega_2(W)_{S^1} = \omega_2(TX)_{S^1}$, $\frac{1}{2} p_1(W - TX)_{S^1} \in H^*_{S^1}(M, \mathbf{Z})$ is well defined. The condition $\omega_2(W)_{S^1} = \omega_2(TX)_{S^1}$ also means $c_1(K_W \otimes K_X^{-1})_{S^1} = 0 \bmod(2)$, by [HaY, Corollary 1.2], the S^1-action on M can be lifted to $(K_W \otimes K_X^{-1})^{1/2}$ and is compatible with the S^1 action on $K_W \otimes K_X^{-1}$.

REMARK 2.2. If we assume $c_1(W)_{S^1} = c_1(TX)_{S^1}$ in $H^*_{S^1}(M, \mathbf{Z})$ instead of $\omega_2(W)_{S^1} = \omega_2(TX)_{S^1}$ in Theorem 2.1, then $K_W \otimes K_X^{-1}$ is a trivial line bundle over M, and S^1 acts trivially on it. In this case, Theorem 2.1 gives the family version of the results of [De].

REMARK 2.3. The interested reader can apply our method to get various rigidity and vanishing theorems, for example, to get a generalization of Theorem1.2 for the elements [W, (65)].

Actually, as in [LiuMaZ], our proof of these theorems works under the following slightly weaker hypothesis. Let us first explain some notations.

For each $n > 1$, consider $\mathbf{Z}_n \subset S^1$, the cyclic subgroup of order n. We have the $\mathbf{Z}_n$ equivariant cohomology of M defined by $H^*_{\mathbf{Z}_n}(M, \mathbf{Z}) = H^*(M \times_{\mathbf{Z}_n} ES^1, \mathbf{Z})$, and there is a natural "forgetful" map $\alpha(S^1, \mathbf{Z}_n) : M \times_{\mathbf{Z}_n} ES^1 \to M \times_{S^1} ES^1$ which induces a pullback $\alpha(S^1, \mathbf{Z}_n)^* : H^*_{S^1}(M, \mathbf{Z}) \to H^*_{\mathbf{Z}_n}(M, \mathbf{Z})$. The arrow which forgets the S^1 action altogether we denote by $\alpha(S^1, 1)$. Thus $\alpha(S^1, 1)^* : H^*_{S^1}(M, \mathbf{Z}) \to H^*(M, \mathbf{Z})$ is induced by the inclusion of M into $M \times_{S^1} ES^1$ as a fiber over BS^1.

Finally, note that if $\mathbf{Z}_n$ acts trivially on a space Y, then there is a new arrow $t^* : H^*(Y, \mathbf{Z}) \to H^*_{\mathbf{Z}_n}(Y, \mathbf{Z})$ induced by the projection $Y \times_{\mathbf{Z}_n} ES^1 = Y \times B\mathbf{Z}_n \xrightarrow{t} Y$.

We let $\mathbf{Z}_\infty = S^1$. For each $1 < n \leq +\infty$, let $i : M(n) \to M$ be the inclusion of the fixed point set of $\mathbf{Z}_n \subset S^1$ in M and so i induces $i_{S^1} : M(n) \times_{S^1} ES^1 \to M \times_{S^1} ES^1$.

In the rest of this paper, we suppose that there exists some integer $e \in \mathbf{Z}$ such that for $1 < n \leq +\infty$,

$$\begin{aligned} \alpha(S^1, \mathbf{Z}_n)^* \circ i^*_{S^1}\Big(\frac{1}{2}p_1(V + W - TX)_{S^1} - e \cdot \overline{\pi}^* u^2\Big) \\ = t^* \circ \alpha(S^1, 1)^* \circ i^*_{S^1}\Big(\frac{1}{2}p_1(V + W - TX)_{S^1}\Big). \end{aligned} \tag{2.4}$$

Remark 2.4. The relation (2.4) clearly follows from the hypothesises of Theorem 2.1 by pulling back and forgetting. Thus it is weaker.

We can now state a slightly more general version of Theorem 2.1.

Theorem 2.2. *Under the hypothesis (2.4), we have*

i) If $e = 0$, then the index bundles of the elliptic operators in Theorem 2.1 are rigid on the equivariant K-theory level for the S^1-action.

ii) If $e < 0$, then the index bundles of the elliptic operators in Theorem 2.1 are zero as elements in $K_{G_y \times S^1}(B)$. In particular, these index bundles are zero in $K_{G_y}(B)$.

The rest of this section is devoted to a proof of Theorem 2.2.

2.2. Two intermediate results. Let $F = \{F_\alpha\}$ be the fixed point set of the circle action. Then $\pi : F \to B$ is a fibration with compact fibre denoted by $Y = \{Y_\alpha\}$.

As in [LiuMaZ, §2], we may and we will assume that

$$\begin{aligned} &TX_{|F} = TY \oplus \textstyle\bigoplus_{0<v} N_v, \\ &TX_{|F} \otimes_{\mathbf{R}} \mathbf{C} = TY \otimes_{\mathbf{R}} \mathbf{C} \textstyle\bigoplus_{0<v} (N_v \oplus \overline{N}_v), \end{aligned} \tag{2.5}$$

where N_v is the complex vector bundle on which S^1 acts by sending g to g^v (Here N_v can be zero). We also assume that

$$\begin{aligned} &V_{|F} = V_0^{\mathbf{R}} \oplus \textstyle\bigoplus_{0<v} V_v, \\ &W_{|F} = \oplus_v W_v, \end{aligned} \tag{2.6}$$

where V_v, W_v are complex vector bundles on which S^1 acts by sending g to g^v, and $V_0^{\mathbf{R}}$ is a real vector bundle on which S^1 acts as identity.

By (2.5), as in (1.10), there is a natural isomorphism between the $\mathbf{Z}_2$-graded $C(TX)$-Clifford modules over F,

$$S(TY, K_X \otimes_{0<v} (\det N_v)^{-1}) \widehat{\otimes}_{0<v} \Lambda N_v \simeq S(TX, K_X)_{|F}. \tag{2.7}$$

For R a complex vector bundle over F, let $D^Y \otimes R$, $D^{Y_\alpha} \otimes R$ be the twisted Spin^c Dirac operator on $S(TY, K_X \otimes_{0<v} (\det N_v)^{-1}) \otimes R$ on F, F_α respectively.

On F, we write

$$\begin{aligned} e(N) &= \textstyle\sum_{0<v} v^2 \dim N_v, & d'(N) &= \textstyle\sum_{0<v} v \dim N_v, \\ e(V) &= \textstyle\sum_{0<v} v^2 \dim V_v, & d'(V) &= \textstyle\sum_{0<v} v \dim V_v, \\ e(W) &= \textstyle\sum_{v} v^2 \dim W_v, & d'(W) &= \textstyle\sum_{v} v \dim W_v. \end{aligned} \tag{2.8}$$

Then $e(N)$, $e(V)$, $e(W)$, $d'(N)$, $d'(V)$ and $d'(W)$ are locally constant functions on F.

By [H, §8], we have the following property,

Lemma 2.1. *If* $c_1(W) = 0 \operatorname{mod}(N)$, *then* $d'(W)$ $\operatorname{mod}(N)$ *is constant on each connected component of* M.

Proof. As $c_1(W) = 0$ $\operatorname{mod}(N)$, $(K_W)^{1/N}$ is well defined. Consider the N-fold covering $S^1 \to S^1$, with $\mu \to \lambda = \mu^N$, then μ acts on M and K_W through λ. This action can be lift to $(K_W)^{1/N}$. On F, μ acts on $(K_W)^{1/N}$ by multiplication by $\mu^{d'(W)}$. However, if $\mu = \zeta = e^{2\pi i/N}$, then it operates trivially on M. So the action of ζ in each fibre of L is by multiplication by ζ^a, and a $\operatorname{mod}(N)$ is constant on each connected component of M.

The proof of Lemma 2.1 is complete. □

Let us write

$$\begin{aligned} &L(N) = \otimes_{0<v} (\det N_v)^v, \qquad L(V) = \otimes_{0<v} (\det V_v)^v, \\ &L(W) = \otimes_{v \neq 0} (\det W_v)^v, \\ &L = L(N)^{-1} \otimes L(V) \otimes L(W). \end{aligned} \tag{2.9}$$

We denote the Chern roots of N_v by $\{x_v^j\}$ (resp. V_v by u_v^j and W_v by w_v^j), and the Chern roots of $TY \otimes_{\mathbf{R}} \mathbf{C}$ by $\{\pm y_j\}$ (resp. $V_0 = V_0^{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ by $\{\pm u_0^j\}$). Then if we take $\mathbf{Z}_\infty = S^1$ in (2.4), we get

$$\begin{aligned} &\tfrac{1}{2}(\Sigma_{v,j}(u_v^j + vu)^2 + \Sigma_{v,j}(w_v^j + vu)^2 - \Sigma_j (y_j)^2 - \Sigma_{v,j}(x_v^j + vu)^2) - eu^2 \\ &= \tfrac{1}{2}(\Sigma_{v,j}(u_v^j)^2 + \Sigma_{v,j}(w_v^j)^2 - \Sigma_j (y_j)^2 - \Sigma_{v,j}(x_v^j)^2). \end{aligned} \tag{2.10}$$

By (2.3), (2.10), we get

$$\begin{aligned} &c_1(L) = \Sigma_{v,j} v u_v^j + \Sigma_{v,j} v w_v^j - \Sigma_{v,j} v x_v^j = 0, \\ &e(V) + e(W) - e(N) \\ &\quad = \textstyle\sum_{0<v} v^2 \dim V_v + \sum_v v^2 \dim W_v - \sum_{0<v} v^2 \dim N_v = 2e, \end{aligned} \tag{2.11}$$

which does not depends on the connected components of F. This means L is a trivial complex line bundle over each component F_α of F, and S^1 acts on L by sending g to g^{2e}, and G_y acts on L by sending y to $y^{d'(W)}$. By Lemma 2.1, we can extend L to a trivial complex line bundle over M, and we extend the S^1-action on it by sending g on the canonical section 1 of L to $g^{2e} \cdot 1$, and G_y acts on L by sending y to $y^{d'(W)}$.

The line bundles in (2.9) will play important roles in the next two sections which consist of the proof of Theorems 2.3, 2.4 to be stated below.

In what follows, if $R(q) = \sum_{m \in \frac{1}{2}\mathbf{Z}} R_m q^m \in K_{S^1}(M)[[q^{1/2}]]$, we will also denote $\operatorname{Ind}(D^X \otimes R_m, h)$ by $\operatorname{Ind}(D^X \otimes R(q), m, h)$. For $k = 1, 2, 3, 4$, set

$$R_{1k} = (K_W \otimes K_X^{-1})^{1/2} \otimes Q_1(W) \otimes R_k(V). \tag{2.12}$$

We first state a result which expresses the global equivariant family index via the family indices on the fixed point set.

Proposition 2.1. *For* $m \in \frac{1}{2}\mathbf{Z}$, $h \in \mathbf{Z}$, $1 \leq k \leq 4$, *we have the following identity in* $K_{G_y}(B)$,

$$\begin{aligned}
&\mathrm{Ind}(D^X \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX) \otimes R_{1k}, m, h) \\
&= \textstyle\sum_\alpha (-1)^{\Sigma_{0<v} \dim N_v} \mathrm{Ind}(D^{Y_\alpha} \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX) \otimes R_{1k} \\
&\qquad\qquad \otimes \mathrm{Sym}(\oplus_{0<v} N_v) \otimes_{0<v} \det N_v, m, h).
\end{aligned} \tag{2.13}$$

Proof. This follows directly from Theorem 1.1 and (2.7). □

For $p \in \mathbf{N}$, we define the following elements in $K_{S^1}(F)[[q]]$: [1]

$$\begin{aligned}
&\mathcal{F}_p(X) = \textstyle\bigotimes_{0<v} \Big(\otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(N_v) \otimes_{n>pv} \mathrm{Sym}_{q^n}(\overline{N}_v)\Big) \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TY), \\
&\mathcal{F}'_p(X) = \textstyle\bigotimes_{\substack{0<v \\ 0 \leq n \leq pv}} \Big(\mathrm{Sym}_{q^{-n}}(N_v) \otimes \det N_v\Big), \\
&\mathcal{F}^{-p}(X) = \mathcal{F}_p(X) \otimes \mathcal{F}'_p(X).
\end{aligned} \tag{2.14}$$

Then, from (2.5), over F, we have

$$\mathcal{F}^0(X) = \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX) \otimes \mathrm{Sym}(\oplus_{0<v} N_v) \otimes_{0<v} \det N_v. \tag{2.15}$$

We now state two intermediate results on the relations between the family indices on the fixed point set. They will be used in the next subsection to prove Theorem 2.2.

Theorem 2.3. *For* $1 \leq k \leq 4$, $h,\ p \in \mathbf{Z}$, $p > 0$, $m \in \frac{1}{2}\mathbf{Z}$, *we have the following identity in* $K_{G_y}(B)$,

$$\begin{aligned}
&\textstyle\sum_\alpha (-1)^{\Sigma_{0<v} \dim N_v} \mathrm{Ind}(D^{Y_\alpha} \otimes \mathcal{F}^0(X) \otimes R_{1k}, m, h) \\
&= \textstyle\sum_\alpha (-1)^{pd'(N) + \Sigma_{0<v} \dim N_v} \mathrm{Ind}(D^{Y_\alpha} \otimes \mathcal{F}^{-p}(X) \otimes R_{1k}, \\
&\qquad\qquad m + \tfrac{1}{2}p^2 e(N) + \tfrac{1}{2}pd'(N), h).
\end{aligned} \tag{2.16}$$

Theorem 2.4. *For each* α, $1 \leq k \leq 4$, $h,\ p \in \mathbf{Z}$, $p > 0$, $m \in \frac{1}{2}\mathbf{Z}$, *we have the following identity in* $K_{G_y}(B)$,

$$\begin{aligned}
&\mathrm{Ind}(D^{Y_\alpha} \otimes \mathcal{F}^{-p}(X) \otimes R_{1k}, m + \tfrac{1}{2}p^2 e(N) + \tfrac{1}{2}pd'(N), h) \\
&= (-1)^{pd'(W)} \mathrm{Ind}(D^{Y_\alpha} \otimes \mathcal{F}^0(X) \otimes R_{1k} \otimes L^{-p}, m + ph + p^2 e, h).
\end{aligned} \tag{2.17}$$

Theorem 2.3 is a direct consequence of Theorem 2.5 to be stated below, which will be proved in Section 4, while Theorem 2.4 will be proved in Section 3.

To state Theorem 2.5, let $J = \{v \in \mathbf{N} |$ There exists α such that $N_v \neq 0$ on $F_\alpha\}$ and

$$\Phi = \{\beta \in]0,1] | \text{There exists } v \in J \text{ such that } \beta v \in \mathbf{Z}\}. \tag{2.18}$$

---

[1] Here by $K_{S^1}(F)$ we also mean the direct sum of the form $\oplus_{n \in Z} E_n$ with each E_n a finite dimensional vector bundle over F of weight n under the S^1-action.

We order the elements in Φ so that $\Phi = \{\beta_i | 1 \leq i \leq J_0, J_0 \in \mathbf{N}$ and $\beta_i < \beta_{i+1}\}$. Then for any integer $1 \leq i \leq J_0$, there exist $p_i,\ n_i \in \mathbf{N},\ 0 < p_i \leq n_i$, with $(p_i, n_i) = 1$ such that

$$\beta_i = p_i/n_i. \tag{2.19}$$

Clearly, $\beta_{J_0} = 1$. We also set $p_0 = 0$ and $\beta_0 = 0$.

For $1 \leq j \leq J_0$, $p \in \mathbf{N}^*$, we write

$$\begin{aligned} &I_0^p = \phi, \text{the empty set},\\ &I_j^p = \{(v,n) | v \in J, (p-1)v < n \leq pv, \frac{n}{v} = p - 1 + \frac{p_j}{n_j}\},\\ &\overline{I}_j^p = \{(v,n) | v \in J, (p-1)v < n \leq pv, \frac{n}{v} > p - 1 + \frac{p_j}{n_j}\}. \end{aligned} \tag{2.20}$$

For $0 \leq j \leq J_0$, set

(2.21)

$$\mathcal{F}_{p,j}(X) = \mathcal{F}_p(X) \otimes \mathcal{F}'_{p-1}(X) \bigotimes_{(v,n) \in \cup_{i=1}^j I_i^p} \Big(\mathrm{Sym}_{q^{-n}}(N_v) \otimes \det N_v\Big) \bigotimes_{(v,n) \in \overline{I}_j^p} \mathrm{Sym}_{q^n}(\overline{N}_v).$$

Then

$$\begin{aligned} \mathcal{F}_{p,0}(X) &= \mathcal{F}^{-p+1}(X),\\ \mathcal{F}_{p,J_0}(X) &= \mathcal{F}^{-p}(X). \end{aligned} \tag{2.22}$$

For $s \in \mathbf{R}$, let $[s]$ denote the greatest integer which is less than or equal to the given number s. For $0 \leq j \leq J_0$, denote by

$$\begin{aligned} &e(p, \beta_j, N) = \tfrac{1}{2} \textstyle\sum_{0<v} (\dim N_v) \Big((p-1)v + [\tfrac{p_j v}{n_j}]\Big)\Big((p-1)v + [\tfrac{p_j v}{n_j}] + 1\Big),\\ &d'(p, \beta_j, N) = \textstyle\sum_{0<v} (\dim N_v)([\tfrac{p_j v}{n_j}] + (p-1)v). \end{aligned} \tag{2.23}$$

Then $e(p, \beta_j, N)$ and $d'(p, \beta_j, N)$ are locally constant functions on F. And

$$\begin{aligned} e(p, \beta_0, N) &= \tfrac{1}{2}(p-1)^2 e(N) + \tfrac{1}{2}(p-1)d'(N),\\ e(p, \beta_{J_0}, N) &= \tfrac{1}{2}p^2 e(N) + \tfrac{1}{2}pd'(N),\\ d'(p, \beta_{J_0}, N) &= d'(p+1, \beta_0, N) = pd'(N). \end{aligned} \tag{2.24}$$

THEOREM 2.5. *For $1 \leq k \leq 4$, $1 \leq j \leq J_0$, $p \in \mathbf{N}^*$, $h \in \mathbf{Z}$, $m \in \frac{1}{2}\mathbf{Z}$, we have the following identity in $K_{G_y}(B)$,*

$$\begin{aligned} &\textstyle\sum_\alpha (-1)^{d'(p,\beta_{j-1},N) + \Sigma_{0<v} \dim N_v} \mathrm{Ind}(D^{Y_\alpha} \otimes \mathcal{F}_{p,j-1}(X) \otimes R_{1k},\\ &\qquad\qquad m + e(p, \beta_{j-1}, N), h)\\ &= \textstyle\sum_\alpha (-1)^{d'(p,\beta_j,N) + \Sigma_{0<v} \dim N_v} \mathrm{Ind}(D^{Y_\alpha} \otimes \mathcal{F}_{p,j}(X) \otimes R_{1k},\\ &\qquad\qquad m + e(p, \beta_j, N), h). \end{aligned} \tag{2.25}$$

Proof. The proof is delayed to Section 4. □

Proof of Theorem 2.3. From (2.22), (2.24), and Theorem 2.5, for $1 \leq k \leq 4$, $h \in \mathbf{Z}$, $p \in \mathbf{N}^*$ and $m \in \frac{1}{2}\mathbf{Z}$, we have the following identity in $K_{G_y}(B)$:

$$\begin{aligned} &\textstyle\sum_\alpha (-1)^{d'(p,\beta_{J_0},N) + \Sigma_{0<v} \dim N_v} \mathrm{Ind}(D^{Y_\alpha} \otimes \mathcal{F}^{-p}(X) \otimes R_{1k},\\ &\qquad\qquad m + \tfrac{1}{2}p^2 e(N) + \tfrac{1}{2}pd'(N), h)\\ &= \textstyle\sum_\alpha (-1)^{d'(p,\beta_0,N) + \Sigma_{0<v} \dim N_v} \mathrm{Ind}(D^{Y_\alpha} \otimes \mathcal{F}^{-p+1}(X) \otimes R_{1k},\\ &\qquad\qquad m + \tfrac{1}{2}(p-1)^2 e(N) + \tfrac{1}{2}(p-1)d'(N), h). \end{aligned} \tag{2.26}$$

From (2.24), (2.26), we get Theorem 2.3. □

2.3. Proof of Theorem 2.2. As $\frac{1}{2}p_1(TX-W)_{S^1} \in H^*_{S^1}(M, \mathbf{Z})$ is well defined, by (2.8), and (2.10),

$$d'(N) + d'(W) = 0 \bmod(2). \tag{2.27}$$

From Proposition 2.1, Theorems 2.3, 2.4, (2.23), (2.27), for $1 \leq k \leq 4$, $h, p \in \mathbf{Z}$, $p > 0$, $m \in \frac{1}{2}\mathbf{Z}$, we get the following identity in $K_{G_y}(B)$,

$$\begin{aligned} &\operatorname{Ind}(D^X \otimes_{n=1}^{\infty} \operatorname{Sym}_{q^n}(TX) \otimes R_{1k}, m, h) \\ &\qquad = \operatorname{Ind}(D^X \otimes_{n=1}^{\infty} \operatorname{Sym}_{q^n}(TX) \otimes R_{1k} \otimes L^{-p}, m', h), \end{aligned} \tag{2.28}$$

with

$$m' = m + ph + p^2 e. \tag{2.29}$$

Note that from (2.1), (2.12), if $m < 0$, or $m' < 0$, then two side of (2.28) are zero in $K_{G_y}(B)$. Also recall that $y \in G_y$ acts on the trivial line bundle L by sending y to $y^{d'(W)}$.

i) Assume that $e = 0$. Let $h \in \mathbf{Z}$, $m_0 \in \frac{1}{2}\mathbf{Z}$, $h \neq 0$ be fixed. If $h > 0$, we take $m' = m_0$, then for p big enough, we get $m < 0$ in (2.29). If $h < 0$, we take $m = m_0$, then for p big enough, we get $m' < 0$ in (2.29).

So for $h \neq 0$, $m_0 \in \frac{1}{2}\mathbf{Z}$, $1 \leq k \leq 4$, we get

$$\operatorname{Ind}(D^X \otimes_{n=1}^{\infty} \operatorname{Sym}_{q^n}(TX) \otimes R_{1k}, m_0, h) = 0 \quad \text{in} \quad K_{G_y}(B). \tag{2.30}$$

ii) Assume that $e < 0$. For $h \in \mathbf{Z}$, $m_0 \in \frac{1}{2}\mathbf{Z}$, we take $m = m_0$, then for p big enough, we get $m' < 0$ in (2.29), which again gives us (2.30).

The proof of Theorem 2.2 is complete. □

Remark 2.5. Under the condition of Theorem 2.2 i), if $d'(W) \neq 0 \bmod(N)$, we can't deduce these index bundles are zero in $K_{G_y}(B)$. If in addition, M is connected, by (2.28), for $1 \leq k \leq 4$, in $K_{G_y}(B)$, we get

$$\begin{aligned} &\operatorname{Ind}(D^X \otimes_{n=1}^{\infty} \operatorname{Sym}_{q^n}(TX) \otimes R_{1k}) \\ &\qquad = \operatorname{Ind}(D^X \otimes_{n=1}^{\infty} \operatorname{Sym}_{q^n}(TX) \otimes R_{1k}) \otimes [d'(W)]. \end{aligned} \tag{2.31}$$

Here we denote by $[d'(W)]$ the one dimensional complex vector space on which $y \in G_y$ acts by multiplication by $y^{d'(W)}$. In particular, if B is a point, by (2.31), we get the vanishing theorem analogue to the result of [H, §10].

Remark 2.6. If we replace $c_1(W) = 0 \bmod(N), y = e^{2\pi i/N}$ by $c_1(W) = 0, y = e^{2\pi c i}$, with $c \in \mathbf{R} \setminus \mathbf{Q}$ in Theorem 2.2, then by Lemma 2.1, $d'(W)$ is constant on each connected component of M. In this case, we still have Theorem 2.2. In fact, we only use $c_1(W) = 0 \bmod(N)$ to insure the action G_y on L is well defined. So we also generalize the main result of [K] to family case.

3. Proof of Theorem 2.4. This section is organized as follows: In Section 3.1, we introduce some notations. In Section 3.2, we prove Theorem 2.4 by introducing some shift operators as in [LiuMaZ, §3].

Throughout this section, we keep the notations of Section 2.

3.1. Reformulation of Theorem 2.4. To simplify the notations, we introduce some new notations in this subsection. For $n_0 \in \mathbf{N}^*$, we define a number operator P on $K_{S^1}(M)[[q^{\frac{1}{n_0}}]]$ in the following way: if $R(q) = \oplus_{n \in \frac{1}{n_0}\mathbf{Z}} q^n R_n \in K_{S^1}(M)[[q^{\frac{1}{n_0}}]]$, then P acts on $R(q)$ by multiplication by n on R_n. From now on, we simply denote $\mathrm{Sym}_{q^n}(TX)$, $\Lambda_{q^n}(V)$ by $\mathrm{Sym}(TX_n)$, $\Lambda(V_n)$ respectively. In this way, P acts on TX_n, V_n by multiplication by n, and the action P on $\mathrm{Sym}(TX_n)$, $\Lambda(V_n)$ is naturally induced by the corresponding action of P on TX_n, V_n. So the eigenspace of $P = n$ is just given by the coefficient of q^n of the corresponding element $R(q)$. For $R(q) = \oplus_{n \in \frac{1}{n_0}\mathbf{Z}} q^n R_n \in K_{S^1}(M)[[q^{\frac{1}{n_0}}]]$, we will also denote

$$\mathrm{Ind}(D^X \otimes R(q), m, h) = \mathrm{Ind}(D^X \otimes R_m, h). \tag{3.1}$$

Let H be the canonical basis of $\mathrm{Lie}(S^1) = \mathbf{R}$, i.e., $\exp(tH) = \exp(2\pi i t)$ for $t \in \mathbf{R}$. If E is an S^1-equivariant vector bundle over M, on the fixed point set F, let J_H be the representation of $\mathrm{Lie}(S^1)$ on $E|_F$. Then the weight of S^1 action on $\Gamma(F, E|_F)$ is given by the action

$$\mathbf{J}_H = \frac{-1}{2\pi}\sqrt{-1}J_H. \tag{3.2}$$

Recall that the $\mathbf{Z}_2$ grading on $S(TX, K_X) \otimes_{n=1}^{\infty} \mathrm{Sym}(TX_n)$ (resp. $S(TY, K_X \otimes \otimes_{0<v}(\det N_v)^{-1}) \otimes \mathcal{F}^{-p}(X))$ is induced by the $\mathbf{Z}_2$-grading on $S(TX, K_X)$ (resp. $S(TY, K_X \otimes \otimes_{0<v}(\det N_v)^{-1})$). Let

$$\begin{aligned} F_V^1 &= S(V) \bigotimes_{n=1}^{\infty} \Lambda(V_n),\\ F_V^2 &= \otimes_{n \in \mathbf{N}+\frac{1}{2}} \Lambda(V_n),\\ Q(W) &= \otimes_{n=0}^{\infty} \Lambda(\overline{W}_n) \otimes_{n=1}^{\infty} \Lambda(W_n). \end{aligned} \tag{3.3}$$

There are two natural $\mathbf{Z}_2$ gradings on F_V^1, F_V^2 (resp. $Q(W)$). The first grading is induced by the $\mathbf{Z}_2$-grading of $S(V)$ and the forms of homogeneous degree in $\bigotimes_{n=1}^{\infty} \Lambda(V_n)$, $\otimes_{n \in \mathbf{N}+\frac{1}{2}} \Lambda(V_n)$ (resp. $Q(W)$). We define $\tau_{e|F_V^{i\pm}} = \pm 1$ (resp. $\tau_{1|Q(W)^{\pm}} = \pm 1$) to be the involution defined by this $\mathbf{Z}_2$-grading. The second grading is the one for which F_V^i $(i = 1,\ 2)$ are purely even, i.e., $F_V^{i+} = F_V^i$. We denote by $\tau_s = \mathrm{Id}$ the involution defined by this $\mathbf{Z}_2$ grading. Then the coefficient of q^n $(n \in \frac{1}{2}\mathbf{Z})$ in (2.1) of $R_1(V)$ or $R_2(V)$ (resp. $R_3(V)$, $R_4(V)$, or $Q_1(W)$) is exactly the $\mathbf{Z}_2$-graded vector subbundle of (F_V^1, τ_s) or (F_V^1, τ_e) (resp. (F_V^2, τ_e), (F_V^2, τ_s) or $(Q(W), \tau_1)$), on which P acts by multiplication by n.

We denote by τ_e (resp. by τ_s) the $\mathbf{Z}_2$-grading on $S(TX, K_X) \otimes_{n=1}^{\infty} \mathrm{Sym}(TX_n) \otimes F_V^k$ $(k = 1,\ 2)$ induced by the above $\mathbf{Z}_2$-gradings. We will denote by τ_{e1} (resp. by τ_{s1}) the $\mathbf{Z}_2$-gradings on $S(TX, K_X) \otimes \otimes_{n=1}^{\infty} \mathrm{Sym}(TX_n) \otimes F_V^k \otimes Q(W)$ defined by

$$\tau_{e1} = \tau_e \otimes 1 + 1 \otimes \tau_1, \qquad \tau_{s1} = \tau_s \otimes 1 + 1 \otimes \tau_1. \tag{3.4}$$

Let h^{V_v} be the metric on V_v induced by the metric h^V on V. In the following, we identify ΛV_v with $\Lambda \overline{V}_v^*$ by using the Hermitian metric h^{V_v} on V_v. By (2.6), as in (1.10), there is a natural isomorphism between $\mathbf{Z}_2$-graded $C(V)$-Clifford modules over F,

$$S(V_0^{\mathbf{R}}, \otimes_{0<v}(\det V_v)^{-1}) \widehat{\otimes}_{0<v} \Lambda V_v \simeq S(V)_{|F}. \tag{3.5}$$

By using the above notations, we rewrite (2.14), on the fixed point set F, for $p \in \mathbf{N}$,

$$\begin{aligned}
&\mathcal{F}_p(X) = \bigotimes_{0<v} \Big(\bigotimes_{n=1}^{\infty} \mathrm{Sym}(N_{v,n}) \bigotimes_{\substack{n\in\mathbf{N},\\ n>pv}} \mathrm{Sym}(\overline{N}_{v,n}) \Big) \bigotimes_{n=1}^{\infty} \mathrm{Sym}(TY_n),\\
&\mathcal{F}'_p(X) = \bigotimes_{\substack{0<v,n\in\mathbf{N},\\ 0\le n\le pv}} \Big(\mathrm{Sym}(N_{v,-n}) \otimes \det N_v\Big),\\
&\mathcal{F}^{-p}(X) = \mathcal{F}_p(X) \otimes \mathcal{F}'_p(X).
\end{aligned} \tag{3.6}$$

Let $V_0 = V_0^{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$. From (2.5), (3.5), we get

$$\begin{aligned}
\mathcal{F}^0(X) &= \bigotimes_{n=1}^{\infty} \mathrm{Sym}\Big(\oplus_{0<v} (N_{v,n} \oplus \overline{N}_{v,n})\Big) \bigotimes_{n=1}^{\infty} \mathrm{Sym}(TY_n)\\
&\qquad \bigotimes \mathrm{Sym}(\oplus_{0<v} N_{v,0}) \otimes \det(\oplus_{0<v} N_v),\\
F_V^1 &= \bigotimes_{n=1}^{\infty} \Lambda(\oplus_{0<v}(V_{v,n} \oplus \overline{V}_{v,n}) \oplus V_{0,n})\\
&\quad \otimes S(V_0^{\mathbf{R}}, \otimes_{0<v}(\det V_v)^{-1}) \otimes_{0<v} \Lambda(V_{v,0}),\\
F_V^2 &= \bigotimes_{0<n\in\mathbf{Z}+1/2} \Lambda(\oplus_{0<v}(V_{v,n} \oplus \overline{V}_{v,n}) \oplus V_{0,n}),\\
Q(W) &= \bigotimes_{n=0}^{\infty} \Lambda(\oplus_v \overline{W}_{v,n}) \bigotimes_{n=1}^{\infty} \Lambda(\oplus_v W_{v,n}).
\end{aligned} \tag{3.7}$$

Now we can reformulate Theorem 2.4 as follows.

Theorem 3.1. *For each α, h, $p \in \mathbf{Z}$, $p > 0$, $m \in \frac{1}{2}\mathbf{Z}$, for $i = 1, 2$, $\tau = \tau_{e1}$ or τ_{s1}, we have the following identity in $K_{G_y}(B)$,*

$$\begin{aligned}
&\mathrm{Ind}_\tau(D^{Y_\alpha} \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathcal{F}^{-p}(X) \otimes F_V^i \otimes Q(W),\\
&\qquad m + \tfrac{1}{2}p^2 e(N) + \tfrac{1}{2}pd'(N), h)\\
&\quad = (-1)^{pd'(W)} \mathrm{Ind}_\tau(D^{Y_\alpha} \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathcal{F}^0(X) \otimes F_V^i\\
&\qquad \otimes Q(W) \otimes L^{-p}, m + ph + p^2 e, h).
\end{aligned} \tag{3.8}$$

Proof. The rest of this section is devoted to a proof of Theorem 3.1. □

3.2. Proof of Theorem 3.1. Inspired by [T, §7], as in [LiuMaZ, §3], for $p \in \mathbf{N}^*$, we define the shift operators,

$$\begin{aligned}
r_* &: N_{v,n} \to N_{v,n+pv}, & r_* &: \overline{N}_{v,n} \to \overline{N}_{v,n-pv},\\
r_* &: W_{v,n} \to W_{v,n+pv}, & r_* &: \overline{W}_{v,n} \to \overline{W}_{v,n-pv},\\
r_* &: V_{v,n} \to V_{v,n+pv}, & r_* &: \overline{V}_{v,n} \to \overline{V}_{v,n-pv}.
\end{aligned} \tag{3.9}$$

Recall that $L(N)$, $L(W)$, $L(V)$ are the complex line bundles over F defined by (2.9). Recall also that $L = L(N)^{-1} \otimes L(W) \otimes L(V)$ is a trivial complex line bundle over F, and $g \in S^1$ acts on it by multiplication by g^{2e}.

Proposition 3.1. *For $p \in \mathbf{Z}$, $p > 0$, $i = 1, 2$, there are natural isomorphisms of vector bundles over F,*

$$\begin{aligned}
&r_*(\mathcal{F}^{-p}(X)) \simeq \mathcal{F}^0(X) \otimes L(N)^p,\\
&r_*(F_V^i) \simeq F_V^i \otimes L(V)^{-p}.
\end{aligned} \tag{3.10}$$

For any $p \in \mathbf{Z}$, $p > 0$, there is a natural $G_y \times S^1$-equivariant isomorphism of vector bundles over F,

$$r_*(Q(W)) \simeq Q(W) \otimes L(W)^{-p}. \tag{3.11}$$

Proof. The equation (3.10) was proved in [LiuMaZ, Prop. 3.1]. To prove (3.11), we only need to consider the shift operator on the following elements,

$$Q_W = \bigotimes_{n=0}^{\infty} \Lambda(\oplus_{v\neq 0}\overline{W}_{v,n}) \bigotimes_{n=1}^{\infty} \Lambda(\oplus_{v\neq 0} W_{v,n}). \tag{3.12}$$

We compute easily that

$$r_* Q_W = \bigotimes_{n=0}^{\infty} \Lambda(\oplus_{v\neq 0}\overline{W}_{v,n-pv}) \bigotimes_{n=1}^{\infty} \Lambda(\oplus_{v\neq 0} W_{v,n+pv}). \tag{3.13}$$

Let h^W be a Hermitian metric on W. Let h^{W_v} be the metric on W_v induced by h^W. As in [LiuMaZ, §3], the hermitian metric h^{W_v} on W_v induces a natural isomorphism of complex vector bundles over F,

$$\Lambda^i \overline{W}_v \simeq \Lambda^{\dim W_v - i} W_v \otimes \det \overline{W}_v. \tag{3.14}$$

- If $v > 0$, for $n \in \mathbf{N}$, $0 \leq n < pv$, $0 \leq i \leq \dim W_v$, (3.14) induces a natural $G_y \times S^1$-equivariant isomorphism of complex vector bundles

$$\Lambda^i \overline{W}_{v,n-pv} \simeq \Lambda^{\dim W_v - i} W_{v,-n+pv} \otimes \det \overline{W}_v. \tag{3.15}$$

- If $v < 0$, for $n \in \mathbf{N}$, $0 < n \leq -pv$, $0 \leq i \leq \dim W_v$, (3.14) induces a natural $G_y \times S^1$-equivariant isomorphism of complex vector bundles

$$\Lambda^i W_{v,n+pv} \simeq \Lambda^{\dim W_v - i} \overline{W}_{v,-n-pv} \otimes (\det \overline{W}_v)^{-1}. \tag{3.16}$$

From (2.9), (3.15) and (3.16), we have

$$\begin{aligned} &\bigotimes_{\substack{n\in\mathbf{N}, v>0,\\ 0\leq n<pv}} \Lambda^{i_n}\overline{W}_{v,n-pv} \bigotimes_{\substack{n\in\mathbf{N}, v<0,\\ 0<n\leq -pv}} \Lambda^{i'_n} W_{v,n+pv} \\ \simeq &\bigotimes_{\substack{n\in\mathbf{N}, v>0,\\ 0\leq n<pv}} \Lambda^{\dim W_v - i_n} W_{v,-n+pv} \bigotimes_{\substack{n\in\mathbf{N}, v<0,\\ 0<n\leq -pv}} \Lambda^{\dim W_v - i'_n}\overline{W}_{v,-n-pv} \otimes L(W)^{-p}. \end{aligned} \tag{3.17}$$

From (3.13), (3.17), we get (3.11).

The proof of Proposition 3.1 is complete. □

PROPOSITION 3.2. *For $p \in \mathbf{Z}$, $p > 0$, $i = 1, 2$, the G_y-equivariant bundle isomorphism induced by (3.10) and (3.11),*

$$\begin{aligned} r_* : S(TY, K_X \otimes_{0<v} (\det N_v)^{-1}) \otimes (K_W \otimes K_X^{-1})^{1/2} \\ \otimes \mathcal{F}^{-p}(X) \otimes F_V^i \otimes Q(W) \\ \to S(TY, K_X \otimes_{0<v} (\det N_v)^{-1}) \otimes (K_W \otimes K_X^{-1})^{1/2} \\ \otimes \mathcal{F}^{0}(X) \otimes F_V^i \otimes Q(W) \otimes L^{-p}, \end{aligned} \tag{3.18}$$

verifies the following identities

$$\begin{aligned} &r_*^{-1} \cdot \mathbf{J}_H \cdot r_* = \mathbf{J}_H, \\ &r_*^{-1} \cdot P \cdot r_* = P + p\mathbf{J}_H + p^2 e - \tfrac{1}{2}p^2 e(N) - \tfrac{p}{2} d'(N). \end{aligned} \tag{3.19}$$

For the $\mathbf{Z}_2$-gradings, we have

$$\begin{aligned} &r_*^{-1}\tau_e r_* = \tau_e, \qquad r_*^{-1}\tau_s r_* = \tau_s, \\ &r_*^{-1}\tau_1 r_* = (-1)^{pd'(W)}\tau_1. \end{aligned} \tag{3.20}$$

Proof. We divide the argument into several steps.

1) The first equation of (3.19) is obvious.

2) a) From [LiuMaZ, (3.23)] and (2.8), for $i = 1, 2$, on F_V^i, we have

$$r_*^{-1} P r_* = P + p\mathbf{J}_H + \frac{1}{2} p^2 e(V). \tag{3.21}$$

b) Note that on $\otimes_{0<v, 0\leq n \leq pv} \det N_v$, $\mathbf{J}_H$ acts as $pe(N) + d'(N)$. On $S(TY, K_X \otimes \det(\oplus_{0<v} N_v)^{-1}) \otimes (K_W \otimes K_X^{-1})^{1/2}$, $\mathbf{J}_H$ acts as $-\frac{1}{2} d'(N) + \frac{1}{2} d'(W)$. From (2.8), (3.6), on $S(TY, K_X \otimes \det(\oplus_{0<v} N_v)^{-1}) \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathcal{F}^{-p}(X)$,

$$r_*^{-1} P r_* = P + p\mathbf{J}_H - p^2 e(N) - \frac{1}{2} p(d'(N) + d'(W)). \tag{3.22}$$

c) From (2.8), (3.17), on $\bigotimes_{\substack{n\in\mathbf{N}, v>0,\\ 0\leq n<pv}} \Lambda^{i_n} \overline{W}_{v,n} \bigotimes_{\substack{n\in\mathbf{N}, v<0,\\ 0<n\leq -pv}} \Lambda^{i'_n} W_{v,n}$, one has

(3.23)

$$\begin{aligned}
r_*^{-1} P r_* &= \sum_{\substack{n\in\mathbf{N}, v>0,\\ 0\leq n<pv}} (\dim W_v - i_n)(-n + pv) + \sum_{\substack{n\in\mathbf{N}, v<0,\\ 0<n\leq -pv}} (\dim W_v - i'_n)(-n - pv) \\
&= P + p\mathbf{J}_H + \sum_{\substack{n\in\mathbf{N}, v>0,\\ 0\leq n<pv}} (\dim W_v)(-n + pv) + \sum_{\substack{n\in\mathbf{N}, v<0,\\ 0<n\leq -pv}} (\dim W_v)(-n - pv) \\
&= P + p\mathbf{J}_H + \frac{1}{2} p^2 e(W) + \frac{1}{2} p d'(W).
\end{aligned}$$

From (2.11), (3.21), (3.22) and (3.23), we get the second equality of (3.19).

3) The first two identities of (3.20) were proved in [LiuMaZ, Proposition 3.2].

For the $\mathbf{Z}_2$-grading τ_1, it changes only on $\bigotimes_{\substack{n\in\mathbf{N}, v>0,\\ 0\leq n<pv}} \Lambda^{i_n} \overline{W}_{v,n} \bigotimes_{\substack{n\in\mathbf{N}, v<0,\\ 0<n\leq -pv}} \Lambda^{i'_n} W_{v,n}$. From (2.8), (3.17), we get the last equality of (3.20).

The proof of Proposition 3.2 is complete. □

Proof of Theorem 3.1. From (2.11), (3.4) and Propositions 3.2, we easily obtain Theorem 3.1. □

4. Proof of Theorem 2.5.

In this section, we prove Theorem 2.5. As in [LiuMaZ, §4], we will construct a family twisted Dirac operator on $M(n_j)$, the fixed point set of the induced $\mathbf{Z}_{n_j}$ action on M. By applying our K-theory version of the equivariant family index theorem to this operator, we prove Theorem 2.5.

This section is organized as follows: In Section 4.1, we construct a family Dirac operator on $M(n_j)$. In Section 4.2, by introducing a shift operator, we will relate both sides of equation (2.25) to the index bundle of the family Dirac operator on $M(n_j)$. In Section 4.3, we prove Theorem 2.5.

In this section, we make the same assumptions and use the same notations as in Sections 2, 3.

4.1. The Spin^c Dirac operator on $M(n_j)$.

Let $\pi : M \to B$ be a fibration of compact manifolds with fiber X and $\dim_{\mathbf{R}} X = 2l$. We assume that S^1 acts fiberwise on M, and TX has an S^1-invariant Spin^c structure. Let $F = \{F_\alpha\}$ be the fixed point set of the S^1-action on M. Then $\pi : F \to B$ is a fibration with compact fiber Y. For $n \in \mathbf{N}$, $n > 0$, let $\mathbf{Z}_n \subset S^1$ denote the cyclic subgroup of order n.

Let V be a real even dimensional vector bundle over M with an S^1-invariant spin structure. Let W be an S^1-equivariant complex vector bundle over M.

For $n_j \in \mathbf{N}$, $n_j > 0$, let $M(n_j)$ be the fixed point set of the induced $\mathbf{Z}_{n_j}$-action on M. Then $\pi : M(n_j) \to B$ is a fibration with compact fiber $X(n_j)$. Let $N(n_j) \to M(n_j)$ be the normal bundle to $M(n_j)$ in M. As in [LiuMaZ, §4.1], we see that $N(n_j)$ and V can be decomposed, as real vector bundles over $M(n_j)$, to

$$\begin{aligned} N(n_j) &\simeq \bigoplus_{0<v<n_j/2} N(n_j)_v \oplus N(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}, \\ V|_{M(n_j)} &\simeq V(n_j)^{\mathbf{R}}_0 \bigoplus_{0<v<n_j/2} V(n_j)_v \oplus V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}} \end{aligned} \tag{4.1}$$

respectively. In (4.1), the last term is understood to be zero when n_j is odd. We also denote by $V(n_j)_0$, $V(n_j)_{\frac{n_j}{2}}$, $N(n_j)_{\frac{n_j}{2}}$ the corresponding complexification of the real vector bundles $V(n_j)^{\mathbf{R}}_0$, $V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}$ and $N(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}$ on $M(n_j)$. Then $N(n_j)_v$, $V(n_j)_v$'s are complex vector bundles over $M(n_j)$ with $g \in \mathbf{Z}_{n_j}$ acting by g^v on it.

Similarly, we also have the following $\mathbf{Z}_{n_j}$-equivariant decomposition of W on $M(n_j)$,

$$W = \oplus_{0\le v<n_j} W(n_j)_v. \tag{4.2}$$

Here $W(n_j)_v$ is a complex vector bundle over $M(n_j)$ with $g \in \mathbf{Z}_{n_j}$ acting by g^v on it.

It is essential for us to know that the vector bundles $TX(n_j)$ and $V(n_j)^{\mathbf{R}}_0$ are orientable. For this we have the following lemma which generalizes [BT, Lemmas 9.4, 10.1] (See also [O]).

Lemma 4.1. *Let R be a real, even dimensional orientable vector bundle over a manifold M. Let G be a compact Lie group. We assume that G acts on M, and lifts to R. We assume that R has a G-invariant Spin^c structure. For $g \in G$, let M^g be the fixed point set of g on M. Let R_0 be the subbundle of R over M^g on which g acts trivially. Then R_0 is even dimensional and orientable.*

Proof. Let h^R be the metric on R which is induced from the Spin^c structure on R. As g preserves the Spin^c structure of R, g is an isometry on R and preserves the orientation of R. On M^g, we have the following decomposition of real vector bundles,

$$R = R_0 \oplus R_1.$$

Since the only possible real eigenvalue of g on R_1 is -1, and $\det(g_{|R_1}) = 1$ on M^g, we know that $\dim_{\mathbf{R}} R_1 = \dim_{\mathbf{R}} R - \dim_{\mathbf{R}} R_0$ must be even. So $\dim_{\mathbf{R}} R_0$ is even.

Let K_R be the G-equivariant complex line bundle over M which is induced by the Spin^c structure of R. Then $E = R \oplus K_R$ has an G-invariant spin structure. On M^g, we have the decomposition of vector bundles $E = E_1 \oplus E_0$, here E_0 is the subbundle of E on which g acts trivially. Now the action of g on the fiber of the spinor bundle $S(E)$ at $x \in M^g$ gives an element $\widetilde{g} \in \mathrm{Spin}(E_x) \subset C(E_x)$, here $C(E_x)$ is the Clifford algebra of E_x. Let $\rho : \mathrm{Spin}(E_x) \to SO(E_x)$ be the standard representation of $\mathrm{Spin}(E_x)$, then $\rho(\widetilde{g}) = g$. So $\widetilde{g}c(a) = c(ga)\widetilde{g}$ for $a \in E_x$. Here we denote by $c(\cdot)$ the Clifford action. This means that $\widetilde{g}$ commutes $c(a)$ for $a \in E_{0x}$, so $\widetilde{g} \in \mathrm{Spin}(E_{1x})$.

Let $e_1, \cdots, e_{2k}$ be an orthonormal basis of E_{1x}, then $e_{i_1} \cdots e_{i_j}$ $(1 \le i_1 < \cdots < i_j \le 2k)$ is an orthonormal basis of the vector space $C(E_{1x})$. We define $\sigma : C(E_{1x}) \to \det(E_{1x})$ by

$$\sigma(e_{i_1} \cdots e_{i_j}) = \begin{cases} e_1 \wedge \cdots \wedge e_{2k} & \text{if} \quad j = 2k = \dim_{\mathbf{R}} E_1, \\ 0 & \text{otherwise.} \end{cases}$$

By [**BGV**, Lemma 3.22],

$$|\sigma(\widetilde{g})| = \det{}^{1/2}((1-g_{|E_1})/2). \tag{4.3}$$

So $\sigma(\widetilde{g})$ is a nonvanishing section of $\det(E_1)$, $\det(E_1)$ is a trivial line bundle on M^g. But E_1 is equal R_1 or $R_1 \oplus K_R$, this means R_1 is orientable. So R_0 is orientable.

This completes the proof of Lemma 4.1. □

By Lemma 4.1, $TX(n_j)$ and $V(n_j)_0^{\mathbf{R}}$ are even dimensional and orientable over $M(n_j)$. Thus $N(n_j)$ is orientable over $M(n_j)$. By (4.1), $N(n_j)_{\frac{n_j}{2}}^{\mathbf{R}}$ and $V(n_j)_{\frac{n_j}{2}}^{\mathbf{R}}$ are also even dimensional and orientable over $M(n_j)$. In the following, we fix the orientations of $N(n_j)_{\frac{n_j}{2}}^{\mathbf{R}}$ and $V(n_j)_{\frac{n_j}{2}}^{\mathbf{R}}$ over $M(n_j)$. We also fix the orientations of $TX(n_j)$ and $V(n_j)_0^{\mathbf{R}}$ which are induced by (4.1) and the orientations on TX, V, $N(n_j)_{\frac{n_j}{2}}^{\mathbf{R}}$ and $V(n_j)_{\frac{n_j}{2}}^{\mathbf{R}}$.

Let

$$r(n_j) = \frac{1}{2}(1+(-1)^{n_j}). \tag{4.4}$$

Lemma 4.2. *Assume that (2.4) holds. Let*

$$\begin{aligned} L(n_j) = \bigotimes_{0<v<n_j/2} \Big(& \det(N(n_j)_v) \otimes \det(\overline{V(n_j)_v}) \\ & \otimes \det(\overline{W(n_j)}_v) \otimes \det(W(n_j)_{n_j-v}) \Big)^{(r(n_j)+1)v} \end{aligned} \tag{4.5}$$

be the complex line bundle over $M(n_j)$. Then we have

i) $L(n_j)$ has an n_j^{th} root over $M(n_j)$.

ii) Let

$$\begin{aligned} L_1 &= K_X \bigotimes_{0<v<n_j/2} \Big(\det(N(n_j)_v) \otimes \det(\overline{V(n_j)_v}) \Big) \\ &\qquad \otimes \det(W(n_j)_{n_j/2}) \otimes L(n_j)^{r(n_j)/n_j}, \\ L_2 &= K_X \bigotimes_{0<v<n_j/2} \Big(\det(N(n_j)_v) \Big) \otimes \det(W(n_j)_{n_j/2}) \otimes L(n_j)^{r(n_j)/n_j}. \end{aligned} \tag{4.6}$$

Let $U_1 = TX(n_j) \oplus V(n_j)_0^{\mathbf{R}}$ and $U_2 = TX(n_j) \oplus V(n_j)_{\frac{n_j}{2}}^{\mathbf{R}}$. Then U_1 (resp. U_2) has a Spin^c structure defined by L_1 (resp. L_2).

Proof. Both statements follow from the proof of [BT, Lemmas 11.3 and 11.4]. □

Lemma 4.2 allows us, as we are going to see, to apply the constructions and results in Section 1.1 to the fibration $M(n_j) \to B$, which is the main concern of this section.

For $p_j \in \mathbf{N}$, $p_j < n_j$, $(p_j, n_j) = 1$, $\beta_j = \frac{p_j}{n_j}$, let us write

$$\begin{aligned} \mathcal{F}(\beta_j) = \otimes_{0<n\in\mathbf{Z}} \mathrm{Sym}(TX(n_j)_n) \bigotimes_{0<v<n_j/2} \mathrm{Sym}\Big(& \bigoplus_{0<n\in\mathbf{Z}+p_jv/n_j} N(n_j)_{v,n} \\ & \bigoplus_{0<n\in\mathbf{Z}-p_jv/n_j} \overline{N(n_j)}_{v,n} \Big) \otimes_{0<n\in\mathbf{Z}+\frac{1}{2}} \mathrm{Sym}(N(n_j)_{\frac{n_j}{2},n}), \end{aligned} \tag{4.7}$$

$$\begin{aligned} F_V^1(\beta_j) = \Lambda\Big(& \oplus_{0<n\in\mathbf{Z}} V(n_j)_{0,n} \bigoplus_{0<v<n_j/2} \Big(\bigoplus_{0<n\in\mathbf{Z}+p_jv/n_j} V(n_j)_{v,n} \\ & \bigoplus_{0<n\in\mathbf{Z}-p_jv/n_j} \overline{V(n_j)}_{v,n} \Big) \oplus_{0<n\in\mathbf{Z}+\frac{1}{2}} V(n_j)_{\frac{n_j}{2},n} \Big), \\ F_V^2(\beta_j) = \Lambda\Big(& \oplus_{0<n\in\mathbf{Z}} V(n_j)_{\frac{n_j}{2},n} \bigoplus_{0<v<n_j/2} \Big(\bigoplus_{0<n\in\mathbf{Z}+p_jv/n_j+\frac{1}{2}} V(n_j)_{v,n} \\ & \bigoplus_{0<n\in\mathbf{Z}-p_jv/n_j+\frac{1}{2}} \overline{V(n_j)}_{v,n} \Big) \oplus_{0<n\in\mathbf{Z}+\frac{1}{2}} V(n_j)_{0,n} \Big), \\ Q_W(\beta_j) = \Lambda\Big(& \bigoplus_{0\le v<n_j} \Big(\bigoplus_{0<n\in\mathbf{Z}+p_jv/n_j} W(n_j)_{v,n} \bigoplus_{0\le n\in\mathbf{Z}-p_jv/n_j} \overline{W(n_j)}_{v,n} \Big)\Big). \end{aligned}$$

We denote by $D^{X(n_j)}$ the S^1-equivariant Spin^c-Dirac operator on $S(U_1, L_1)$ or $S(U_2, L_2)$ along the fiber $X(n_j)$ defined as in Section 1.1. We denote by $D^{X(n_j)} \otimes \mathcal{F}(\beta_j) \otimes F_V^i(\beta_j) \otimes Q_W(\beta_j)$ ($i = 1,\ 2$) the corresponding twisted Spin^c Dirac operator on $S(U_i, L_i) \otimes \mathcal{F}(\beta_j) \otimes F_V^i(\beta_j) \otimes Q_W(\beta_j)$ along the fiber $X(n_j)$.

REMARK 4.1. In fact, to define an S^1 (resp. G_y)-action on $L(n_j)^{r(n_j)/n_j}$, one must replace the S^1-action by its n_j-fold action (resp. the G_y-action by $G_{y^{1/n_j}}$-action). Here by abusing notation, we still say an S^1 (resp. G_y)-action without causing any confusion.

In the rest of this subsection, we will reinterpret all of the above objects when we restrict ourselves to F, the fixed point set of the S^1 action. We will use the notation of Sections 1.1 and 2.

Let $N_{F/M(n_j)}$ be the normal bundle to F in $M(n_j)$. Then by (2.5),

$$\begin{aligned} &N_{F/M(n_j)} = \bigoplus\nolimits_{0<v: v\in n_j\mathbf{Z}} N_v, \\ &TX(n_j) \otimes_{\mathbf{R}} \mathbf{C} = TY \otimes_{\mathbf{R}} \mathbf{C} \oplus_{0<v, v\in n_j\mathbf{Z}} (N_v \oplus \overline{N}_v). \end{aligned} \tag{4.8}$$

By (2.5), (2.6) and (4.1), the restriction to F of $N(n_j)_v$, $V(n_j)_v$ $(1 \le v \le n_j/2)$ is given by

$$\begin{aligned} N(n_j)_v &= \bigoplus_{0<v': v'=v \text{ mod}(n_j)} N_{v'} \bigoplus_{0<v': v'=-v \text{ mod}(n_j)} \overline{N}_{v'}, \\ V(n_j)_v &= \bigoplus_{0<v': v'=v \text{ mod}(n_j)} V_{v'} \bigoplus_{0<v': v'=-v \text{ mod}(n_j)} \overline{V}_{v'}. \end{aligned} \tag{4.9}$$

And

$$V(n_j)_0 = V_0^{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C} \bigoplus_{0<v, v=0 \text{ mod}(n_j)} (V_v \oplus \overline{V}_v). \tag{4.10}$$

By (4.8)-(4.10), we have the following identifications of real vector bundles over F,

$$\begin{aligned} &N(n_j)^{\mathbf{R}}_{\frac{n_j}{2}} = \bigoplus\nolimits_{0<v, v=\frac{n_j}{2} \text{ mod}(n_j)} N_v, \\ &TX(n_j) = TY \bigoplus\nolimits_{0<v, v=0 \text{ mod}(n_j)} N_v, \\ &V(n_j)^{\mathbf{R}}_0 = V_0^{\mathbf{R}} \bigoplus\nolimits_{0<v, v=0 \text{ mod}(n_j)} V_v, \\ &V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}} = \bigoplus\nolimits_{0<v, v=\frac{n_j}{2} \text{ mod}(n_j)} V_v. \end{aligned} \tag{4.11}$$

By (2.6) and (4.2), the restriction to F of $W(n_j)_v$ $(0 \le v < n_j)$ is given by

$$W(n_j)_v = \oplus_{v'=v \text{ mod } (n_j)} W_{v'}. \tag{4.12}$$

We denote by $V_0 = V_0^{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ the complexification of $V_0^{\mathbf{R}}$ over F. As $(p_j, n_j) = 1$, we know that for $v \in \mathbf{Z}$, $p_j v/n_j \in \mathbf{Z}$ iff $v/n_j \in \mathbf{Z}$. Also, $p_j v/n_j \in \mathbf{Z} + \frac{1}{2}$ iff $v/n_j \in \mathbf{Z} + \frac{1}{2}$. From (4.8)-(4.12), we then get

(4.13)

$$\begin{aligned} \mathcal{F}(\beta_j) = \otimes_{0<n\in\mathbf{Z}} \text{Sym}(TY_n) &\bigotimes\nolimits_{0<v, v=0, \frac{n_j}{2} \text{ mod}(n_j)} \bigotimes\nolimits_{0<n\in\mathbf{Z}+\frac{p_j v}{n_j}} \text{Sym}(N_{v,n} \oplus \overline{N}_{v,n}) \\ &\bigotimes\nolimits_{0<v'<n_j/2} \text{Sym}\Big(\oplus_{v=v' \text{ mod}(n_j)} \Big(\oplus_{0<n\in\mathbf{Z}+\frac{p_j v}{n_j}} N_{v,n} \oplus_{0<n\in\mathbf{Z}-\frac{p_j v}{n_j}} \overline{N}_{v,n} \Big) \\ &\qquad \oplus_{v=-v' \text{ mod}(n_j)} \Big(\oplus_{0<n\in\mathbf{Z}+\frac{p_j v}{n_j}} N_{v,n} \oplus_{0<n\in\mathbf{Z}-\frac{p_j v}{n_j}} \overline{N}_{v,n} \Big) \Big), \end{aligned}$$

$$\begin{aligned}
F_V^1(\beta_j) = \Lambda\Big[& \oplus_{0<n\in\mathbf{Z}} V_{0,n} \bigoplus_{0<v, v=0, \frac{n_j}{2} \bmod(n_j)} \Big(\oplus_{0<n\in\mathbf{Z}+\frac{p_j v}{n_j}} V_{v,n} \oplus_{0<n\in\mathbf{Z}-\frac{p_j v}{n_j}} \overline{V}_{v,n}\Big) \\
& \bigoplus_{0<v'<n_j/2} \Big(\bigoplus_{v=v',-v' \bmod(n_j)} \Big(\oplus_{0<n\in\mathbf{Z}+\frac{p_j v}{n_j}} V_{v,n} \oplus_{0<n\in\mathbf{Z}-\frac{p_j v}{n_j}} \overline{V}_{v,n}\Big)\Big)\Big], \\
F_V^2(\beta_j) = \Lambda\Big[& \oplus_{0<n\in\mathbf{Z}+\frac{1}{2}} V_{0,n} \oplus_{0<v, v=0, \frac{n_j}{2} \bmod(n_j)} \\
& \Big(\oplus_{0<n\in\mathbf{Z}+\frac{p_j v}{n_j}+\frac{1}{2}} V_{v,n} \oplus_{0<n\in\mathbf{Z}-\frac{p_j v}{n_j}+\frac{1}{2}} \overline{V}_{v,n}\Big) \\
& \bigoplus_{0<v'<n_j/2} \Big(\oplus_{v=v',-v' \bmod(n_j)} \Big(\oplus_{0<n\in\mathbf{Z}+\frac{p_j v}{n_j}+\frac{1}{2}} V_{v,n} \oplus_{0<n\in\mathbf{Z}-\frac{p_j v}{n_j}+\frac{1}{2}} \overline{V}_{v,n}\Big)\Big)\Big], \\
Q_W(\beta_j) = \Lambda\Big(& \bigoplus_v \Big(\bigoplus_{0<n\in\mathbf{Z}+p_j v/n_j} W_{v,n} \bigoplus_{0\le n\in\mathbf{Z}-p_j v/n_j} \overline{W}_{v,n}\Big)\Big).
\end{aligned}$$

Now, we want to compare the spinor bundles over F. From (4.5), (4.6), (4.9) and (4.12), we get that over F we have the identities

$$\begin{aligned}
L(n_j)^{\frac{r(n_j)}{n_j}} &= \bigotimes_{0<v'<n_j/2} \Big(\bigotimes_{v=v' \bmod(n_j)} (\det N_v \otimes \det \overline{V}_v \otimes \det \overline{W}_v)^{2v'} \\
&\qquad \bigotimes_{v=-v' \bmod(n_j)} (\det N_v \otimes \det \overline{V}_v \otimes \det \overline{W}_v)^{-2v'}\Big)^{r(n_j)/n_j}, \\
L_1 &= K_X \otimes L(n_j)^{r(n_j)/n_j} \bigotimes_{0<v'<n_j/2} \Big(\bigotimes_{v=v' \bmod(n_j)} (\det N_v \otimes \det \overline{V}_v) \\
&\qquad \bigotimes_{v=-v' \bmod(n_j)} (\det N_v \otimes \det \overline{V}_v)^{-1}\Big) \bigotimes_{v=\frac{n_j}{2} \bmod(n_j)} \det W_v, \\
L_2 &= K_X \otimes L(n_j)^{r(n_j)/n_j} \bigotimes_{0<v'<n_j/2} \Big(\bigotimes_{v=v' \bmod(n_j)} \det N_v \\
&\qquad \bigotimes_{v=-v' \bmod(n_j)} (\det N_v)^{-1}\Big) \bigotimes_{v=\frac{n_j}{2} \bmod(n_j)} \det W_v.
\end{aligned} \tag{4.14}$$

From (4.11), we have, over F,

$$\begin{aligned}
TX(n_j) \oplus V(n_j)_0^{\mathbf{R}} &= TY \oplus V_0^{\mathbf{R}} \oplus_{0<v, v=0 \bmod(n_j)} (N_v \oplus V_v), \\
TX(n_j) \oplus V(n_j)_{\frac{n_j}{2}}^{\mathbf{R}} &= TY \oplus_{0<v, v=0 \bmod(n_j)} N_v \oplus_{0<v, v=\frac{n_j}{2} \bmod(n_j)} V_v.
\end{aligned} \tag{4.15}$$

Recall that the Spin^c vector bundles U_1, U_2 have been defined in Lemma 4.2. Denote by

(4.16)

$$\begin{aligned}
S(U_1, L_1)' &= S\Big(TY \oplus V_0^{\mathbf{R}}, L_1 \bigotimes_{\substack{0<v,\\ v=0 \bmod(n_j)}} (\det N_v \otimes \det V_v)^{-1}\Big) \bigotimes_{\substack{0<v,\\ v=0 \bmod(n_j)}} \Lambda V_v, \\
S(U_2, L_2)' &= S\Big(TY, L_2 \bigotimes_{\substack{0<v,\\ v=0 \bmod(n_j)}} (\det N_v)^{-1} \bigotimes_{\substack{0<v,\\ v=\frac{n_j}{2} \bmod(n_j)}} (\det V_v)^{-1}\Big) \bigotimes_{\substack{0<v,\\ v=\frac{n_j}{2} \bmod(n_j)}} \Lambda V_v.
\end{aligned}$$

Then from (1.10) and (4.16), for $i = 1,\ 2$, we have the following isomorphism of Clifford modules over F,

$$S(U_i, L_i) \simeq S(U_i, L_i)' \otimes \Lambda(\oplus_{0<v, v=0 \bmod(n_j)} N_v). \tag{4.17}$$

We define the $\mathbf{Z}_2$ gradings on $S(U_i, L_i)'$ ($i = 1,\ 2$) induced by the $\mathbf{Z}_2$-gradings on $S(U_i, L_i)$ ($i = 1,\ 2$) and on $\Lambda(\oplus_{0<v, v=0 \bmod(n_j)} N_v)$ such that the isomorphism (4.17) preserves the $\mathbf{Z}_2$-grading.

We introduce formally the following complex line bundles over F,

$$\begin{aligned}
L_1' &= \Big[L_1^{-1} \otimes_{\substack{0<v,\\ v=0 \bmod(n_j)}} (\det N_v \otimes \det V_v) \otimes_{0<v} (\det N_v \otimes \det V_v)^{-1} \otimes K_X\Big]^{1/2}, \\
L_2' &= \Big[L_2^{-1} \otimes_{\substack{0<v,\\ v=0 \bmod(n_j)}} \det N_v \otimes_{\substack{0<v,\\ v=n_j/2 \bmod(n_j)}} \det V_v \otimes_{0<v} (\det N_v)^{-1} \otimes K_X\Big]^{1/2}.
\end{aligned}$$

From (1.10), Lemma 4.2 and the assumption that V is spin, one verifies easily that $c_1({L'_i}^2) = 0 \bmod(2)$ for $i = 1,\ 2$. Thus $L'_1,\ L'_2$ are well defined complex line bundles over F. For the later use, we also write down the following expressions of L'_i $(i = 1,\ 2)$ which can be deduced from (4.14):

$$\begin{aligned} L'_1 &= \Big[L(n_j)^{-r(n_j)/n_j} \otimes_{v=\frac{n_j}{2} \bmod(n_j)} (\det N_v \otimes \det \overline{V}_v \otimes \det \overline{W}_v)\Big]^{\frac{1}{2}} \\ &\qquad \otimes_{0<v\leq \frac{n_j}{2} \bmod(n_j)} (\det N_v)^{-1} \otimes_{\frac{n_j}{2}<v<n_j \bmod(n_j)} (\det V_v)^{-1}, \\ L'_2 &= \Big[L(n_j)^{-r(n_j)/n_j} \otimes_{v=\frac{n_j}{2} \bmod(n_j)} (\det N_v \otimes \det V_v \otimes \det \overline{W}_v)\Big]^{\frac{1}{2}} \\ &\qquad \otimes_{0<v\leq \frac{n_j}{2} \bmod(n_j)} (\det N_v)^{-1}. \end{aligned} \tag{4.18}$$

From (4.14), (4.16), and the definition of L'_i $(i = 1,\ 2)$, we get the following identifications of Clifford modules over F,

$$\begin{aligned} S(U_1, L_1)' \otimes L'_1 &= S(TY, K_X \otimes_{0<v} (\det N_v)^{-1}) \otimes S(V_0^{\mathbf{R}}, \otimes_{0<v} (\det V_v)^{-1}) \\ &\qquad \otimes \Lambda(\oplus_{0<v, v=0 \bmod(n_j)} V_v), \\ S(U_2, L_2)' \otimes L'_2 &= S(TY, K_X \otimes_{0<v} (\det N_v)^{-1}) \otimes \Lambda(\oplus_{0<v, v=\frac{n_j}{2} \bmod(n_j)} V_v). \end{aligned} \tag{4.19}$$

Let

$$\begin{aligned} \Delta(n_j, N) &= \sum_{\frac{n_j}{2}<v'<n_j} \ \sum_{0<v=v' \bmod(n_j)} \dim N_v + o(N(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}), \\ \Delta(n_j, V) &= \sum_{\frac{n_j}{2}<v'<n_j} \ \sum_{0<v=v' \bmod(n_j)} \dim V_v + o(V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}), \end{aligned} \tag{4.20}$$

with $o(N(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}) = 0$ or 1 (resp. $o(V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}) = 0$ or 1), depending on whether the given orientation on $N(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}$ (resp. $V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}$) agrees or disagrees with the complex orientation of $\oplus_{v=\frac{n_j}{2} \bmod(n_j)} N_v$ (resp. $\oplus_{v=\frac{n_j}{2} \bmod(n_j)} V_v$).

By [LiuMaZ, §4.1], (4.12) and (4.17), for the $\mathbf{Z}_2$-gradings induced by τ_s, the difference of the $\mathbf{Z}_2$-gradings of (4.19) is $(-1)^{\Delta(n_j,N)}$; for the $\mathbf{Z}_2$-gradings induced by τ_e, the difference of the $\mathbf{Z}_2$-gradings of the first (resp. second) equation of (4.19) is $(-1)^{\Delta(n_j,N)+\Delta(n_j,V)}$ (resp. $(-1)^{\Delta(n_j,N)+o(V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}})}$).

4.2. The Shift operators. Let $p \in \mathbf{N}^*$ be fixed. For any $1 \leq j \leq J_0$, inspired by [T, §9], as in [LiuMaZ, §4], we define the following shift operators r_{j*}:

$$\begin{aligned} &r_{j*} : N_{v,n} \to N_{v,n+(p-1)v+p_jv/n_j}, \quad && r_{j*} : \overline{N}_{v,n} \to \overline{N}_{v,n-(p-1)v-p_jv/n_j}, \\ &r_{j*} : W_{v,n} \to W_{v,n+(p-1)v+p_jv/n_j}, \quad && r_{j*} : \overline{W}_{v,n} \to \overline{W}_{v,n-(p-1)v-p_jv/n_j}, \\ &r_{j*} : V_{v,n} \to V_{v,n+(p-1)v+p_jv/n_j}, \quad && r_{j*} : \overline{V}_{v,n} \to \overline{V}_{v,n-(p-1)v-p_jv/n_j}. \end{aligned} \tag{4.21}$$

If E is a combination of the above bundles, we denote by $r_{j*}E$ the bundle on which the action of P is changed in the above way.

Recall that the vector bundles F_V^i $(i = 1,\ 2)$ have been defined in (3.7). From (2.21), we see that

$$\begin{aligned} \mathcal{F}_{p,j}(X) = \mathcal{F}_p(X) \otimes \mathcal{F}'_{p-1}(X) \bigotimes_{(v,n)\in \cup_{i=1}^j I_i^p} \Big(\mathrm{Sym}(N_{v,-n}) \otimes \det N_v\Big) \\ \bigotimes_{(v,n)\in \overline{I}_j^p} \mathrm{Sym}(\overline{N}_{v,n}). \end{aligned} \tag{4.22}$$

Proposition 4.1. *There are natural isomorphisms of vector bundles over F,*

$$\begin{aligned} r_{j*}\mathcal{F}_{p,j-1}(X) &\simeq \mathcal{F}(\beta_j) \bigotimes\nolimits_{0<v,v=0 \ \mathrm{mod}(n_j)} \mathrm{Sym}(\overline{N}_{v,0}) \\ &\quad \otimes_{0<v} (\det N_v)^{[\frac{p_j v}{n_j}]+(p-1)v+1} \bigotimes\nolimits_{0<v,v=0 \ \mathrm{mod}(n_j)} (\det N_v)^{-1}, \\ r_{j*}\mathcal{F}_{p,j}(X) &\simeq \mathcal{F}(\beta_j) \bigotimes\nolimits_{0<v,v=0 \ \mathrm{mod}(n_j)} \mathrm{Sym}(N_{v,0}) \otimes_{0<v} (\det N_v)^{[\frac{p_j v}{n_j}]+(p-1)v+1}, \\ r_{j*}F_V^1 &\simeq S(V_0^{\mathbf{R}}, \otimes_{0<v} (\det V_v)^{-1}) \otimes F_V^1(\beta_j) \bigotimes\nolimits_{0<v,v=0 \ \mathrm{mod}(n_j)} \Lambda(V_{v,0}) \\ &\quad \otimes_{0<v} (\det \overline{V}_v)^{[\frac{p_j v}{n_j}]+(p-1)v}, \\ r_{j*}F_V^2 &\simeq F_V^2(\beta_j) \bigotimes\nolimits_{0<v,v=\frac{n_j}{2} \ \mathrm{mod}(n_j)} \Lambda(V_{v,0}) \otimes_{0<v} (\det \overline{V}_v)^{[\frac{p_j v}{n_j}+\frac{1}{2}]+(p-1)v}, \\ r_{j*}Q(W) &\simeq Q_W(\beta_j) \otimes_{0<v} (\det \overline{W}_v)^{[\frac{p_j v}{n_j}]+(p-1)v+1} \bigotimes\nolimits_{0<v,v=0 \ \mathrm{mod}(n_j)} (\det \overline{W}_v)^{-1} \\ &\quad \otimes_{v<0} (\det W_v)^{[-\frac{p_j v}{n_j}]-(p-1)v}. \end{aligned} \tag{4.23}$$

Proof. The proof is similar to the proof of Proposition 3.1.

Note that, by (2.19), for $v \in J = \{v \in \mathbf{N} |$ There exists α such that $N_v \neq 0$ on $F_\alpha\}$, there are no integer in $]\frac{p_{j-1}v}{n_{j-1}}, \frac{p_j v}{n_j}[$. So for $v \in J$, the elements $(v,n) \in \cup_{i=1}^{i_0} I_i^p$ are $(v,(p-1)v+1), \cdots, (v,(p-1)v+[\frac{p_{i_0}v}{n_{i_0}}])$ for $i_0 = j-1, \ j$. Furthermore,

$$\begin{aligned} [\frac{p_{j-1}v}{n_{j-1}}] &= [\frac{p_j v}{n_j}] - 1 \quad \text{if} \quad v = 0 \quad \mathrm{mod}(n_j), \\ [\frac{p_{j-1}v}{n_{j-1}}] &= [\frac{p_j v}{n_j}] \quad \text{if} \quad v \neq 0 \quad \mathrm{mod}(n_j). \end{aligned} \tag{4.24}$$

By using (3.7), (4.21), (4.22), (4.24), we can prove the first four equalities of (4.23) as in the proof of [LiuMaZ, Proposition 4.1].

From (3.14), we have the natural $G_y \times S^1$-equivariant isomorphisms of complex vector bundles over F,

$$\begin{aligned} \bigotimes_{\substack{n\in\mathbf{N}, v>0, \\ 0\le n<(p-1)v+\frac{p_j v}{n_j}}} \Lambda^{i_n} \overline{W}_{v,n-(p-1)v-\frac{p_j v}{n_j}} &\simeq \bigotimes_{\substack{n\in\mathbf{N}, v>0, \\ 0\le n<(p-1)v+\frac{p_j v}{n_j}}} \Lambda^{\dim W_v - i_n} W_{v,-n+(p-1)v+\frac{p_j v}{n_j}} \\ &\quad \bigotimes_{0<v} (\det \overline{W}_v)^{[\frac{p_j v}{n_j}]+(p-1)v+1} \bigotimes_{0<v,v=0 \ \mathrm{mod}(n_j)} (\det \overline{W}_v)^{-1}, \\ \bigotimes_{\substack{n\in\mathbf{N}, v<0, \\ 0<n\le-(p-1)v-\frac{p_j v}{n_j}}} \Lambda^{i_n} W_{v,n+(p-1)v+\frac{p_j v}{n_j}} &\simeq \bigotimes_{\substack{n\in\mathbf{N}, v<0, \\ 0<n\le-(p-1)v-\frac{p_j v}{n_j}}} \Lambda^{\dim W_v - i_n} \overline{W}_{v,-n-(p-1)v-\frac{p_j v}{n_j}} \\ &\quad \bigotimes_{v<0} (\det W_v)^{[-\frac{p_j v}{n_j}]-(p-1)v}. \end{aligned} \tag{4.25}$$

From (3.7), (4.13), (4.25), we get the last equation of (4.23).

The proof of Proposition 4.1 is complete. □

Lemma 4.3. *Let us write*

$$\begin{aligned} L(\beta_j)_1 = L_1' &\otimes_{0<v} (\det N_v)^{[\frac{p_jv}{n_j}]+(p-1)v+1} \otimes_{0<v} (\det \overline{V}_v)^{[\frac{p_jv}{n_j}]+(p-1)v} \\ &\otimes_{0<v, v=0 \bmod (n_j)} (\det N_v)^{-1} \otimes_{v<0} (\det W_v)^{[-\frac{p_jv}{n_j}]-(p-1)v} \\ &\otimes_{0<v} (\det \overline{W}_v)^{[\frac{p_jv}{n_j}]+(p-1)v+1} \bigotimes_{0<v, v=0 \bmod (n_j)} (\det \overline{W}_v)^{-1}, \\ L(\beta_j)_2 = L_2' &\otimes_{0<v} (\det N_v)^{[\frac{p_jv}{n_j}]+(p-1)v+1} \otimes_{0<v} (\det \overline{V}_v)^{[\frac{p_jv}{n_j}+\frac{1}{2}]+(p-1)v} \\ &\otimes_{0<v, v=0 \bmod (n_j)} (\det N_v)^{-1} \otimes_{v<0} (\det W_v)^{[-\frac{p_jv}{n_j}]-(p-1)v} \\ &\otimes_{0<v} (\det \overline{W}_v)^{[\frac{p_jv}{n_j}]+(p-1)v+1} \bigotimes_{0<v, v=0 \bmod (n_j)} (\det \overline{W}_v)^{-1}. \end{aligned} \tag{4.26}$$

Then $L(\beta_j)_1,\ L(\beta_j)_2$ can be extended naturally to $G_y \times S^1$-equivariant complex line bundles which we will still denote by $L(\beta_j)_1,\ L(\beta_j)_2$ respectively over $M(n_j)$.

Proof. Write

$$[\frac{p_jv}{n_j}] = \frac{p_jv}{n_j} - \frac{\omega(v)}{n_j}. \tag{4.27}$$

Note that for $v = \frac{n_j}{2} \bmod (n_j)$, $\frac{\omega(v)}{n_j} = \frac{1}{2}$.

We introduce the following line bundle over $M(n_j)$,

$$\begin{aligned} L^\omega(\beta_j) = \bigotimes_{0<v<\frac{n_j}{2}} \Big(&\det(N(n_j)_v) \otimes \det(\overline{V(n_j)}_v) \\ &\otimes \det(\overline{W(n_j)}_v) \otimes \det(W(n_j)_{n_j-v}) \Big)^{-\omega(v)-r(n_j)v}. \end{aligned} \tag{4.28}$$

As in [LiuMaZ, (4.38)], Lemma 4.2 implies $L^\omega(\beta_j)^{1/n_j}$ is well defined over $M(n_j)$.

The contributions of N and V in $L(\beta_j)_1, L(\beta_j)_2$ are the same as given in [LiuMaZ, Lemma 4.2], we only need to calculate the contribution of W in $L(\beta_j)_1, L(\beta_j)_2$. Actually from [LiuMaZ, (4.37), (4.44)], (2.9), (4.12), (4.18), (4.26), (4.27) and (4.28), we get

$$\begin{aligned} L(\beta_j)_1 &= L^{-(p-1)-p_j/n_j} \otimes L^\omega(\beta_j)^{1/n_j} \bigotimes_{0<v\le\frac{n_j}{2}} \det(\overline{W(n_j)}_v), \\ L(\beta_j)_2 &= L^{-(p-1)-\frac{p_j}{n_j}} \otimes L^\omega(\beta_j)^{\frac{1}{n_j}} \bigotimes_{0<v\le\frac{n_j}{2}} \det(\overline{W(n_j)}_v) \\ &\qquad \bigotimes_{1\le m\le p_j/2} \bigotimes_{m-\frac{1}{2}<p_jv'/n_j<m} \det(\overline{V(n_j)}_{v'}). \end{aligned} \tag{4.29}$$

The proof of Lemma 4.3 is complete. □

Let us write

$$\begin{aligned} \varepsilon(W) = &-\tfrac{1}{2} \textstyle\sum_{0<v} (\dim W_v) \Big[([\tfrac{p_jv}{n_j}] + (p-1)v)([\tfrac{p_jv}{n_j}] + (p-1)v + 1) \\ &\qquad -(\tfrac{p_jv}{n_j} + (p-1)v)\Big(2\Big([\tfrac{p_jv}{n_j}] + (p-1)v\Big) + 1\Big)\Big] \\ &-\tfrac{1}{2} \textstyle\sum_{v<0} (\dim W_v) \Big[([-\tfrac{p_jv}{n_j}] - (p-1)v)([-\tfrac{p_jv}{n_j}] - (p-1)v + 1) \\ &\qquad +(\tfrac{p_jv}{n_j} + (p-1)v)\Big(2\Big([-\tfrac{p_jv}{n_j}] - (p-1)v\Big) + 1\Big)\Big], \end{aligned} \tag{4.30}$$

$$\begin{aligned}
\varepsilon_1 = \tfrac{1}{2}\textstyle\sum_{0<v}(\dim N_v - \dim V_v)\Big[&([\tfrac{p_j v}{n_j}] + (p-1)v)([\tfrac{p_j v}{n_j}] + (p-1)v + 1) \\
&-(\tfrac{p_j v}{n_j} + (p-1)v)\Big(2\Big([\tfrac{p_j v}{n_j}] + (p-1)v\Big) + 1\Big)\Big], \\
\varepsilon_2 = \tfrac{1}{2}\textstyle\sum_{0<v}(\dim N_v)\Big[&([\tfrac{p_j v}{n_j}] + (p-1)v)([\tfrac{p_j v}{n_j}] + (p-1)v + 1) \\
&-(\tfrac{p_j v}{n_j} + (p-1)v)\Big(2([\tfrac{p_j v}{n_j}] + (p-1)v) + 1\Big)\Big] \\
-\tfrac{1}{2}\textstyle\sum_{0<v}(\dim V_v)\Big[&([\tfrac{p_j v}{n_j} + \tfrac{1}{2}] + (p-1)v)^2 \\
&-2(\tfrac{p_j v}{n_j} + (p-1)v)([\tfrac{p_j v}{n_j} + \tfrac{1}{2}] + (p-1)v)\Big].
\end{aligned}$$

Then $\varepsilon(W), \varepsilon_1,\ \varepsilon_2$ are locally constant functions on F.

Recall that the involutions τ_e, τ_s and τ_1 were defined in Section 3.1. Also recall that if E is an S^1-equivariant vector bundle over M, then the weight of the S^1-action on $\Gamma(F, E)$ is given by the action $\mathbf{J}_H$ (cf. §3.1).

Proposition 4.2. *For $i = 1,\ 2$, the G_y-equivariant isomorphisms induced by (4.19) and (4.23),*

$$\begin{aligned}
r_{i1} : S(TY, K_X \otimes_{0<v} (\det N_v)^{-1}) \otimes (K_W \otimes K_X^{-1})^{1/2} \\
\otimes \mathcal{F}_{p,j-1}(X) \otimes F_V^i \otimes Q(W) \to \\
S(U_i, L_i)' \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathcal{F}(\beta_j) \otimes F_V^i(\beta_j) \\
\otimes Q_W(\beta_j) \otimes L(\beta_j)_i \otimes_{\substack{0<v,\\ v=0\mathrm{mod}(n_j)}} \mathrm{Sym}(\overline{N}_{v,0}), \\
r_{i2} : S(TY, K_X \otimes_{0<v} (\det N_v)^{-1}) \otimes (K_W \otimes K_X^{-1})^{1/2} \\
\otimes \mathcal{F}_{p,j}(X) \otimes F_V^i \otimes Q(W) \to \\
S(U_i, L_i)' \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathcal{F}(\beta_j) \otimes F_V^i(\beta_j) \\
\otimes Q_W(\beta_j) \otimes L(\beta_j)_i \otimes_{\substack{0<v,\\ v=0 \ \mathrm{mod}(n_j)}} (\mathrm{Sym}(N_{v,0}) \otimes \det N_v),
\end{aligned} \tag{4.31}$$

have the following properties : 1) for $i = 1,\ 2$, $\gamma = 1,\ 2$,

$$\begin{aligned}
r_{i\gamma}^{-1}\mathbf{J}_H r_{i\gamma} &= \mathbf{J}_H, \\
r_{i\gamma}^{-1} P r_{i\gamma} &= P + (\tfrac{p_j}{n_j} + (p-1))\mathbf{J}_H + \varepsilon_{i\gamma},
\end{aligned} \tag{4.32}$$

where

$$\begin{aligned}
\varepsilon_{i1} &= \varepsilon_i + \varepsilon(W) - e(p, \beta_{j-1}, N), \\
\varepsilon_{i2} &= \varepsilon_i + \varepsilon(W) - e(p, \beta_j, N).
\end{aligned} \tag{4.33}$$

2) Recall that $o(V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}})$ was defined in (4.20). Let

$$\begin{aligned}
&\mu_1 = -\textstyle\sum_{0<v}[\tfrac{p_j v}{n_j}] \dim V_v + \Delta(n_j, N) + \Delta(n_j, V) \quad \mathrm{mod}(2), \\
&\mu_2 = -\textstyle\sum_{0<v}[\tfrac{p_j v}{n_j} + \tfrac{1}{2}] \dim V_v + \Delta(n_j, N) + o(V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}) \quad \mathrm{mod}(2), \\
&\mu_3 = \Delta(n_j, N) \quad \mathrm{mod}(2), \\
&\mu_4 = \textstyle\sum_v (\dim W_v)([\tfrac{p_j v}{n_j}] + (p-1)v) + \dim W + \dim W(n_j)_0 \quad \mathrm{mod}(2).
\end{aligned} \tag{4.34}$$

Then for $i = 1,\ 2$; $\gamma = 1,\ 2$,

$$\begin{aligned}
&r_{i\gamma}^{-1}\tau_e r_{i\gamma} = (-1)^{\mu_i}\tau_e, \quad r_{i\gamma}^{-1}\tau_s r_{i\gamma} = (-1)^{\mu_3}\tau_s, \\
&r_{i\gamma}^{-1}\tau_1 r_{i\gamma} = (-1)^{\mu_4}\tau_1.
\end{aligned} \tag{4.35}$$

Proof. The first equality of (4.32) is trivial. From (2.23) and (4.24), one has

$$e(p, \beta_j, N) = e(p, \beta_{j-1}, N) + \sum_{0<v, v=0 \ \mathrm{mod}(n_j)} \Big((p-1)v + \frac{p_j v}{n_j}\Big) \dim N_v. \tag{4.36}$$

Denote by $\varepsilon_i(V)$ $(i = 1,\ 2)$ the contribution of $\dim V$ in ε_i $(i = 1,\ 2)$ respectively. Then from [LiuMaZ, (4.52), (4.53)], on F^i_V, we have

$$r_{j*}^{-1} P r_{j*} = P + ((p-1) + \tfrac{p_j}{n_j})\mathbf{J}_H + \varepsilon_i(V). \tag{4.37}$$

From (4.25), as in (3.23), on $Q(W)$, we get

$$r_{j*}^{-1} P r_{j*} = P + ((p-1) + \frac{p_j}{n_j})\mathbf{J}_H + \varepsilon(W) + \frac{1}{2}\Big((p-1) + \frac{p_j}{n_j}\Big) d'(W). \tag{4.38}$$

From (4.36), (4.37), (4.38), and by proceeding as in the proof of Proposition 3.2, as in [LiuMaZ, Proposition 4.2], one deduces easily the second equation of (4.32).

Finally from the discussion following (4.20), and [LiuMaZ, (4.50)], we get the first two equations of (4.35). By (4.12) and (4.25), we get the last equation of (4.35).

The proof of Proposition 4.2 is complete. □

Lemma 4.4. *For each connected component M' of $M(n_j)$, $\varepsilon_1 + \varepsilon(W)$, $\varepsilon_2 + \varepsilon(W)$ are independent on the connected component of F in M'.*

Proof. From (2.11), (4.10), (4.12), (4.27) and (4.30), we have

$$\begin{aligned} \varepsilon_1 = {} & \frac{1}{2} \sum_{0 \le v' < n_j} \sum_{v = v' \ \mathrm{mod}(n_j)} (\dim N_v - \dim V_v - \dim W_v) \\ & \qquad \Big[-(\frac{p_j v}{n_j} + (p-1)v)^2 - \frac{\omega(v')(n_j - \omega(v'))}{n_j^2} \Big] \\ = {} & (p - 1 + \frac{p_j}{n_j})^2 e - \frac{1}{16}\Big(\dim_{\mathbf{R}} N(n_j)^{\mathbf{R}}_{\frac{n_j}{2}} - \dim_{\mathbf{R}} V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}} - 2\dim W(n_j)_{\frac{n_j}{2}} \Big) \\ & - \frac{1}{2} \sum_{0 < v' < n_j/2} \Big(\dim N(n_j)_{v'} - \dim V(n_j)_{v'} - \dim W(n_j)_{v'} \\ & \qquad - \dim W(n_j)_{n_j - v'} \Big) \frac{\omega(v')(n_j - \omega(v'))}{n_j^2}. \end{aligned} \tag{4.39}$$

By (4.30), $\varepsilon_2 - \varepsilon_1$ was given in [LiuMaZ, (4.49)], it is independent on the connected component of F in M'.

The proof of Lemma 4.4 is complete. □

The following Lemma was proved in [BT, Lemma 9.3] and [T, Lemma 9.6] (cf. [LiuMaZ, Lemma 4.6]).

Lemma 4.5. *Let M be a smooth manifold on which S^1 acts. Let M' be a connected component of $M(n_j)$, the fixed point set of the subgroup $\mathbf{Z}_{n_j}$ of S^1 on M. Let F be the fixed point set of the S^1-action on M. Let $V \to M$ be a real, oriented, even dimensional vector bundle to which the S^1-action on M lifts. Assume that V is Spin over M. Let $p_j \in]0, n_j[$, $p_j \in \mathbf{N}$ and $(p_j, n_j) = 1$, then*

$$\begin{aligned} &\textstyle\sum_{0<v} (\dim V_v)[\frac{p_j v}{n_j}] + \Delta(n_j, V) \quad \mathrm{mod}(2), \\ &\textstyle\sum_{0<v} (\dim V_v)[\frac{p_j v}{n_j} + \frac{1}{2}] + o(V(n_j)^{\mathbf{R}}_{n_j/2}) \quad \mathrm{mod}(2) \end{aligned} \tag{4.40}$$

are independent on the connected components of F in M'.

Recall that the number $d'(p, \beta_j, N)$ has been defined in (2.23).

Lemma 4.6. *For each connected component M' of $M(n_j)$, $d'(p, \beta_j, N) + \mu_i + \mu_4$ $\mathrm{mod}(2)$ $(i = 1,\ 2,\ 3)$ are independent on the connected component of F in M'.*

Proof. By (4.34), and Lemma 4.5, to prove Lemma 4.6, we only need to prove

$$\sum_{0<v}(\dim N_v)([\frac{p_j v}{n_j}] + (p-1)v) + \Delta(n_j, N) + \mu_4 \mod(2)$$

is independent on the connected components of F in M'. But by [BT, Lemma 9.3], as $\omega_2(TX \oplus W)_{S^1} = 0$, we know that, mod(2),

$$\sum_{0<v}(\dim N_v)[\frac{p_j v}{n_j}] + \Delta(n_j, N) + \sum_{v}(\dim W_v)[\frac{p_j v}{n_j}] \tag{4.41}$$

is independent on the connected components of F in M'. From (2.23), (2.27), (4.41), we get Lemma 4.6.

The proof of Lemma 4.6 is complete. □

4.3. Proof of Theorem 2.5. From (2.23), (4.9), (4.12) and (4.24), we see that

(4.42)

$$\begin{aligned}&\sum_{0<v}\dim N_v = \sum_{0<v<\frac{n_j}{2}}\dim N(n_j)_v + \frac{1}{2}\dim_{\mathbf{R}} N(n_j)^{\mathbf{R}}_{n_j/2} + \sum_{0<v, v=0 \mod(n_j)}\dim N_v,\\ &d'(p,\beta_j,N) = d'(p,\beta_{j-1},N) + \sum_{0<v, v=0 \mod(n_j)}\dim N_v.\end{aligned}$$

By Lemma 4.6, (4.42), $d'(p,\beta_{j-1},N) + \sum_{0<v}\dim N_v + \mu_i + \mu_4 \mod(2)$ $(i = 1, 2, 3)$ are constant functions on each connected component of $M(n_j)$.

From Lemma 4.3, one knows that the Dirac operator $D^{X(n_j)} \otimes F(\beta_j) \otimes F^i_V(\beta_j) \otimes Q_W(\beta_j) \otimes L(\beta_j)_i$ $(i = 1,\ 2)$ is well-defined on $M(n_j)$. Thus, by using Proposition 4.2, Lemma 4.4, (4.17) and (4.42), for $i = 1,\ 2$, $h \in \mathbf{Z}$, $m \in \frac{1}{2}\mathbf{Z}$, $\tau = \tau_{e1}$ or τ_{s1}, and by applying both the first and the second equations of Theorem 1.1 to each connected component of $M(n_j)$ separately, we get the following identity in $K_{G_y}(B)$,

(4.43)

$$\begin{aligned}&\textstyle\sum_\alpha(-1)^{d'(p,\beta_{j-1},N)+\sum_{0<v}\dim N_v}\mathrm{Ind}_\tau(D^{Y_\alpha}\otimes(K_W\otimes K_X^{-1})^{1/2}\otimes\mathcal{F}_{p,j-1}(X)\\ &\qquad\qquad \otimes F^i_V\otimes Q(W), m + e(p,\beta_{j-1},N), h)\\ &= \textstyle\sum_\beta(-1)^{d'(p,\beta_{j-1},N)+\sum_{0<v}\dim N_v+\mu}\mathrm{Ind}_\tau(D^{X(n_j)}\otimes(K_W\otimes K_X^{-1})^{1/2}\otimes F(\beta_j)\\ &\qquad\qquad \otimes F^i_V(\beta_j)\otimes Q_W(\beta_j)\otimes L(\beta_j)_i, m+\varepsilon_i+\varepsilon(W)+(\frac{p_j}{n_j}+(p-1))h, h)\\ &= \textstyle\sum_\alpha(-1)^{d'(p,\beta_j,N)+\sum_{0<v}\dim N_v}\mathrm{Ind}_\tau(D^{Y_\alpha}\otimes(K_W\otimes K_X^{-1})^{1/2}\otimes\mathcal{F}_{p,j}(X)\\ &\qquad\qquad \otimes F^i_V\otimes Q(W), m+e(p,\beta_j,N), h).\end{aligned}$$

Here $\sum_\beta$ means the sum over all connected components of $M(n_j)$. In (4.43), if $\tau = \tau_{s1}$, then $\mu = \mu_3 + \mu_4$; if $\tau = \tau_{e1}$, then $\mu = \mu_i + \mu_4$.

The proof of Theorem 2.5 is complete. □

REFERENCES

[AH] M. F. Atiyah and F. Hirzebruch, *Spin manifolds and groups actions*, in Essays on topology and Related Topics, Memoires dédié à Georges de Rham, A. Haefliger and R. Narasimhan, ed., Springer-Verlag, New York-Berlin, 1970, pp. 18–28.

[AS] M. F. Atiyah and I. M. Singer, *The index of elliptic operators IV*, Ann. of Math., 93 (1971), pp. 119–138.

[BGV] N. Berline, E. Getzler, and M. Vergne, *Heat Kernels and the Dirac Operator*, Grundl. Math. Wiss. 298, Springer, Berlin-Heidelberg-New York, 1992.

[BT] R. Bott and C. Taubes, *On the rigidity theorems of Witten*, J.A.M.S., 2 (1989), pp. 137–186.

[De] A. Dessai, *Rigidity theorem for Spin^c-manifolds*, Topology, 39 (2000), pp. 239–258.

[DeJ] A. Dessai and R. Jung, *On the rigidity theorem for elliptic genera*, Trans. Amer. Math. Soc., 350 (1998), pp. 4195–4220.

[Ha] A. Hattori, *Spin^c-structure and S^1-actions*, Invent. Math., 48 (1978), pp. 7–31.

[HaY] A. Hattori and T. Yosida, *Lifting compact group actions in fiber bundles*, Japan J. Math., 2 (1976), pp. 13–25.

[H] F. Hirzebruch, *Elliptic genera of level N for complex manifolds,* in Differential Geometric Methods in Theoretic Physics, Kluwer, Dordrecht, 1988, pp. 37–63.

[K] I. Krichever, *Generalized elliptic genera and Baker-Akhiezer functions*, Math. Notes, 47 (1990), pp. 132–142.

[LaM] H. B. Lawson and M. L. Michelsohn, *Spin Geometry*, Princeton Univ. Press, Princeton, 1989.

[Liu1] K. Liu, *On elliptic genera and theta-functions*, Topology, 35 (1996), pp. 617–640.

[Liu2] K. Liu, *On modular invariance and rigidity theorems*, J. Diff. Geom., 41 (1995), pp. 343–396.

[LiuMa1] K. Liu and X. Ma, *On family rigidity theorems I*, Duke Math. J., 102 (2000), pp. 451–474.

[LiuMa2] K. Liu and X. Ma, *On family rigidity theorems for Spin^c Manifolds*, Preprint, 1999.

[LiuMaZ] K. Liu, X. Ma, and W. Zhang, *Rigidity and vanishing theorems in K-theory*, math.KT/9912108.

[LiuMaZ1] K. Liu , X. Ma, and W. Zhang, *Rigidity and vanishing theorems in K-theory*, C. R. Acad. Sci. Paris, Série A, 330 (2000), pp. 301–306.

[O] K. Ono, *On a theorem of Edmonds*, in Progress in differential geometry, K. Shiohama, ed., Adv. Stud. Pure Math. 22, Kinokuniya Company Ltd., Tokyo, 1993, pp. 243–245.

[T] C. Taubes, *S^1 actions and elliptic genera*, Comm. Math. Phys., 122 (1989), pp. 455–526.

[W] E. Witten, *The index of the Dirac operator in loop space*, in SLNM 1326, Springer, Berlin, pp. 161–186.

[Z] W. Zhang, *Symplectic reduction and family quantization*, Inter. Math. Res. Notices, 19 (1999), pp. 1043–1055.