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A Mod 2 Index Theorem for Pin$^-$ Manifolds

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A mod 2 index theorem for pin$^-$ manifolds

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Abstract. We establish a mod 2 index theorem for real vector bundles over $8k+2$ dimensional compact pin$^-$ manifolds. The analytic index is the reduced η invariants of (twisted) Dirac operators and the topological index is defined through KO-theory. Our main result extends the mod 2 index theorem of Atiyah and Singer [AS] to non-orientable manifolds.

Introduction

Let B be an $8k+2$ dimensional compact pin$^-$ manifold. By this we always assume a pin$^-$ structure has been chosen on TB. Let E be a real vector bundle over B. By introducing suitable metrics and connections on TB and E, one can define a self-adjoint 'twisted' Dirac operator $\tilde{D}_{B,E}$ on B with coefficient E. The reduced η invariant [APS] of $\tilde{D}_{B,E}$, denoted by $\bar{\eta}(\tilde{D}_{B,E})$, turns out to be mod 2 independent of the metrics and connections appeared in the definition of $\tilde{D}_{B,E}$. Thus $\bar{\eta}(\tilde{D}_{B,E})$ is a mod 2 topological invariant (in fact a pin$^-$ cobordism invariant, as we will see in the main text).

The purpose of this paper is to give a purely topological formula for this analytically defined invariant. Our motivation of proving such a formula comes from the Rokhlin type congruence formulas we proved in Zhang [Z1], where pin$^-$ manifolds appear as obstructions to the existence of spin structures on oriented manifolds.

Now suppose B is orientable and carries an orientation. Then B is a spin manifold carrying a spin structure induced from the pin$^-$ structure. The reduced η invariant turns out to be the mod 2 analytic index defined by Atiyah and Singer [AS].

Recall that in this case a topological index was defined by Atiyah-Singer [AS] and an equality between the analytic and topological indices was established in [AS].

Our topological interpretation of the reduced η invariants for pin^- manifolds is inspired by Atiyah-Singer's construction. The topological index we will define will turn out to lie in $Z[\frac{1}{2}]$ (mod 2).

After making clear what should be proved, we find the result follows from an easy modification of the paper by Bismut-Zhang [BZ] where a Riemann-Roch property for reduced η invariants on odd dimensional manifolds was formulated and proved. In fact, a direct proof of the Atiyah-Singer mod 2 index theorem [AS] along the lines of [BZ] has already been worked out in Zhang [Z2]. One thing to be remarked is that while in [Z2], one need not use the local index techniques in [BZ], here for non-orientable manifolds, the full strength of the techniques in [BZ], which in turn relie on Bismut-Lebeau [BL], should be used.

Twisted Dirac operators and their reduced η invariants were first studied by Gilkey [G] for pin^c manifolds. In [St], Stolz studied the reduced η invariants on pin^+ 4-manifolds and used them to detect for example the exotic RP^4 constructed by Cappell and Shaneson [CS]. The same method here can be used to give unified topological formulas for these reduced η invariants too. The modifications are fairly easy and will not be carried out in this paper.

Our results suggest that one can use the reduced η invariants to detect pin^- cobordism classes. This will be carried out elsewhere (see [LiuZ]).

Also our definition of the topological index seems to be closely related to the KR-theory developed by Atiyah [A] and hopefully will find applications in real algebraic geometry. In fact one of the first applications of the original Rokhlin congruence [R] lies in real algebraic geometry.

This paper is organized as follows. In the first section, we recall some algebaic preliminaries which will be used in the rest of this paper. In Section 2 we define twisted Dirac operators and the associated analytic index. Section 3 contains the definition of the topological index. In Section 4 we establish an equality between the analytic and topological indices defined in Sections 2 and 3 respectively, based on a Riemann-Roch property for analytic indices. This Riemann-Roch property will be proved in Section 5. There is also an Appendix in which we prove an extended Rokhlin congruence formula not included in Zhang [Z1]. The mod 2 indices studied in the main text appear most naturally in this version of Rokhlin congruences.

1. Algebraic preliminaries

In this Section, we recall some elementary algebraic facts for the completeness of this paper. A standard reference is the paper of Atiyah, Bott and Shapiro [ABS]. One can also consult Lawson-Michelsohn's book [LM].

This section is organized as follows. In a), we recall the basic definitions of pin$^-$ groups and their representations. We pay special attention to dimensions $8k+2$ and $8k+3$ which are essential for this paper. In b), we recall the real structure of the spinor representations in dimension $8k$. This plays the basic role in our definition of the topological index in Section 3. In c), we recall a factorization formula for pin$^-$ representations.

a). Pin$^-$ groups and their representations

Let E be an n dimensional oriented Euclidean space. Let $c(E)$ be the real Clifford algebra of E. That is, $c(E)$ is spanned over R by 1, e, $e \in E$ and the commutation relations $ee' + e'e = -2 < e, e' >$. The pin$^-$ group in dimension n, pin$^-(n)$, is the multiplication group generated by $e \in E \subset c(E), \|e\| = 1$. Let χ be the representation $\chi : \text{pin}^-(n) \to O(1)$ given by

$$(1.1) \qquad \chi : e_{i_1} ... e_{i_j} \longmapsto (-1)^{i_1 + ... + i_j}, i_l \neq i_k, l \neq k.$$

Let γ : pin$^-(n) \to O(n)$ be the canonical representation defined by $\gamma(e)(\omega) = -e\omega e$ for $\omega, e \in E \subset c(E), \|e\| = 1$. Let Δ be a pin$^-(n)$ module. Then one verifies that

$$(1.2) \qquad \omega(ew) = \omega e \omega^{-1}(\omega)w = \chi(\omega)(\gamma(\omega)e)(\omega w)$$

for $\omega \in \text{pin}^-(n), e \in E, w \in \Delta$.

Thus, the Clifford action

$$(1.3) \qquad c : \; E \otimes \Delta \longrightarrow \chi \otimes \Delta$$

is pin$^-(n)$ invariant.

We now assume $n = 8k+3$. Then by Atiyah-Bott-Shapiro [ABS], $c(E) = \text{End}_H(S_+) \oplus \text{End}_H(S_-)$, $\dim S_+ = \dim S_- = 2^{4k}$. Let $e_1, ..., e_n$ be an oriented orthonormal base of E. Set $s_n = e_1 ... e_n$. Then $S_\pm$ are characterized by

s_n, acting on $S_\pm$ as $\pm Id$. We will fix S_+ as *the* irreducible module of $c(E)$ as well as $\text{pin}^-(n)$. One has $S_- = \chi \otimes S_+$, and that the Clifford action $c : E \otimes S_+ \to S_-$ is $\text{pin}^-(n)$ invariant. Also $S_\pm$ carry naturally induced metrics.

Now let G be a Euclidean space of dimension $8k+2$. Let $E = R \oplus G$. Then viewing $G \subset E$, we can view $S_+(E)$ as a $\text{pin}^-(G)$ module and have the Clifford action

$$(1.4) \qquad c: \ G \otimes S_+ \longrightarrow \chi S_+.$$

Let $e \in G^\perp \subset E$, $\|e\| = 1$. One then composes (1.4) to a pin^- invariant action

$$(1.5) \qquad c(e)c \ : G \otimes S_+ \longrightarrow S_+.$$

Without any confusion, we will note $S_+(G)$ from now on for this S_+. This is sometimes called a tangential representation. It plays a fundamental role in the construction of 'twisted' Dirac operators in Section 2.

b). Spin representations in dimension $8k$

Now let E be an $8k$ dimensional oriented Euclidean space. Then by [ABS],

$$(1.6) \qquad c(E) = \text{End}_R(F) = F \otimes F^*,$$

where $F = F_+ \oplus F_-$ is the Z_2-graded Eulidean space of E-spinors. Let $e_1, ..., e_{8k}$ be an oriented orthonormal base of E. Then $F_\pm$ are characterized by $s_{8k} = e_1 ... e_{8k}$ acts on $F_\pm$ as $\pm Id$.

The real space structure of an $8k$ dimensional Clifford algebra and the corresponding irreducible spin representations play important roles in the definition of the topological index in Section 3.

Now let E^* be the dual of E carrying the dual metric. If $e \in E$, let $e^* \in E^*$ corresponds to e by the scalar product of E. If $e \in E$, let $c(e)$, $\hat{c}(e)$ be the operators acting on $\wedge(E^*)$,

$$(1.7) \qquad c(e) = e^* \wedge - i_e,$$

$$\hat{c}(e) = e^* \wedge + i_e.$$

Recall that we have the identification of Z-graded real vector spaces

$$(1.8) \qquad c(E) \cong \wedge(E^*).$$

Let σ be $+1$ on $c^{even}(E)$, -1 on $c^{odd}(E)$. Then σ also acts in the obvious way on $\wedge(E^*)$. Under the identification (1.8), $c(e)$ is exactly the left Clifford multiplication by e and $\hat{c}(e)\sigma$ is the right Clifford action by e. Also, as F is a $c(E)$ module, (1.6) is an identification of left and right Clifford modules.

Let $\tau = \pm 1$ on $F_\pm$, $\tau^* = \pm 1$ on $F^*_\pm$. Then τ, τ^* act on $F \otimes F^*$ as $\tau \otimes 1$, $1 \otimes \tau^*$. One easily verifies that

$$\tau^* = \sigma\tau. \tag{1.9}$$

c). Decomposition of pin⁻ invariant representations

Now let G be an $8k+2$ dimensional oriented Euclidean space and E an $8l$ dimensional oriented Euclidean space. Then by using the notation as in the previous subsections one gets

$$c(G \oplus E) = c(G)\hat{\otimes}c(E), \tag{1.10}$$

where $\hat{\otimes}$ is the standard notation for super tensor product. By (1.10) one gets easily the following pin$^-$ invariant factorization of representations,

$$S_+(G \oplus E) = S_+(G)\hat{\otimes}F(E). \tag{1.11}$$

2. Twisted Dirac operators on pin$^-$ manifolds and η invariants

In this Section, we define the twisted Dirac operators on $8k+2$ dimensional pin$^-$ manifolds with coefficients in real vector bundles. The word 'twisted' reflects the fact that these operators are defined as 'tangential operators' of Dirac operators on associated $8k+3$ dimensional manifolds. Then we recall the definition of the reduced η invariants of twisted Dirac operators and show that these invariants are mod 2 topological invariants.

This Section is organized as follows. In a), we recall the definition of pin$^-$ manifolds. In b), we define the twisted Dirac operators associated to vector bundles over pin$^-$ manifolds. In c), we recall the definition of reduced η invariants and its basic properties.

a). Pin⁻ structures on vector bundles and manifolds.

A basic reference for this subsection is Kirby-Taylor [KT].

Let B be a compact manifold. Let E be an n dimensional real vector bundle over B.

Definition 2.1. If there is a $\text{pin}^{-}(n)$ principle bundle P over B and a representation $\rho : \text{pin}^{-}(n) \to \text{End}(R^n)$ such that $E = P \times_\rho R^n$, then we call E a $\text{pin}^{-}(n)$ vector bundle over E carrying a pin^{-} structure determined by (P, ρ).

The following characterization of a vector bundle to be a pin^{-} bundle can be found for example in Stolz [St].

Proposition 2.2. A real vector bundle E over a compact manifold B is a pin^{-} vector bundle if and only if it's Stiefel-Whitney classes satisfy the condition that $w_1^2(E) + w_2(E) = 0$.

Definition 2.3. If the tangent bundle TB of a compact manifold B carries a pin^{-} structure, then B is called a pin^{-} manifold. And in general by this we will mean B has been equipped with a pin^{-} structure.

b). The twisted Dirac operators on an $8k+2$ dimensional pin^{-} manifold

Let B be a compact $8k+2$ dimensional pin^{-} manifold. Then $M := (-1, 0] \times B$ is an $8k+3$ dimensional pin^{-} manifold with boundary $\partial M = \{0\} \otimes B$, and carries an induced pin^{-} structure. Note $\pi : (-1, 0] \times B \to B$ the projection map.

Let g^{TB} be a metric on TB. Let $(-1, 0]$ carry the standard metric dt^2. Let $g^{TM} = dt^2 \oplus g^{TB}$ be the product metric on M.

Recall that an $8k+3$ dimensional pin^{-} representation has been specified in Section 1 a).

Let $S_+(M)$ be the associated pinor bundle over M for (M, g^{TM}). Then $S_+(M)$ carries a canonical metric $g^{S_+(M)}$ induced from g^{TM}. Also the Levi-Civita connection ∇^{TM} of g^{TM} lifts to a Euclidean connection $\nabla^{S_+(M)}$ on $S_+(M)$.

Let E be a real vector bundle over B. Let g^E be a metric on E. Let ∇^E be a Euclidean connection on E. Then $\pi^*\nabla^E$ is a Euclidean connection on (π^*E, π^*g^E).

Let $\nabla^{S_+(M)\otimes\pi^*E}$ be the connection on $S_+(M)\otimes\pi^*E$ defined by

$$(2.1)\qquad \nabla^{S_+(M)\otimes\pi^*E}(u\otimes v)=\nabla^{S_+(M)}u\otimes v+u\otimes\pi^*\nabla^E v$$

for $u\in\Gamma(S_+(M)), v\in\Gamma(\pi^*E)$. Then $\nabla^{S_+(M)\otimes\pi^*E}$ is a Euclidean connection on $(S_+(M)\otimes\pi^*E, g^{S_+(M)}\otimes\pi^*g^E)$.

Let $o(TM)$ be the orientation bundle of TM.

Let $e_1,...,e_{8k+2}$ be an orthornomal base of TB. Then $\frac{\partial}{\partial t},\pi^*e_1,...,\pi^*e_{8k+2}$ is an orthonormal base of TM.

Definition 2.4. The Dirac operator D_{M,π^*E} is a differential operator from $\Gamma(S_+(M)\otimes\pi^*E)$ to $\Gamma(o(TM)\otimes S_+(M)\otimes\pi^*E)$ defined by

$$(2.2)\qquad D_{M,\pi^*E}=c(\frac{\partial}{\partial t})\nabla^{S_+(M)\otimes\pi^*E}_{\frac{\partial}{\partial t}}+\sum_1^{8k+2}c(\pi^*e_i)\nabla^{S_+(M)\otimes\pi^*E}_{\pi^*e_i}.$$

Clearly, D_{M,π^*E} is a first order elliptic differential operator.

Now since M is of product structure, one has $\nabla^{S_+(M)\otimes\pi^*E}_{\frac{\partial}{\partial t}}=\frac{\partial}{\partial t}$ and we can write (2.2) as

$$(2.3)\qquad D_{M,\pi^*E}=-c(\frac{\partial}{\partial t})(-\frac{\partial}{\partial t}+c(\frac{\partial}{\partial t})\sum_1^{8k+2}c(\pi^*e_i)\nabla^{S_+(M)\otimes\pi^*E}_{\pi^*e_i})$$

$$=-c(\frac{\partial}{\partial t})(-\frac{\partial}{\partial t}+\pi^*\tilde{D}_{B,E}),$$

where $\tilde{D}_{B,E}$ is a uniquely determined differential operator on $\Gamma(S_+(M)\mid_B\otimes E)$.

Notation 2.5. From now on, we will denote by $S_+(B)$ the bundle $S_+(M)\mid_B$ over B.

Definition 2.6. The operator $\tilde{D}_{B,E}$ is called the twisted Dirac operator on B with coefficient E.

One verifies easily that $\tilde{D}_{B,E}:\Gamma(S_+(B)\otimes E)\to\Gamma(S_+(B)\otimes E)$ is a self adjoint first order elliptic differential operator.

c). Reduced η invariants as analytic indices

The η and reduced η invariants were introduced by Atiyah, Patodi and Singer [APS] in their study of index theorems for manifolds with boundary. We recall the definitions in our context.

The η function of $\tilde{D}_{B,E}$ is defined by

$$(2.4) \qquad \eta(\tilde{D}_{B,E}, s) = \sum \frac{\mathrm{sign}\lambda}{|\lambda|^s}, \quad \mathrm{Re(s)} >> 0,$$

where λ runs over non-zero eigenvalues of $\tilde{D}_{B,E}$.

Standard methods showes that $\eta(\tilde{D}_{B,E}, s)$ is a holomorphic function on the half plane $\mathrm{Re}(s) >> 0$, and can be extended to a meromorphic function on the whole complex plane and is holomorphic at $s = 0$ [APS].

The value of $\eta(\tilde{D}_{B,E}, s)$ at $s = 0$ is called the η invariant of $\tilde{D}_{B,E}$ and is denoted by $\eta(\tilde{D}_{B,E})$. The reduced η invariant of $\tilde{D}_{B,E}$ is defined by [APS]

$$(2.5) \qquad \bar{\eta}(\tilde{D}_{B,E}) = \frac{1}{2}(\dim \ker \tilde{D}_{B,E} + \eta(\tilde{D}_{B,E})).$$

Now assume B bounds an $8k+3$ dimensional compact pin$^-$ manifold K, and assume E extends to a real vector bundle $\hat{E}$ over K. Let g^{TK}, $g^{\hat{E}}$ be the metrics on TK, $\hat{E}$ respectively such that they restrict to g^{TB}, g^E on the boundary and are of product structures near the boundary. Then the constructions in the previous subsection can be applied here.

Let $\nabla^{\hat{E}}$ be a Euclidean connection on $\hat{E}$ which is of product structure near the boundary.

Let $D_{K,\hat{E}}$ be the Dirac operator associated to $(K, g^{TK}, \hat{E}, g^{\hat{E}}, \nabla^{\hat{E}}, S_+(K))$, which is defined similarly as in Definition 2.4. We impose the Atiyah-Patodi-Singer boundary condition [APS] on $D_{K,\hat{E}}$. Then by the index theorem for manifolds with boundary of Atiyah, Patodi and Singer [APS], one has

$$(2.6) \qquad \mathrm{ind} D_{K,\hat{E}} = -\bar{\eta}(\tilde{D}_{B,E}).$$

The local index term disappeares in (2.6) simply because $\dim K$ is odd.

Now by Section 1a), we know that $S_+(K)$ carries a quaternionic structure. Furthermore, $D_{K,\hat{E}}$, as well as it's Atiyah-Patodi-Singer boundary condition, is easily seen to be H linear. From (2.6), one gets the following important result.

Proposition 2.7. If B bounds a compact pin$^-$ manifold K and E extends to K. Then $\bar{\eta}(\tilde{D}_{B,E})$ is an even interger.

By Proposition 2.7, we know that for any $8k+2$ dimensional compact pin$^-$ manifold B and a real vector bundle E over B. The quantity $\bar{\eta}(\tilde{D}_{B,E})$ (mod 2) is a well defined pin$^-$ cobordism invariant. In particular, it does not depend on the metrics and connections used to define it.

Definition 2.8. The quantity $\bar{\eta}(\tilde{D}_{B,E})$ (mod 2) is called the mod 2 analytic index of E and is denoted by $\mathrm{ind}^a(E)$.

It is easy to verify that ind^a provides a homomorphism $\mathrm{ind}^a : \widetilde{KO}(B) \to R$. The purpose of this paper is to give a unified topological formula for this analytically defined homomorphism.

3. The topological index for vector bundles over pin$^-$ manifolds

The purpose of this Section is to define what we call the topological index of a real vector bundle over an $8k+2$ pin$^-$ manifold.

Our construction is inspired by a construction of Atiyah and Singer [AS] (cf. also [LM]), but is different in several aspects. The reason is that we here should take more care of the situation where our manifold is no longer orientable and that the group $\widetilde{KO}(RP^l)$ satisfies no periodicity of Bott type.

This Section is organized as follows. In a), we give a geometric construction of the direct image in KO-theory for emdeddings between manifolds. In b), we discuss the pin$^-$ structures on real projective spaces. In c), we present our definition of the mod 2 topological index.

a). A geometric construction of the direct image for emdeddings

Let $i : Y \hookrightarrow X$ be an embedding of compact manifolds without boundary. Let $\pi : N \to Y$ be the normal bundle to Y in X.

We make the assumption that $\dim N \equiv 0$ (mod 8) and that N is an oriented spin vector bundle over Y, carrying a fixed spin structure.

Let E be a real vector bundle over Y. Then by the standard construction of Atiyah and Hirzebruch [AH] and of Atiyah, Bott and Shapiro [ABS] (cf.

also Lawson-Michelsohn [LM]), the direct image of E under i is a well defined element $i!E \in \widetilde{KO}(X)$.

In what follows we will give a concrete geometric realization of $i!E$. This construction is inspired by Quillen's superconnection [Q]. A similar construction for complex vector bundles has already appeared in Bismut-Zhang [BZ].

Let g^{TX} be a metric on TX. Let g^{TY} be the metric on TY determined by g^{TX}. Let g^N be the induced metric on N such that we have the orthogonal decompositions of vector bundles and metrics,

$$TX \mid_Y = TY \oplus N, \tag{3.1}$$

$$g^{TX} = g^{TY} \oplus g^N. \tag{3.2}$$

Set for any $r > 0$,

$$D_r(N) = \{n \in N \mid \|n\| \leq r\}, \ S_r(N) = \partial D_r(N). \tag{3.3}$$

Let $F(N) = F_+(N) \oplus F_-(N)$ be the bundle of N-spinors. Since $\dim N \equiv 0 \pmod 8$, $F_\pm$ are real vector bundles by Section 1b). Let $F^*(N) = F_+^*(N) \oplus F_-^*(N)$ be the dual of $F(N)$. We are mainly interested in the Z_2-graded vector bundle $F^*(N) \otimes E = F_+^*(N) \otimes E \oplus F_-^*(N) \otimes E$.

Let $n \in N$. Let $\tilde{c}(n)$ be the transpose of the Clifford action $c(n)$ on $F(N)$. Then $\tilde{c}(n)$ extends to an action on $F^*(N) \otimes E$ as $\tilde{c}(n) \otimes Id_E$ which we still note by $\tilde{c}(n)$.

Let now G be a real vector bundle over Y such that $F_-^*(N) \otimes E \oplus G$ is a trivial vector bundle over Y. The existence of G is clear.

Consider the pair of vector bundles $\pi^*(F_\pm^*(N) \otimes E \oplus G)$ over $D_r(N)$. For any $n \in D_r(N), n \neq 0$, the map

$$\tilde{c}(n) \oplus Id_{\pi^* G} : \pi^*(F_+^*(N) \otimes E \oplus G) \longrightarrow \pi^*(F_-^*(N) \otimes E \oplus G) \tag{3.4}$$

is a linear isomorphism. In other words, $\pi^*(F_-^*(N) \otimes E \oplus G)$ and $\pi^*(F_+^*(N) \otimes E \oplus G)$ are equivalent over $D_r(N) - Y$ for any $r > 0$. In particular, over $S_1(N)$, we have two trivial vector bundles $\pi^*(F_\pm^* \otimes E \oplus G)$ and an identifying map given by $v(n) = \tilde{c}(n) \oplus Id_{\pi^* G}$. This map v can be trivially extended to the whole $X - D_1(N)$ as an invertible map between two trivial vector bundles extending $\pi^*(F_\pm^*(N) \otimes E \oplus G)$ respectively.

Denote by $\xi_\pm$ the two resulting vector bundles over X. Then it is clear that $\xi_+ - \xi_-$ is a representative of the direct image $i!E \in \widetilde{KO}(X)$ constructed in [AH] and [ABS]. This is our geometric realization of the direct image. We emphasis again that we have also constructed a map $v : \xi_+ \to \xi_-$ between ξ_+ and ξ_- such that v is invertible on $X - Y$ and that when restricted to $D_1(N)$, as $\xi_\pm = \pi^*(F^*_\pm(N) \otimes E \oplus G)$, v takes the form

$$(3.6) \qquad v(n) = \tilde{c}(n) \oplus Id_{\pi^* G}$$

for $n \in D_1(N)$.

In the rest of this paper, whenever we refer to a direct image for an embedding, we will mean a geometric realization in the sense of this subsection.

b). The pin$^-$ structures on real projective spaces

Let q be a positive integer. Let RP^q be the q dimensional real projective space. Recall that the total Stiefel-Whitney class of RP^q is given by (cf. Milnor-Stasheff [MS])

$$(3.7) \qquad w(RP^q) = (1+a)^{q+1},$$

where a is the generator of $H^1(RP^q, Z_2)$.

From (3.7) one deduces that

$$(3.8) \qquad w_1^2(RP^q) + w_2(RP^q) = \frac{(q+1)(3q+2)}{2}a^2.$$

By (3.8), $w_1^2(RP^q) + w_2(RP^q) = 0$ occurs when i), q is odd and $q \equiv 3 \pmod 4$ or ii), q is even and $q \equiv 2 \pmod 4$.

In the first case we get a spin manifold while in the second case we get a non-orientable pin$^-$ manifold. We will concentrate on the non-orientable case.

On RP^{4k+2}, there are exactly two different pin$^-$ structures. We will fix one as follows.

The orientation cover S^{4k+2} of RP^{4k+2} bounds a disk D^{4k+3} with its antipodal involution. We can extend this involution on $P = D^{4k+3} \times \text{pin}^-(4k+3)$ by left multiplication with $s_{8k+3} = e_1 ... e_{8k+3}$ on the second factor, where $e_1, ..., e_{8k+3}$ is an oriented orthonormal base of R^{4k+3}. There is an obvious isomorphism between the associated bundle $P \times_\gamma R^{8k+3}$ and the tangent bundle of D^{4k+3}. This $Z/2$ equivariant pin$^-$ structure induces a pin$^-$ structure on RP^{4k+2}. In the rest of this paper, we will always assume that RP^{4k+2} has this pin$^-$ structure.

c). The definition of the topological index

Let B be an $8k+2$ dimensional compact pin$^-$ manifold with a fixed pin$^-$ structure. Let RP^{8k+2} be the real projective space with the pin$^-$ structure specified in the last subsection. Let γ be the canonical line bundle over RP^{8k+2}. That is $\gamma = o(TRP^{8k+2})$, the orientation bundle of TRP^{8k+2}.

By a classical result of Steenrod-Whitney [S], there is a classifying map $f : B \to RP^{8k+2}$, uniquely determined up to homotopy, such that

$$f^*(\gamma) = o(TB), \tag{3.9}$$

where $o(TB)$ is the orientation bundle of TB.

On the otherhand, one can find easily a sufficiently large integer m so that there is an embedding $g : B \hookrightarrow S^{8m}$.

The maps f and g together give an embedding

$$h = (f, g) : B \hookrightarrow RP^{8k+2} \times S^{8m}, \quad x \mapsto (f(x), g(x)). \tag{3.10}$$

Now as S^{8m} is an oriented spin manifold carrying the unique spin structure, $RP^{8k+2} \times S^{8m}$ is a pin$^-$ manifold carrying an induced pin$^-$ structure.

Let N be the normal bundle to B in $RP^{8k+2} \times S^{8m}$. From (3.9), one verifies that

$$\begin{aligned} w_1(N) &= h^*(w_1(RP^{8k+2} \times S^{8m})) - w_1(B) \\ &= f^*w_1(RP^{8k+2}) - w_1(B) = 0 \end{aligned} \tag{3.11}$$

and that

$$\begin{aligned} w_2(N) &= h^*w_2(RP^{8k+2} \times S^{8m}) - w_2(B) \\ &= f^*w_1^2(RP^{8k+2}) + w_2(B) \\ &= w_1^2(B) + w_2(B) = 0. \end{aligned} \tag{3.12}$$

From (3.11) and (3.12), N is a spin vector bundle over B. The orientation of TS^{8m} induces an orientation on N. Furthermore the pin$^-$ structures on B and $RP^{8k+2} \times S^{8m}$ determine a spin structure on N.

Thus the constructions in Section 3a) applies here.

Let E be a real vector bundle over B, we then get an element $i!E \in \widetilde{KO}(RP^{8k+2} \times S^{8m})$.

Now let $i_{8k+2} : RP^{8k+2} \hookrightarrow RP^{8k+2}$ be the identity map. Let p be a point in S^{8m} and let $i_p : RP^{8k+2} \to S^{8m}$ be the collapsing map with image p. Then $i_{8k+2,m} = (i_{8k+2}, i_p) : RP^{8k+2} \hookrightarrow RP^{8k+2} \times S^{8m}$ is an embedding. Furthermore, $i_{8k+2,m}$ determines a map

$$(3.13) \qquad i_{8k+2,m}! : KO(RP^{8k+2}) \to \widetilde{KO}(RP^{8k+2} \times S^{8m}).$$

By the Bott periodicity theorem (cf. [LM]), $i_{8k+2,m}!$ is an isomorphism not depending on the choice of the point p.

Notation 3.1. If $a, b, c, ... \in R$ is a series of real numbers, then we denote by $Z\{a, b, c, ...\}$ the abelian subgroup of R generated by the numbers $a, b, c, ...$

The following result on the structure of $\widetilde{KO}(RP^{8k+2})$ is essential to our definition of the topological index. It replaces the Bott periodicity theorem in the spin case.

Lemma 3.2 (Adams [Ad, Theorem 7.4], Atiyah-Bott-Shapiro [ABS]). The group $\widetilde{KO}(RP^{8k+2})$ is an abelian group of order 2^{4k+2} generated by $1 - \gamma$.

By this Lemma, any element α of $KO(RP^{8k+2})$ can be represented as

$$(3.14) \qquad \alpha = m_\alpha + n_\alpha(1 - \gamma),\ m_\alpha, n_\alpha \in Z,\ 0 \leq n_\alpha < 2^{4k+2} - 1.$$

Let $q_{8k+2} : KO(RP^{8k+2}) \to Z\{\frac{1}{2^{4k+2}}\}/2Z$ be the homomorphism defined by

$$(3.15) \qquad q_{8k+2}(\alpha) = \frac{m_\alpha}{2^{4k+2}} + \frac{n_\alpha}{2^{4k+1}} \pmod 2.$$

We can now give our definition of the topological index.

Definition 3.3. Let E be a real vector bundle over an $8k+2$ dimensional compact pin$^-$ manifold B. The topological index of E, denoted by $\mathrm{ind}^t(E)$, is an element in $Z\{\frac{1}{2^{4k+2}}\}/2Z$ given by $q_{8k+2}(i_{8k+2,m}!)^{-1}h!(E)$.

Remark 3.4. Using the standard methods in KO-theory and the fact that induced vector bundles are independent of the homotopy type of the induced maps, one sees easily that $\text{ind}^t(E)$ does not depend on the integer m, the classifying map f and the embedding g appeared in the process of it's definition. Thus $\text{ind}^t(E)$ is a well defined object.

Remark 3.5. Similarly, one can also define the topological index for real vector bundles over an $8k+6$ dimensional compact pin$^-$ manifold. In this case the value of the index will lie in $Z\{\frac{1}{2^{4k+4}}\}/Z$.

4. A mod 2 index theorem for $8k+2$ dimensional pin$^-$ manifolds

In this Section, we establish a mod 2 index theorem for real vector bundles over $8k+2$ dimensional compact pin$^-$ manifolds, which is the main concern of this paper. The proof relies on a Riemann-Roch formula for the analytic index which we state in Theorem 4.1. The proof of this Riemann-Roch property will be carried out in Section 5.

This Section is organized as follows. In a), we state the Riemann-Roch formula to be proved in Section 5. In b), we will state and prove the mod 2 index theorem.

a). A Riemann-Roch formula for the analytic index

Let X, Y be two compact pin$^-$ manifolds of dimensions $8m+2$, $8n+2$ respectively, such that there is an embedding $i : Y \hookrightarrow X$.

We make the assumption that

$$i^*w_1(TX) = w_1(TY). \tag{4.1}$$

Let N be the normal bundle to Y in X. By (4.1), N is an orientable spin vector bundle of dimension $8(m-n)$. We fix an orientation on N. Then N carries a spin structure canonically induced from the given pin$^-$ structures on TX and TY.

We can then apply the direct image construction in Section 3a).

The following Riemann-Roch type formula is essential.

Theorem 4.1. Let E be a real vector bundle over Y. Then the following identity holds,

$$\text{ind}^a(E) = \text{ind}^a(i!E). \tag{4.2}$$

Proof. The proof of (4.2) will be given in Section 5. □

b). An equality between the analytic and topological indices
We now use the notation as in Section 3c).

Again, B is a compact $8k+2$ dimensional pin$^-$ manifold and E is a real vector bundle over B.

Recall that we have constructed various direct images in the course of the definition of the topological index.

By using Theorem 4.1 two times, we have the following equality between analytic indices,

$$(4.3) \qquad \operatorname{ind}^a(E) = \operatorname{ind}^a((i_{8k+2,m}!)^{-1}h!E).$$

So in order to prove an equality between the analytic and topological indices, we need only to check it on real projective spaces.

Lemma 4.2. Let α be a real vector bundle over the pin$^-$ manifold RP^{8k+2}. Then the following identity holds,

$$(4.4) \qquad \bar{\eta}(\tilde{D}_{RP^{8k+2},\alpha}) \equiv q_{8k+2}(\alpha) \pmod 2.$$

Proof. By Lemma 3.2, we need only to check (4.4) for $\alpha = 1$ and $\alpha = \gamma$. This can be done in exactly the same way as in Gilkey [G] and Stolz [St]. All what one need to do is the trivial modification of [St, Corollary 5.4] by relacing 'pin$^+$' there by 'pin$^-$', and by replacing 'RP^{8k+4}' there by 'RP^{8k+2}'.
We leave the details to the interested reader. □
We can now state the main result of this paper.

Theorem 4.3. The following identity holds for a real vector bundle E over a compact $8k+2$ dimensional pin$^-$ manifold,

$$(4.5) \qquad \operatorname{ind}^a(E) = \operatorname{ind}^t(E).$$

Proof. By (4.3) and Lemma 4.2, we need finally check that for any real vector bundle α over RP^{8k+2}, we have

$$(4.6) \qquad q_{8k+2}(\alpha) = \operatorname{ind}^t(\alpha).$$

In fact, if we take B in Section 3a) to be the RP^{8k+2} and the classifying map to be the identity, then by deforming the embedding $g : RP^{8k+2} \hookrightarrow S^{8m}$

to the constant map g_p, $h = (f, g)$ deformes to an embedding $h_p = (f, g_p) : RP^{8k+2} \hookrightarrow RP^{8k+2} \times S^{8m}$. In particular, the deformed maps in this process from h to h_p remain to be embeddings. Thus, we have

$$(4.7) \qquad h!(\alpha) = h_p!(\alpha) = i_{8k+2,m}!(\alpha).$$

The proof of Theorem 4.3 is completed. □

Remark 4.4. By Theorem 4.3, the definition of the topological index does not depend on the specific choice of the pin$^-$ structure on RP^{8k+2}. In fact, this can be verified directly from the definition in Section 3c).

The following consequence is of independent interest.

Corollary 4.5. For any real vector bundle B over a compact $8k+2$ dimensional pin$^-$ manifold B, if $\tilde{D}_{B,E}$ is a twisted Dirac operator defined by using suitable metrics and connections on B, E respectively, then one has $\bar{\eta}(\tilde{D}_{B,E}) \in Z\{\frac{1}{2^{4k+2}}\}$.

This extends a result of Gilkey [G].

Remark 4.6. There is an analogues mod Z index theorem for real vector bundles over $8k+6$ dimensional pin$^-$ manifolds. Details are easy to carry out and are left to the interested reader.

5. A Riemann-Roch formula for pin$^-$ manifolds

The purpose of this Section is to prove Theorem 4.1. Recall that a similar result for η invariants on odd dimensional spin manifolds has already been proved in Bismut-Zhang [BZ]. All what we need to do is to modify the argument in [BZ] to fit our specific situation here.

We recall that the techniques in [BZ] depend heavily on the paper of Bismut and Lebeau [BL].

This Section is organized as follows. In a), we restate Theorem 4.1 in terms of reduced η invariants. In b), we employ some simplifying assumptions on certain metrics and connections. In c), we state six technical results. The

Riemann-Roch property is proved in d), based on the intermediary results in c). These intermediary results are then proved in e).

a). A Riemann-Roch formula for reduced η invariants

Let $i : Y \hookrightarrow X$ be an embedding of a pair of compact pin$^-$ manifolds of dimensions $8m + 2$ and $8n + 2$ respectively.

As in Section 4a), we make the assumption that

$$(5.1) \qquad i^* w_1(TX) = w_1(TY).$$

Let $\pi : N \to Y$ be the normal bundle to Y in X. From (5.1), one sees easily that N is an orientable spin bundle over Y (compare with (3.11) and (3.12)). We fix an orientation on N. Then the pin$^-$ structures on TX and TY determine a spin structure on N. And we can apply the direct image construction of Section 3a) to real vector bundles over Y.

We will use the notation of Section 3a).

Let E be a real vector bundle over Y.

Recall that in Section 3a), starting with a metric on g^{TX}, we constructed two real vector bundles $\xi_\pm$ of a same dimension on X such that $\xi_+ - \xi_- = i!E \in \widetilde{KO}(X)$.

Let g^{TY} be the restriction of g^{TX} on TY.

We introduce metrics and Euclidean connections on E and $\xi_\pm$ respectively.

The main result of this Section can be stated as follows for the twisted Dirac operators constructed as in Section 2.

Theorem 5.1. The following identity holds,

$$(5.2) \qquad \bar{\eta}(\tilde{D}_{X,\xi_+}) - \bar{\eta}(\tilde{D}_{X,\xi_-}) \equiv \bar{\eta}(\tilde{D}_{Y,E}) \pmod 2.$$

b). Some geometric simplifying assumptions

Recall that by Proposition 2.7, the reduced η invariants in (5.2) do not depend on the metrics and connections used to define the twisted Dirac operators. So in order to prove Theorem 5.1, we can and we will make these metrics and connections as simple as possible.

First of all, we assume that the embedding $i : (Y, g^{TY}) \hookrightarrow (X, g^{TX})$ is totally geodesic.

Let g^N be the metric on N so that we have the orthogonal decompositions of vector bundles and metrics

$$(5.3) \qquad TX|_Y = TY \oplus N,$$

$$g^{TX}|_Y = g^{TY} \oplus g^N.$$

Then g^N lifts to metrics $g^{F_\pm(N)}$, $g^{F_\pm^*(N)}$ on $F_\pm(N)$, $F_\pm^*(N)$ respectively.

Let P^{TY}, P^N be the orthogonal projection maps from TX to TY and N respectively.

Let ∇^{TX}, ∇^{TY} be the Levi-Civita connections associated to g^{TX}, g^{TY} respectively. Then

$$(5.4) \qquad \nabla^{TY} = P^{TY}\nabla^{TX|_Y}P^{TY}.$$

Let ∇^N be the connection defined by

$$(5.5) \qquad \nabla^N = P^N\nabla^{TX|_Y}P^N.$$

Then ∇^N is a Euclidean connection on N. It lifts to Euclidean connections $\nabla^{F_\pm(N)}$ and $\nabla^{F_\pm^*(N)}$ on $F_\pm(N)$ and $F_\pm^*(N)$ accordingly.

Let g^G be a metric on the vector bundle appeared in the construction of $\xi_\pm$ in Section 3a). Let ∇^G be a Euclidean connection on G. Then $\xi_\pm|_Y = F_\pm^*(N)\otimes E\oplus G$ carry the metrics $g^{F_\pm^*(N)\otimes g^E}\oplus g^G$ such that $F_\pm^*(N)\otimes E$ and G are orthogonal to each other, and corresponding connections $\nabla^{F_\pm^*(N)\otimes E}\oplus\nabla^G$. We can and we do lift these metrics and connections to $\pi^*(F_\pm^*(N)\otimes E\oplus G)|_{D_1(N)}$ and then extend them to metrics $g^{\xi_\pm}$ and Euclidean connections $\nabla^{\xi_\pm}$ on $\xi_\pm$.

Let $\xi = \xi_+\oplus\xi_-$ be the Z_2-graded vector space. Let $g^\xi = g^{\xi_+}\oplus g^{\xi_-}$ be the metric on ξ so that ξ_+ and ξ_- are orthogonal to each other. Let $\nabla^\xi = \nabla^{\xi_+}\oplus\nabla^{\xi_-}$ be the corresponding Euclidean connection on ξ.

Then on $D_1(N)$, one has

$$(5.6) \qquad \xi = \pi^*(F_+^*(N)\otimes E\oplus G)\oplus\pi^*(F_-^*(N)\otimes E\oplus G),$$

$$g^\xi = \pi^*(g^{F_+^*(N)}\otimes g^E\oplus g^G)\oplus\pi^*(g^{F_-^*(N)}\otimes g^E\oplus g^G),$$

$$\nabla^\xi = \pi^*(\nabla^{F_+^*(N)\otimes E}\oplus\nabla^G)\oplus\pi^*(\nabla^{F_-^*(N)\otimes E}\oplus\nabla^G).$$

Furthermore, by Section 3a), there is a map $v : \xi_+ \to \xi_-$, which is invertible on $X - Y$, and, when restricted to $D_1(N)$, takes the form

$$v(n) = \tilde{c}(n) \oplus Id_{\pi^* G}, \quad n \in D_1(N), \tag{5.7}$$

where $\tilde{c}(n)$ is the Clifford action of n on $\pi^*(F_+^*(N) \otimes E)$. The Clifford action also acts on $\pi^*(F_-^*(N) \otimes E)$.

Let τ^N be the action on $F(N)$ such that $\tau^N |_{F_\pm(N)} = \pm Id_{F_\pm(N)}$. Let τ^{N*} be the transpose of τ^N. Then τ^{N*} extends to an action on $F^*(N) \otimes E$ as $\tau^{N*} \otimes Id_E$ which we still note by τ^{N*}.

Let v^* be the adjoint of v with respect to g^ξ. Set $V = v + v^*$. Then V is an invertible element in $\mathrm{End}^{odd}(\xi)$, and one has on $D_1(N)$ that

$$V(n) = \tau^{N*} \tilde{c}(n) + Id_{\pi^* G}. \tag{5.8}$$

Remark 5.2. All the simplifying conditions in Bismut-Zhang [BZ, Section 1c) and 2a)] for direct images of complex vector bundles have now real analogues. Compare also with Zhang [Z2, (1.1)].

c). Six intermediary results

We use the same assumptions and notation as in the previous two subsections. In particular, we assume that the simplifying conditions made in the last subsection hold.

Let $\tilde{D}_{X,\xi}$ be the twisted Dirac operator defined by

$$\tilde{D}_{X,\xi} = \tilde{D}_{X,\xi_+} - \tilde{D}_{X,\xi_-}. \tag{5.9}$$

Then

$$\bar{\eta}(\tilde{D}_{X,\xi}) = \bar{\eta}(\tilde{D}_{X,\xi_+}) - \bar{\eta}(\tilde{D}_{X,\xi_-}) + \dim \ker \tilde{D}_{X,\xi_-}. \tag{5.10}$$

Now since $S_+(X)$ has a quaternionic structure, $\dim \ker \tilde{D}_{X,\xi_-}$ is an even integer. From (5.10), one gets

$$\bar{\eta}(\tilde{D}_{X,\xi}) \equiv \bar{\eta}(\tilde{D}_{X,\xi_+}) - \bar{\eta}(\tilde{D}_{X,\xi_-}), \quad (\mathrm{mod}\ 2). \tag{5.11}$$

Let $\tilde{V}$ be the operator acting on $\Gamma(S_+(X) \otimes \xi)$ defined by

$$\tilde{V} : \alpha \otimes \beta \mapsto \alpha \otimes V\beta \tag{5.12}$$

for $\alpha \in \Gamma(S_+(X)), \beta \in \Gamma(\xi)$.

Then $\tilde{V}$ is a pin$^-$ invariant selfadjoint element in $\mathrm{End}^{odd}(S_+(X) \otimes \xi)$.

For any $T \geq 0$, set

$$\tilde{D}_{X,\xi,T} = \tilde{D}_{X,\xi} + T\tilde{V}. \tag{5.13}$$

For $a > 0, T \geq 0$, let K_T^a be the direct sum of the eigenspaces of the operator $\tilde{D}_{X,\xi,T}$ which are associated to eigenvalues whose absolute value is strictly smaller than a. Let P_T^a be the orthogonal projection operator from $\Gamma(S_+(X) \otimes \xi)$ on K_T^a. Set $P_T^{a,\perp} = 1 - P_T^a$.

The following results, which are similar to those of Bismut-Zhang [BZ, Theorems 3.7-3.12] for odd dimensional manifolds and complex bundles case, will play essential roles in the proof of Theorem 5.1 in the next subsection. The proof of these results will be given in Section 5e).

Theorem 5.3. For any $\alpha_0 > 0$, there exists $C > 0$ such that for $\alpha \geq \alpha_0$, $T \geq 1$,

$$| \operatorname{Tr}[\tilde{D}_{X,\xi,T} \exp(-\alpha(\tilde{D}_{X,\xi,T})^2)] - \operatorname{Tr}[\tilde{D}_{Y,E} \exp(-\alpha(\tilde{D}_{Y,E})^2)] | \leq \frac{C}{T^{1/2}}. \tag{5.14}$$

Theorem 5.4. For any $a > 0$, there exist $c > 0$, $C > 0$ such that for $\alpha \geq 1$, $T \geq 1$, then,

$$| \operatorname{Tr}[P_T^{a,\perp} \tilde{D}_{X,\xi,T} \exp(-\alpha(\tilde{D}_{X,\xi,T})^2)] | \leq c \exp(-C\alpha). \tag{5.15}$$

Take now $a_0 > 0$ such that the operator $\tilde{D}_{Y,E}$ has no nonzero eigenvalues in the interval $[-2a_0, 2a_0]$.

Theorem 5.5. For T large enough, then

$$\dim K_T^{a_0} = \dim \ker(\tilde{D}_{Y,E}). \tag{5.16}$$

Moreover,

$$\lim_{T \to \infty} \operatorname{Tr}[| \tilde{D}_{X,\xi,T} | P_T^{a_0}] = 0. \tag{5.17}$$

Theorem 5.6. There exist $c > 0,\ \gamma \in]0,1]$ such that for $u \in]0,1],\ 0 \leq T \leq 1/u$, then

$$(5.18)\qquad |\sup(T,1)\mathrm{Tr}[\tilde{V}\exp(-(u\tilde{D}_{X,\xi}+T\tilde{V})^2)]| \leq c(u(1+T))^{\gamma}.$$

Theorem 5.7. For any $T > 0$, the following identity holds,

$$(5.19)\qquad \lim_{u\to 0}\mathrm{Tr}[\frac{T}{u}\tilde{V}\exp(-(u\tilde{D}_{X,\xi}+\frac{T}{u}\tilde{V})^2)] = 0.$$

Theorem 5.8. There exist $c > 0,\ \delta \in]0,1]$ such that for $u \in]0,1],\ T \geq 1$, then

$$(5.20)\qquad |\mathrm{Tr}[\frac{T}{u}\tilde{V}\exp(-(u\tilde{D}_{X,\xi}+\frac{T}{u}\tilde{V})^2)] \leq \frac{c}{T^{\delta}}.$$

d). Proof of Theorem 5.1

We construct a closed one form on $R_+^* \times R_+$ and then use it to prove Theorem 5.1, in exactly the same way as in Bismut-Zhang [BZ].

Theorem 5.9. Let $u > 0,\ T \geq 0$. Let $\beta_{u,T}$ be the 1-form on $R_+^* \times R_+$,

$$(5.21)\qquad \beta_{u,T} = du\mathrm{Tr}[\tilde{D}_{X,\xi,T}\exp(-u^2\tilde{D}^2_{X,\xi,T})] + dT\mathrm{Tr}[u\tilde{V}\exp(-u^2\tilde{D}^2_{X,\xi,T})].$$

Then the 1-form $\beta_{u,T}$ is closed.

Proof. Theorem 5.9 can be proved in exactly the same way as in [BZ, Theorem 3.4]. □

Proof of Theorem 5.1. Fix constants $\epsilon,\ A,\ T_0$ such that $0 < \epsilon < 1 \leq A < +\infty,\ 0 \leq T_0 < +\infty$. Let Γ_{ϵ,A,T_0} be the oriented contour in $R_+^* \times R_+$ as constructed in [BZ], consisting of four oriented pieces,

$$\Gamma_1 : T = T_0;\ \epsilon \leq u \leq A,$$

$$\Gamma_2 : 0 \leq T \leq T_0 :\ u = A,$$

$$\Gamma_3 : T = 0;\ \epsilon \leq u \leq A,$$

$$\Gamma_4 : 0 \leq T \leq T_0;\ u = \epsilon,$$

with the counterclockwise orientation.

For $1 \leq k \leq 4$, set

$$I_k = \int_{\Gamma_k} \beta_{u,T}.$$

Then by Theorem 5.9, one gets the identity,

$$\sum_{k=1}^{4} I_k = 0. \tag{5.22}$$

Theorem 5.1 then follows by making in (5.22), $A \to +\infty$, $T_0 \to +\infty$, and $\epsilon \to 0$ in this order, and proceeding in exactly the same strategy as in Bismut-Zhang [BZ, Section 3e)-g)]. All one need to notify is the following two points,

i). We use the intermediary results Theorems 5.3-5.8 here, instead of those in [BZ, Section 3d)];

ii). Since $S_+(X)$ and $S_+(Y)$ are H linear spaces and the twisted Dirac operators $\tilde{D}_{X,\xi_\pm}$, $\tilde{D}_{Y,E}$ as well as the map $\tilde{V}$ are H linear, all the mod Z terms in [BZ, Section 3e)-g)] can and will be replaced by mod $2Z$.

By noting these two points and by proceeding in exactly the same way as in [BZ, Section 3e)-g)], one gets Theorem 5.1. □

e). Proof of Theorems 5.3-5.8

The methods of Bismut-Zhang [BZ, Section 4], which goes back to Bismut-Lebeau [BL], can be adapted here with little change to prove Theorems 5.3-5.8. All one need to take care is the following two points,

i). We should modify the harmonic oscillator construction in [BZ, Section 4a)] for complex spinor spaces in order to fit the real situatuion here;

ii). Since we are now in the even dimensional situation, the local index techniques in [BZ] should be modified. But the even dimensional case turns out to be much simpler here, and does not cause any extra difficulty than [BZ]. Details are faily easy to fill and are left to the reader.

So we now concentrate on the modification of the harmonic oscillator construction, which is also easy. It is included here only for completeness.

We use the notation of Section 1b).

Set $m = \dim E \equiv 0 \pmod 8$. Let $e_1, ..., e_m$ be an orthonormal base of E and let $e_1^*, ..., e_m^*$ be the dual base of E^*.

Let $\Gamma(\wedge(E^*))$ be the vector space of smooth sections of $\wedge(E^*)$ over E. Let D^E be the operator acting on $\Gamma(\wedge(E^*))$ defined by

$$(5.23) \qquad D^E = \sum_1^m (c(e_i) \otimes \tau^*)\nabla_{e_i}.$$

Let Z be the generic point of E. Then $\tau^*\tilde{c}(Z)$ acts on $\Gamma(\wedge(E^*))$.
Set

$$(5.24) \qquad S = \sum_1^m c(e_i)\tilde{c}(e_i).$$

Proposition 5.10. The following identity holds,

$$(5.25) \qquad (D^E + \tau^*\tilde{c}(Z))^2 = -\Delta + |Z|^2 + S.$$

Proof. The proof of (5.25) is trivial. □

We now give another expression for S. Let N be the number operator of $\wedge(E^*)$, i.e., N acts on $\wedge^p(E^*)$ by multilication by p.

Proposition 5.11. The following identity holds,

$$(5.26) \qquad S = (2N - m)\sigma.$$

Proof. By the considerations in Section 1b), we know that

$$(5.27) \qquad \tilde{c}(e_i) = \hat{c}(e_i)\sigma.$$

Also one verifies easily that

$$(5.28) \qquad \sum_1^m c(e_i)\hat{c}(e_i) = 2N - m.$$

(5.26) follows from (5.27), (5.28) and (5.24). □

Proposition 5.12. The lowest eigenvalue of the operator S is $-m$. The correponding eigenspace is one dimensional and is spanned by 1.
Proof. For any p and $\alpha \in \wedge^p(E^*)$, one has

$$(5.29) \qquad (2N - m)\sigma\alpha = (-1)^p(2p - m)\alpha.$$

Now for any $0 \leq p \leq m$, one has $(-1)^p(2p-m) \geq -m$ with equality holds only for $p = 0$ (for m is even).

Proposition 5.12 follows immediately. □

From (5.25), the operator $(D^E + \tau^* \tilde{c}(Z))^2$ is of harmonic oscillator type. Therefore it has discret spectrum and compact resolvent.

We now have the following analogue of [BZ, Theorem 4.5].

Theorem 5.13. The kernel of $(D^E + \tau^* \tilde{c}(Z))^2$ is one dimensional and is spanned by

$$\beta = \exp(-\frac{|Z|^2}{2}). \tag{5.30}$$

Also

$$(D^E + \tau^* \tilde{c}(Z))\beta = 0. \tag{5.31}$$

Proof. The kernel of the operator $-\Delta + |Z|^2 - m$ is spanned by $\exp(-\frac{|Z|^2}{2})$. The first part of the theorem follows from Propositions 5.10 and 5.12.

Equation (5.31) is a consequence of the fact that $\beta \in \ker(D^E + \tau^* \tilde{c}(Z))^2$. One can also check it directly. □

We now come back to the proof of Theorems 5.3-5.8. Note that near the embedded manifold Y, we have the following pin$^-$ equivariant factorizations via (1.10),

$$c(TX)|_Y = c(TY) \hat{\otimes} c(N), \tag{5.32}$$

and

$$S_+(X)|_Y = S_+(Y) \hat{\otimes} F(N). \tag{5.33}$$

The proof of Theorem 5.3-5.8 can then be proceeded with little change as in [BZ, Section 4b)-e)].

As we have remarked, the difference in the local index calculation causes no difficulty and is even simpler here.

The other details are easy to fill and we will not make a line by line copy of [BZ, Section 4b)-e)]. □

Appendix. An extended Rokhlin congruence formula

In this Appendix, we prove a new Rokhlin type congruence not included in Zhang [Z1]. As mentioned in the Introduction, the mod 2 indices studied in the main text appear most naturally in this version of congruences.

Let K be an $8k+4$ dimensional compact oriented manifold such that $w_2(TK) \neq 0$. Let B be a compact connected codimension two submanifold of K such that $[B] \in H_{8k+2}(K, Z_2)$ is the Poincaré dual of $w_2(TK) \in H^2(K, Z_2)$. We assume the existence of such a submanifold and consider the case where B is non-orientabe.

We fix a spin structure on $K - B$. Then B carries a canonically induced pin$^-$ structure (cf. Kirby-Taylor [KT]).

Remark A.1. The case where B is orientable has already been considered in Zhang [Z3].

Let E be a real vector bundle over K. Let E_C be the complexification of E.

Let N be the normal bundle to B in K. Let $e \in H^2(TB, o(N)) = H^2(TB, o(TB))$ be the Euler class of N.

Denote by $i : B \hookrightarrow K$ the embedding of B in K.

Let $\mathrm{ind}^t(i^*E)$ be the mod 2 topological index of the real vector bundle i^*E over B.

The main result of this Appendix can be stated as follows.

Theorem A.2. The following identity holds,

$$(A.1) \qquad < \hat{A}(TK)\mathrm{ch}(E_C), [K] > \equiv \mathrm{ind}^t(i^*E)$$

$$- < \hat{A}(TB)\frac{1}{2}\tanh(\frac{e}{4})\mathrm{ch}(i^*E_C), [B] > \quad (\mathrm{mod}\ 2).$$

Remark A.3. Since $\tanh(\frac{e}{4})$ is an odd function in e, the characteristic number in the right hand side of (A.1) is well defined.

The proof of Theorem A.2 is almost the same as in Zhang [Z1] with minor modifications. So we will only give a sketch.

Proof of Theorem A.2 Let g^{TB} be a metric on TB. Let g^N be a metric on N. Let $\pi : N \to B$ be the projection map of the normal bundle.

Set $N_1 = \{n \in N \mid \|n\|_{g^N} \leq 1\}$, $M = \partial N_1$. Then N_1 is a disc bundle over B with fibre D, M is a circle bundle over B with fibre S^1. The metric g^N restricts to each fibre S^1 a metric g^{S^1}.

One constructs easily a metric g^{TD} on TD and a series of metrics $g^{TK,\epsilon}$, $\epsilon > 0$ on TK such that i). $g^{TK,\epsilon}$ is product near M, ii). $g^{TK,\epsilon} |_M = \frac{1}{\epsilon}\pi^* g^{TB} \oplus g^{TS^1}$ and iii). $g^{TK,\epsilon} |_{N_1} = \frac{1}{\epsilon}\pi^* g^{TB} \oplus g^{TD}$. Note $R^{K,\epsilon}$ the curvature of the Levi-Civita connection of $g^{TK,\epsilon}$.

Let g^{i^*E} be a metric on i^*E. Let ∇^{i^*E} be a Euclidean connection on i^*E. g^{i^*E} and ∇^{i^*E} extends to i^*E_C accordingly.

We can then construct a metric g^E and a Euclidean connection ∇^E on E such that i). g^E and ∇^E are of product structure near M, ii). $g^E |_{N_1} = \pi^* g^{i^*E}$ and iii). $\nabla^E |_{N_1} = \pi^* \nabla^{i^*E}$.

Let $D_{M,\pi^*i^*E,\epsilon}$ be the Dirac operator associated to (M, g^{TM}, π^*i^*E). Then, as in [Z1, Lemma 3.3], one can apply the Atiyah-Patodi-Singer index theorem for manifolds with boundary [APS] to $K - N_1$ to get the following formula,

$$(A.3) \qquad < \hat{A}(TK)\mathrm{ch}(E_C), [K] > \equiv \bar{\eta}(D_{M,\pi^*i^*E,\epsilon})$$
$$+ \left(\frac{1}{2\pi}\right)^{\frac{\dim K}{2}} \int_{N_1} \hat{A}(R^{K,\epsilon})\mathrm{ch}(E_C, \nabla^{E_C}) \pmod 2.$$

Now since the coupled connection on π^*i^*E does not depend on ϵ, the formula of [Z1, (3.6)] will take the following form in our situation here,

$$(A.4) \qquad \lim_{\epsilon \to 0} \bar{\eta}(D_{M,\pi^*i^*E,\epsilon}) \equiv \bar{\eta}(\tilde{D}_{B,i^*E})$$
$$+ < \hat{A}(TB)\mathrm{ch}(i^*E_C)\frac{\tanh(\frac{e}{2}) - \frac{e}{2}}{e\tanh(\frac{e}{2})}, [B] > \pmod 2.$$

And the analogue of [Z1, (3.7)] turns out to be

$$(A.5) \quad \lim_{\epsilon \to 0} \left(\frac{1}{2\pi}\right)^{\frac{\dim N_1}{2}} \int_{N_1} \hat{A}(R^{K,\epsilon})\mathrm{ch}(E_C)$$
$$= < \hat{A}(TB)\frac{1}{e}\left(\frac{\frac{e}{2}}{\sinh \frac{e}{2}} - 1\right)\mathrm{ch}(i^*E_C), [B] > .$$

(A.1) now follows from (A.3)-(A.5) and our index theorem Theorem 4.3.

The proof of Theorem A.2 is completed. □

Remark A.4. While in general the mod 2 topological index is difficult to calculate, Theorem A.2 and [Z1, Theorem 3.2] show that they can be computed through characteristic numbers in some cases.

Remark A.5. Comparing with Zhang [Z3], where we treated the case of orientable B, we are not satisfied that in [Z3], a formula corresponding to (A.2) here implies the formula corresponding to [Z1, Theorem 3.2]. While here for the case where B is non-orientable, we don't see such an implication, at least at this moment.

Remark A.6. As we now have formulated the Rokhlin type congruence in a pure topological way, we would like to see a proof of (A.2) similar to what we have done in [Z3] for the case where B is orientable.

Now set $E = TK$ in (A.2). One gets

$$(A.6) \qquad < \hat{A}(TK)\mathrm{ch}(T_C K), [TK] >= \mathrm{ind}^t(TB \oplus N)$$
$$- < \hat{A}(TB)\frac{1}{2}\tanh(\frac{e}{4})\mathrm{ch}(T_C B \oplus N_C), [B] > .$$

On the otherhand, by [Z1, Theorem 3.2] and the index theorem Theorem 4.3, one has

$$(A.7) \qquad < \hat{A}(TK)\mathrm{ch}(T_C K), [K] > \equiv \mathrm{ind}^t(TB \oplus R \oplus o(TB))$$
$$- < \hat{A}(TB)\frac{1}{2}\tanh(\frac{e}{4})\mathrm{ch}(T_C B), [B] >$$
$$+ < \hat{A}(TB)\frac{\mathrm{ch}(N_C) - 2\cosh\frac{e}{2}}{e\sinh(\frac{e}{2})}, [B] > \pmod 2.$$

The following result is a direct consequence of (A.7) and (A.6).

Theorem A.7. The following identity holds,

$$(A.8) \quad \mathrm{ind}^t(N) - \mathrm{ind}^t(R \oplus o(TB)) =< \hat{A}(TB)\sinh(e), [B] > \pmod 2.$$

Corollary A.8. For any two dimensional vector bundle N over an $8k+2$ dimendional compact pin$^-$ manifold B verifying that $w_1(N) = w_1(TB)$, the following identity holds,

$$(A.9) \qquad 2\mathrm{ind}^t(N) \equiv 2\mathrm{ind}^t(R \oplus o(TB)) \pmod{Z}.$$

Proof. Let $\tilde{B}$ be the orientation cover of B. Then N lifts to an orientable vector bundle $\tilde{N}$ over $\tilde{B}$ with Euler class $\tilde{e}$. Now $\tilde{B}$ is a spin manifold, so by the classical theorem of Atiyah and Hirzebruch [AH], the number $< \hat{A}(T\tilde{B}) \sinh(\tilde{e}), [\tilde{B}] >= 2 < \hat{A}(TB) \sinh(e), [B] >$ is an integer. (A.9) follows from this fact and (A.8). □

Remark A.9. A formula of form (A.8) for the case where B is a spin has been proved in [Z3, Corollary 13] before. This later result has been extended by Tianjun Li [Li] to the case where N can be any complex vector bundle over an $8k+2$ dimensional compact spin manifold. It would be interesting to formulate and prove an analogues generalization for (A.8) here.

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