Clifford Asymptotics and the Local Lefschetz Index

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In this note we indicate a direct proof of the Lefschetz fixed point formulas of Atiyah, Bott, Segal, and Singer for isometries.

Consider the standard setup of the Dirac operator D acting on sections of the spin bundle associated with a compact and smooth spin manifold M. Thinking of $D^2 = \Delta$ as a perturbed Laplacian via the Lichnerowicz formula, we are led to consider the parametrix $H_N(t, x, y)$ of Δ given by

$$H_N(t,x,y) = \frac{\exp\left(\frac{-\rho^2(x,y)}{4t}\right)}{(4\pi t)^n}\left(\sum_{i=0}^{N} t^i U^{(i)}(y,x)\right) \tag{1}$$

as a parametrized family of endomorphisms defined in a neighborhood of the diagonal in $M \times M$. Here $2N > 2n = \dim(M)$, $\rho(x,y)$ is the Riemannian distance and $U^{(i)}(y,x) : \pi^{-1}(y) \to \pi^{-1}(x)$ are endomorphisms with $U^{(0)}(x,x) = Id$ and

$$\left(\frac{\partial}{\partial t} + \Delta\right) H_N(t,\cdot,y)v = -\frac{\exp\left(\frac{-\rho^2}{4t}\right)}{(4\pi t)^n} t^N \Delta U^{(N)}(y,\cdot)v. \tag{2}$$

It was precisely this construction that was used by Patodi [5] in the case of the de Rham complex as a means of matching the local asymptotics, as t approaches zero, to evaluate the Euler-Poincaré characteristic in terms of the Chern polynomial. In [6], this program was carried through for the spin complex using the above parametrix and a detailed analysis of the associated Clifford asymptotics.

The first observation to make in such an approach is that a quick asymptotic match gives the order of the term contributing to the final answer. For the spin complex, this is of course the observation that only the term of top order $2n$ in the generators $\{e_i\}$ of the Clifford algebra $C_{2n} = \mathrm{End}(\mathcal{S}_+ \oplus \mathcal{S}_-)$ contributes to the evaluation of the index of D. In terms of the parametrix, this reads

$$\int_M Tr_s\, U^{(i)}(x,x)\, dx = 0 \tag{3}$$

for $i < n$, and

$$\mathrm{index}(D) = \left(\frac{1}{4\pi}\right)^n \int_M Tr_s U^{(n)}(x,x)\, dx. \tag{4}$$

The second observation to make is that choosing the geodesic moving frame greatly simplifies the analysis. Thus, working locally in normal coordinates y_i at $x \in M$, let E_{norm} be an orthonormal frame at x which is moved parallelly along geodesics through x, yielding a local frame field. One then identifies x with zero and proceeds with the analysis by taking local Taylor expansions of operators with respect to this frame field and, essentially, matching terms on both sides of equation (1). Toward this end, the following construction is extremely useful. Let multi-indices $\alpha, \beta \in \mathbf{Z}^{2n}$ and $\gamma \in (\mathbf{Z}_2)^{2n}$ be given and define

$$\chi(y^\alpha D_y^\beta e^\gamma) = |\beta| - |\alpha| + |\gamma| \tag{5}$$

when $y^\alpha = y_1^{\alpha_1} \cdots y_{2n}^{\alpha_{2n}}$, $D_y^\beta = (\partial/\partial y_1)^{\beta_1} \cdots (\partial/\partial y_{2n})^{\beta_{2n}}$ and $e^\gamma = e_1^{\gamma_1} \cdots e_{2n}^{\gamma_{2n}}$. Thus, for example, the Lichnerowicz formula then takes the form

$$D^2 = -\sum_i \frac{\partial^2}{\partial y_i^2} + \frac{1}{4} \sum_{i,j,\alpha,\beta} R_{ij\alpha\beta} y_i \frac{\partial}{\partial y_j} e_\alpha e_\beta + \\ + \frac{1}{64} \sum_{i,j,k} \sum_{\alpha_1\alpha_2\alpha_3\alpha_4} y_i y_j R_{ik\alpha_1\alpha_2} R_{kj\alpha_3\alpha_4} e_{\alpha_1} e_{\alpha_2} e_{\alpha_3} e_{\alpha_4} + O(\chi < 2).$$

The point, of course, is that if ϕ (a local section of some bundle) has a zero of order m at x then there is no contribution from the supertrace $Tr_s(y^\alpha D_y^\beta e^\gamma \phi)$ at x in case $\chi(y^\alpha D_y^\beta e^\gamma) - m < 2n$. The above thus provides an efficient scheme for throwing away terms.

Now, let T act as an isometry on M, and assume that the action of dT lifts to an action of $Spin(M)$ which commutes with the $Spin(2n)$ action, thereby inducing a map on cohomology. Then a map T^* is induced as a linear operator on sections of the spin bundle $E = Spin(M) \times_{Spin(2n)} S$. As a heat problem, the evaluation of the Lefschetz number

$$L(T) = Tr\, T^*|_{ker D_+} - Tr\, T^*|_{ker D_-} \tag{6}$$

localizes on the fixed point set $F = \{x | \, Tx = x\}$, which consists of the disjoint union of a finite number of even-dimensional totally geodesic submanifolds $F_1, F_2, \ldots F_r$. There is thus no harm in assuming that $r = 1$. Let ν be the normal bundle of F and $\nu(\epsilon) = \{v \in \nu | \; \|v\| < \epsilon\}$ for $\epsilon > 0$. The bundle ν is invariant under dT and $dT|_\nu$ is nondegenerate.

We denote by $P_t^\pm(x, y) : E_\pm|_y \to E_\pm|_x$ the fundamental solutions for the heat operators $\partial/\partial t + \Delta_\pm$. The standard heat equation argument yields

$$L(T) = \int_M (Tr\, T^* P_t^+(Tx, x) - Tr\, T^* P_t^-(Tx, x))\, dx, \;\; t > 0 \tag{7}$$

where dx is the Riemannian volume element. Denote the integrand by

$$\mathcal{L}(t, x) = Tr\, T^* P_t^+(Tx, x) - Tr\, T^* P_t^-(Tx, x). \tag{8}$$

Then straightforward pseudodifferential operator and parametrix estimates allow us to write

$$L(T) = \int_F L_{\mathrm{loc}}(T)(\xi)\, d\xi \tag{9}$$

where the local Lefschetz number, defined by the limit

$$L_{\mathrm{loc}}(T)(\xi) = lim_{t\to 0} \int_{\nu_\xi(\epsilon)} \mathcal{L}(t, exp\, v)\, dv \quad , \tag{10}$$

exists and is independent of ϵ.

Patodi's parametrix strategy for evaluating the local index is equally appropriate here. However, a more delicate treatment of the Clifford asymptotics is now required near the fixed-point submanifold.

We have already observed that in normal coordinates, and with respect to the geodesic moving frame E_{norm}, the expression of the parametrix as a parametrized family of endomorphisms is particularly tractable. The next important observation is that in what we call "orthogonal" coordinates near F the action of our isometry T has a particularly nice form. Such coordinates are expressed in terms of geodesics in F and transversals normal to F; and, as such, they trivialize the normal bundle. In particular, the action of dT is constant along fibers of ν in this trivialization. The key to evaluating the local Lefschetz index lies in relating the geodesic moving frame E_{norm} and the moving frame E_{orthog} obtained from orthogonal coordinates (by moving parallelly along geodesics in F and then along geodesics normal to F) in terms of an infinitesimal holonomy.

Let us now fix $\xi \in F$ and work locally near ξ. Then in our orthogonal coordinates the map dT acts as the identity in directions tangential to F and as, say, $e^{-\Theta(x')}$ in the normal fiber over $x' \in F$, where $\Theta(x') \in so(2n-2n')$ and $2n' = dim(F)$. Of course, we may arrange things so that

$$\Theta(\xi) = \begin{bmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \theta_{n-n'} \\ & & & -\theta_{n-n'} & 0 \end{bmatrix} . \tag{11}$$

It turns out that the infinitesimal holonomy relating the frames E_{norm} and E_{orthog}, expressed as a Lie algebra-valued map $\Phi : U \to so(2n)$ defined in a neighborhood U of ξ by $E_{norm}(x) = E_{orthog}(x)e^{\Phi(x)}$ has the property that for $x = (x', v)$ (in terms of the local trivialization of ν)

$$\Phi_{ij}(x) = -\frac{1}{2} \sum_{\alpha,\beta=1}^{2n-2n'} (ve^{-\Theta(x')})_\alpha v_\beta R_{\alpha+2n',\, \beta+2n',\, i,\, j}(x') + o(|v|^2) \tag{12}$$

where the curvature R is computed with respect to the frame E_{orthog}.

Now, to investigate the Clifford asymptotics, it is best to scale the metric in the normal directions, setting $v = \sqrt{t}w$. It is also helpful to consider a modified χ operator, setting

$$\bar{\chi}\left(\phi(t)e_{i_1}\cdots e_{i_s}\right) = s - sup\{k \in \mathbf{Z} \mid \lim_{t\to 0^+} \frac{|\phi(t)|}{t^{k/2}} < \infty\} \tag{13}$$

for a monoid $\phi(t)e_{i_1}\cdots e_{i_s}$ with real-valued ϕ.

The notation is simplified by letting $\bar{P}_t(x), \bar{T}^*(x) \in \mathrm{Hom}(\mathcal{S}_\pm, \mathcal{S}_\pm)$ be defined locally through the equivalence relations

$$P_t(Tx,x)[(\sigma(x),v)] = [(\sigma(Tx),\bar{P}_t(x)v] \tag{14}$$

and

$$T^*[(\sigma(Tx),u)] = [(\sigma(x),\bar{T}^*(x)u]. \tag{15}$$

so that

$$\mathcal{L}(t,x) = Tr\,\bar{T}^*(x)\bar{P}_t(x)|_{\mathcal{S}_+} - Tr\,\bar{T}^*(x)\bar{P}_t(x)|_{\mathcal{S}_-} = Tr_s\bar{T}^*\bar{P}(x)\,. \tag{16}$$

It may then be shown that

$$\begin{aligned}\bar{T}^*(x) = (-1)^{n-n'}(\prod_{\alpha=1}^{n-n'} \sin\frac{\theta_\alpha}{2}\exp\{-t/4\sum_{\alpha,\beta=1}^{2n-2n'} b_\alpha b_\beta(e^{-\Theta(\xi)}A^\perp)_{\alpha\beta}\})e_{2n'+1}\cdots e_{2n}+ \\ + O(\bar{\chi}<2(n-n')).\end{aligned} \tag{17}$$

where $A^\perp$ is the $(2n-2n')\times(2n-2n')$ matrix whose (α,β) element is given by

$$(A^\perp)_{\alpha\beta} = -\frac{1}{2}\sum_{i,j=1}^{2n'} R_{\alpha+2n',\,\beta+2n',\,i,\,j}(\xi)e_ie_j. \tag{18}$$

($A^\top$ is defined in the obvious analogous fashion, replacing $\alpha+2n'$ and $\beta+2n'$ by indices ranging between 1 and $2n'$.) Finally, let $\tilde{A}$ be the $2n\times 2n$ matrix given by

$$\tilde{A}_{ij} = -\frac{1}{2}\sum_{k,l=1}^{2n}\tilde{R}_{ijkl}e_ke_l, \tag{19}$$

where $\tilde{R}_{ijkl}$ are the components of the Riemannian curvature tensor now computed with respect to the frame field E_{norm}, and set

$$\tilde{A}^k(y) = \sum_{i,j=1}^{2n} y_iy_j(\tilde{A}^k)_{ij},\quad k=1,2,\ldots \tag{20}$$

Then there is a operator $P(t;z_1,z_2,\ldots;w_1,w_2,\ldots)$ which is a power series in t with coefficients polynomials in z_i and w_i such that

$$\begin{aligned}\bar{P}_t(x) = \frac{\exp\frac{-\rho(x,Tx)^2}{4t}}{(4\pi t)^n}P(t;Tr\tilde{A}^2,\ldots,Tr\tilde{A}^{2k},\ldots,Tr\tilde{A}^{2n};\tilde{A}^2(y),\ldots,\tilde{A}^{2k}(y),\ldots,\tilde{A}^{2n}(y)) \\ + \sum_{m\geq 0} t^mO(\bar{\chi}<2m)\,.\end{aligned} \tag{21}$$

Furthermore, in diagonal form we have, by solving harmonic oscillator-type equations,

$$\begin{aligned}&P(t;((-1)^k 2(u_1^{2k}+\cdots+u_n^{2k}));((-1)^k\sum_{\alpha=1}^{n}(v_{2\alpha-1}^2+v_{2\alpha}^2)u_\alpha^{2k})) = \\ &= (4\pi t)^n\, e^{\|v\|^2/4t}\prod_{\alpha=1}^{n}\left(\frac{iu_\alpha}{8\pi\sinh\frac{iu_\alpha t}{2}}\exp\left(\frac{-iu_\alpha}{8}(v_{2\alpha-1}^2+v_{2\alpha}^2)\coth\frac{iu_\alpha t}{2}\right)\right).\end{aligned} \tag{22}$$

These then, are all of the pieces necessary to a final calculation of the local Lefschetz index. It remains simply to note that to compute the supertrace it suffices to compute the coefficient of $e_1 \cdots e_{2n}$ and that since $e_i e_j = -e_j e_i + O(\bar{\chi} < 1)$, if we formally replace e_i by ω_i, where $\omega = (\omega_1, \ldots, \omega_{2n})$ is the frame dual to E_{norm}, and then substitute the associated forms $\Omega^\top$ and $\Omega^\perp$ for $A^\top$ and $A^\perp$, where

$$\begin{aligned} \Omega^\top &= -\frac{1}{2} \sum_{k,l=1}^{2n} R_{ijkl} \omega_k \wedge \omega_l \qquad 1 \le i, j \le 2n' \\ \Omega^\perp &= -\frac{1}{2} \sum_{k,l=1}^{2n} R_{ijkl} \omega_k \wedge \omega_l \qquad 2n' + 1 \le i, j \le 2n' \end{aligned} \tag{23}$$

then computing the supertrace is equivalent to computing the form of the top order $2n'$ on F, if we multiply by $(\frac{2}{\sqrt{-1}})^n$, which is the so-called Berezin-Patodi constant. (To explain the appearance of this term, simply note that $Tr|_{S_\pm}(e_1 \cdots e_{2n}) = \frac{2^{n-1}}{\sqrt{-1}^n}$.)

Letting $\Omega = \begin{bmatrix} \Omega^\top & 0 \\ 0 & \Omega^\perp \end{bmatrix}$ be given formally as

$$\Omega^\top = \begin{bmatrix} 0 & u_1 & & & \\ -u_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & u_{n'} \\ & & & -u_{n'} & 0 \end{bmatrix} \quad \Omega^\perp = \begin{bmatrix} 0 & v_1 & & & \\ -v_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & v_{n-n'} \\ & & & -v_{n-n'} & 0 \end{bmatrix} \tag{24}$$

where u_i and v_i are indeterminates, a straightforward (however tedious) calculation gives at last that as a $2n'$ form on F,

$$L_{\text{loc}}(T) = \prod_{\alpha=1}^{n'} \frac{u_\alpha/4\pi}{\sinh u_\alpha/4\pi} \left(\prod_{\beta=1}^{n-n'} 2\sinh(\frac{v_\beta}{4\pi} + \frac{\sqrt{-1}\theta_\beta}{2}) \right)^{-1} \tag{25}$$

whence the main result

Theorem. *The Lefschetz number $L(T)$ of the isometry T acting on the spin manifold M is expressed by*

$$L(T) = \sum_i \int_{F_i} [L_{\text{loc}}(T)]_i \tag{26}$$

where in the notation used above

$$\begin{aligned} [L_{\text{loc}}(T)]_i &= \sqrt{\det \frac{\Omega^\top/4\pi}{\sin \Omega^\top/4\pi}} \text{Pf} \left(2\sin(\Omega^\perp/4\pi + \sqrt{-1}\frac{\Theta}{2}) \right)^{-1} \\ &= \hat{A}(TF_i) \left[\text{Pf} \left(2\sin(\Omega/4\pi + \sqrt{-1}\Theta/2) \right) (\nu(F_i)) \right]^{-1} . \end{aligned} \tag{27}$$

Clearly the introduction of a twisting bundle results in only minor changes necessary in the above approach. For details of the above direct and purely elementary geometrical program, as well as comments on related approaches to the local index theory, we refer to [4].

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